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SELF-TUNING CONTROLS OF LINEAR STOCHASTIC SYSTEMS IN PRESENCE OF DRIFT

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This paper deals with self-tuning controls constructed by inserting the estimates for the unknown parameters. The model of linear controlled system (5) containing a constant drift is considered. The unknown parameters are estimated by the least squares method. Recursive formula for the estimate is introduced and a sufficient condition for its consistency is presented. Assuming the consistency the asymptotic distributions of the estimate and of the quadratic functionals are investigated. From the asymptotic distributions the quality of the self-tuning can be assessed. At the end two examples are included for illustration.

1. INTRODUCTION

One of the methods for constructing self-tuning controls consists in expressing the calculated feedback gains as function of the unknown parameters and in substituting for the parameters their on-line estimates. This approach has been named the Principle of Estimation and Control. The monographs [4], [10] and the survey [3] contain the information of the present state of the subject. Generally speaking the approach can be identified with the Certainty Equivalence Principle. The latter is however mostly connected with the separation of the filtering and of the control in linear/quadratic control problems under indirect observation (see [2]) or with the Beyesian method.

The Estimation and Control Principle has been developed using the asymptotic theory of parameter estimation. The verification of the self-tuning property is of primary importance. Since the class of the self-tuning controls is usually extensive, it is advisable to apply additional criteria. By analogy with mathematical statistics asymptotic distribution of the parameter estimates or of certain cost functionals etc. are used to this purpose.

The subject of this paper are self-tuning controls of systems with constant drift. The paper continues the work done in [5], [6], [7]. When investigating controlled systems with known parameters the drift can be eliminated by transforming the variables. Systems with unknown parameters require certain modifications of the basic methods. The modifications exhibit some additional properties of linear systems.

A typical example of the constant drift is the reference input r, function of which is to keep state vector near a prescribed quantity (see Fig. 1).

Let us present the following elementary model. The controlled system in Figure 2 is written in equations for the trajectories as

$$\mathrm{d}X_t = -aX_t\,\mathrm{d}t + \mathrm{d}Y_t + U_t\,\mathrm{d}t\,, \tag{1}$$

$$\mathrm{d}Y_t = -bY_t\,\mathrm{d}t + \mathrm{d}W_t\,. \tag{2}$$

Inserting from the second equation into the first one we obtain

$$\mathrm{d}X_t = -(aX_t + bY_t)\,\mathrm{d}t + U_t\,\mathrm{d}t + \mathrm{d}W_t\,. \tag{3}$$

From (1) it follows

$$Y_t = X_t - X_0 - \int_0^t (-aX_s + U_s)\,\mathrm{d}s + Y_0\,. \tag{4}$$

The integral in (4) is denoted by Z_t, hence,

$$\mathrm{d}Z_t = (-aX_t + U_t)\,\mathrm{d}t\,.$$

Substituting (4) for Y_t in (3) we get

$$\mathrm{d}X_t = -(a + b)\,X_t\,\mathrm{d}t - bZ_t\,\mathrm{d}t + U_t\,\mathrm{d}t + \mathrm{d}W_t + c\,\mathrm{d}t\,,$$

where

$$c = b(Y_0 - X_0)\,.$$

In the case that only X_0 is observable, c is an unknown parameter and represents a constant drift.

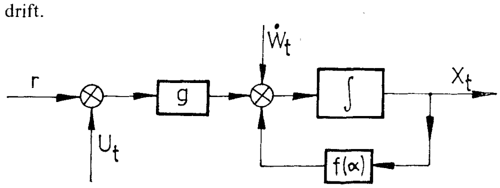

Fig. 1.

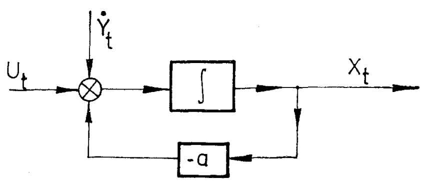

Fig. 2.

2. LEAST SQUARES METHOD

Let us consider the model of linear controlled system

$$\mathrm{d}X_t = f(\alpha)\,X_t\,\mathrm{d}t + e(\alpha)\,\mathrm{d}t + gU_t\,\mathrm{d}t + \mathrm{d}W_t\,, \quad t \geqq 0\,, \tag{5}$$

where

$$\alpha = (\alpha^0, \alpha^1, \ldots, \alpha^q)'$$

is the $(q + 1)$-dimensional vector of parameters,

$$f(\alpha) = f_0 + \alpha^0 f_1\,,$$
$$e(\alpha) = e_0 + \alpha^1 e_1 + \ldots + \alpha^q e_q\,.$$

Let the dimension of X_t and U_t be n and m, respectively. Let f_0, f_1 be $(n \times n)$-matrices, $e_0, e_1, \ldots, e_q$ be n-dimensional vectors and let $e_1, \ldots, e_q$ be linearly independent. g is a constant matrix. $W = \{W_t, t \geqq 0\}$ is the n-dimensional Wiener process with incremental variance matrix h, i.e.

$$\mathsf{E}(W_t - W_s)(W_t - W_s)' = h(t - s)\,, \quad t > s\,.$$

The parameter α is assumed to be unknown to the controller in this paper. The true value of α will be denoted by $\alpha_0 = (\alpha_0^0, \alpha_0^1, \ldots, \alpha_0^q)'$. The estimate α_T^* of α_0 from the observations of X_t, $t \in [0, T]$ is obtained by the least squares method.

To define α_T^* we introduce the discretized version of (5) and minimize the weighted sum of squares

$$\sum_k \frac{1}{\Delta t_k}(\Delta X_{t_k} - f(\alpha)\,X_{t_k}\,\Delta t_k - e(\alpha)\,\Delta t_k - gU_{t_k}\,\Delta t_k)'$$
$$l(\Delta X_{t_k} - f(\alpha)\,X_{t_k}\,\Delta t_k - e(\alpha)\,\Delta t_k - gU_{t_k}\,\Delta t_k)\,, \tag{6}$$

where l is a positively semidefinite symmetric matrix. Equating the derivatives of (6) with respect to α^i, $i = 0, \ldots, q$, to zero we obtain the following relations

$$\sum_k X'_{t_k} f'_1 l f_1 X_{t_k}\,\Delta t_k\,\alpha^0 + \sum_k \sum_{j=1}^q X'_{t_k} f'_1 l e_j\,\Delta t_k\,\alpha^j =$$
$$= \sum_k X'_{t_k} f'_1 l(\Delta X_{t_k} - f_0 X_{t_k}\,\Delta t_k - e_0\,\Delta t_k - gU_{t_k}\,\Delta t_k)\,,$$
$$\sum_k e'_i l f_1 X_{t_k}\,\Delta t_k \alpha^0 + \sum_k \sum_{j=1}^q e'_i l e_j\,\Delta t_k \alpha^j =$$
$$= \sum_k e'_i l(\Delta X_{t_k} - f_0 X_{t_k}\,\Delta t_k - e_0\,\Delta t_k - gU_{t_k}\,\Delta t_k)\,, \quad i = 1, \ldots, q\,.$$

From here we get letting $\Delta t_k \to 0$ the system of equations for α_T^*

$$\int_0^T X'_t f'_1 l f_1 X_t\,\mathrm{d}t\,\alpha_T^{0*} + \sum_{j=1}^q \int_0^T X'_t f'_1 l e_j\,\mathrm{d}t\,\alpha_T^{j*} =$$
$$= \int_0^T X'_t f'_1 l(\mathrm{d}X_t - f_0 X_t\,\mathrm{d}t - e_0\,\mathrm{d}t - gU_t\,\mathrm{d}t)\,, \tag{7}$$

$$\int_0^T e_i' l f_1 X_t \,\mathrm{d}t\, \alpha_T^{0*} + \sum_{j=1}^{q} \int_0^T e_i' l e_j \,\mathrm{d}t\, \alpha_T^{j*} =$$

$$= \int_0^T e_i' l(\mathrm{d}X_t - f_0 X_t \,\mathrm{d}t - e_0 \,\mathrm{d}t - gU_t \,\mathrm{d}t)\,, \quad i = 1, \ldots, q\,.$$

The matrix of system (7) is denoted by A_T. It holds

$$A_T = \int_0^T Z_t' l Z_t \,\mathrm{d}t\,, \tag{8}$$

where

$$Z_t = (f_1 X_t, e_1, \ldots, e_q)'\,.$$

From

$$\mathrm{d}X_t - f_0 X_t \,\mathrm{d}t - e_0 \,\mathrm{d}t - gU_t \,\mathrm{d}t = \alpha_0^0 f_1 X_t \,\mathrm{d}t + \sum_{j=1}^{q} \alpha_0^j e_j \,\mathrm{d}t + \mathrm{d}W_t\,,$$

and from (7) it follows

$$A_T(\alpha_T^* - \alpha_0) = \int_0^T Z_t' l \,\mathrm{d}W_t\,. \tag{9}$$

Next we demonstrate that α_t^* is a recursive estimate. Assume that the matrix A_t is nonsingular for $t \geqq 0$. Set

$$P_t = (A_t)^{-1}\,.$$

From (8) it follows that

$$\mathrm{d}P_t = -P_t Z_t' l Z_t P_t \,\mathrm{d}t\,. \tag{10}$$

Using P_t the solution of equations (7) is expressed as

$$\alpha_T^* = P_T \int_0^T Z_t' l(\mathrm{d}X_t - f_0 X_t \,\mathrm{d}t - e_0 \,\mathrm{d}t - gU_t \,\mathrm{d}t)\,.$$

Differentiating this equality and using (10) we get after rearrangements

$$\mathrm{d}\alpha_t^* = P_t Z_t' l(\mathrm{d}X_t - f(\alpha_t^*)\, X_t \,\mathrm{d}t - e(\alpha_t^*) \,\mathrm{d}t - gU_t \,\mathrm{d}t)\,. \tag{11}$$

The differential $\mathrm{d}(\alpha_t^* - \alpha_0)$ is obtained by addition and subtraction of the term $f(\alpha_0) + e(\alpha_0)$ on the right-hand side of (11),

$$\mathrm{d}(\alpha_t^* - \alpha_0) = P_t Z_t' l(\mathrm{d}W_t - f(\alpha_t^* - \alpha_0)\, X_t \,\mathrm{d}t - e(\alpha_t^* - \alpha_0) \,\mathrm{d}t)\,.$$

(5), (10), and (11) are differential equations for the trajectories of process X_t, the matrix P_t, and the estimate α_t^*.

Applying the recursive least squares estimation method (see [4]) to (6) and then letting $t_k \to 0$ we get the same result.

3. THE SELF-TUNING PROPERTY

Next we investigate the consistency of the estimate α_t^*. We recall equation (5) and assume that a design method, which yields the control U_t in the feedback form

$$U_t = k(\alpha)\, X_t + k_0(\alpha)\,,$$

has been selected. The pole assignment method or optimal stationary controls with respect to quadratic cost can be mentioned as examples.

Since the true value α_0 is unknown, α_0 is replaced by the least squares estimate

α_t^*. This leads to

(12) $$U_t = k(\alpha_t^*)\,X_t + k_0(\alpha_t^*) = k_t^* X_t + k_{0t}^* .$$

The sets $\mathscr{K} = \{k(\alpha), \alpha \in \mathbb{R}^{q+1}\}$, $\mathscr{K}_0 = \{k_0(\alpha), \alpha \in \mathbb{R}^{q+1}\}$ are supposed to be bounded. This can be usually achieved by slightly modifying the design method. To guarantee the stability of the system under control (12) we make a global Liapunov type hypothesis.

Assumption 1. There exists a positively definite matrix z such that the matrices

$$z(f + gk) + (f + gk)'\,z + I\,, \quad k \in \mathscr{K}\,,$$

are negatively semidefinite.

Further some consequences of this assumption are derived.

Considering any nonanticipative control in the form

$$U_t = K_t X_t + K_{0t}\,, \quad K_t \in \mathscr{K}\,, \quad K_{0t} \in \mathscr{K}_0\,,$$

the equation (5) is rewritten as

$$\mathrm{d}X_t = (f + gK_t)\,X_t\,\mathrm{d}t + (e + gK_{0t})\,\mathrm{d}t + \mathrm{d}W_t\,.$$

Let $X_0 = x$. From the Itô formula it follows

(13) $$X_T' z X_T - x' z x = 2\int_0^T X_t' z(f + gK_t)\,X_t\,\mathrm{d}t + \\ + 2\int_0^T X_t' z(e + gK_{0t})\,\mathrm{d}t + 2\int_0^T X_t' z\,\mathrm{d}W_t + T\,\mathrm{tr}\,(zh)\,.$$

Assumption 1 implies that the first term on the right-hand side of (13) is smaller than

$$-\int_0^T |X_t|^2\,\mathrm{d}t\,.$$

Since $|z(e + gK_{0t})|$ is bounded by a constant z_0, we have

(14) $$\mathsf{E}X_T' z X_T + \mathsf{E}\int_0^T |X_t|^2\,\mathrm{d}t - 2z_0\int_0^T |X_t|\,\mathrm{d}t \leqq T\,\mathrm{tr}\,(zh) + x' z x\,.$$

By the Schwarz inequality it follows from (14)

$$\mathsf{E}\frac{1}{T}\int_0^T |X_t|^2\,\mathrm{d}t - 2z_0\left(\frac{1}{T}\int_0^T |X_t|^2\,\mathrm{d}t\right)^{1/2} \leqq \mathrm{tr}\,(zh) + o_p(1)\,.$$

Hence,

(15) $$\mathsf{E}\frac{1}{T}\int_0^T |X_t|^2\,\mathrm{d}t \leqq C_0\,,$$

where

(16) $$C_0 = (z_0^2 + \sqrt{(z_0^2 + \mathrm{tr}\,zh)})^2$$

independently of the control and the initial state.

Analogously from (13) we obtain

$$\frac{1}{T}\left(X_T' z X_T + \int_0^T |X_t|^2\,\mathrm{d}t - 2z_0\int_0^T |X_t|\,\mathrm{d}t\right) \leqq \mathrm{tr}\,(zh) + \frac{1}{T}\left(x' z x + 2\int_0^T X_t' z\,\mathrm{d}W_t\right).$$

(15) implies that the last term on the right-hand side converges to zero a.s. Letting $T \to \infty$ we get

$$(17) \qquad \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T |X_t|^2 \, \mathrm{d}t \leqq C_0 \, .$$

Similar reasoning applied to $(X_T' z X_T)^2$ yields

$$(18) \qquad \lim_{T\to\infty} |X_T|^2/T = 0 \quad \text{a.s.}$$

The relations (17) and (18) are used in the proof of the following proposition.

Proposition 1. Let Assumption 1 holds, let $\sqrt{(l)}\, e_i$, $i = 1, \ldots, q$, be linearly independent. If $f_1 \neq 0$ (i.e. the parameter α_0 is not absent), let $\sqrt{(l)} f_1 \sqrt{(h)} \neq 0$. Then, as $t \to \infty$,

$$\alpha_t^* \to \alpha_0 \quad \text{a.s.}$$

Proof. Equation (9) is multiplied from left by $(\alpha_T^* - \alpha_0)'/T$. This yields

$$(19) \qquad (\alpha_T^* - \alpha_0)' \frac{1}{T} \int_0^T Z_t' l Z_t \, \mathrm{d}t (\alpha_T^* - \alpha_0) = (\alpha_T^* - \alpha_0)' \frac{1}{T} \int_0^T Z_t' l \, \mathrm{d}W_t \, .$$

Denote for $\mu = (\mu^0, \ldots, \mu^q)'$

$$L_T(\mu) = \mu' \frac{1}{T} \int_0^T Z_t' l Z_t \, \mathrm{d}t \, \mu \, .$$

$L_T(\mu)/T$ is a quadratic-linear functional of the trajectory of the form

$$L_T(\mu) = \frac{1}{T} \int_0^T (X_t' \, q(\mu) \, X_t + X_t' \, q_0(\mu)) \, \mathrm{d}t + q_1(\mu) \, ,$$

where

$$(20) \qquad \begin{aligned} q(\mu) &= (\mu^0)^2 f_1' l f_1 \, , \\ q_0(\mu) &= 2\mu^0 f_1' \sqrt{(l)} \, (\sqrt{(l)} \sum_{j=1}^q e_j \mu^j) \, , \\ q_1(\mu) &= (\sqrt{(l)} \sum_{j=1}^q e_j \mu^j)' \, (\sqrt{(l)} \sum_{j=1}^q e_j \mu^j) \, . \end{aligned}$$

If $f_1 = 0$, then $L_T(\mu) = q_1(\mu)$, (34) holds and the proof of the proposition is simple. Let $f_1 \neq 0$. The equation of the system can be considered in the form

$$\mathrm{d}X_t = f(\alpha_0) \, X_t \, \mathrm{d}t + e(\alpha_0) \, \mathrm{d}t + g U_t \, \mathrm{d}t + \mathrm{d}W_t = S_t \, \mathrm{d}t + \mathrm{d}W_t \, .$$

Set for fixed μ

$$Q_T(\mu) = \textstyle\int_0^T (X_t' \, q(\mu) \, X_t + X_t' \, q_0(\mu)) \, \mathrm{d}t + c \int_0^T |S_t|^2 \, \mathrm{d}t \, .$$

We shall deal with the problem of minimizing the average cost $Q_T(\mu)/T$, as $T \to \infty$. The minimum is denoted by $\Theta_c(\mu)$. To obtain $\Theta_c(\mu)$ we solve the stationary Bellman equation

$$(21) \qquad \inf_s \left[\nabla V(y) \, s + \tfrac{1}{2} \operatorname{tr} (h \, \nabla' \nabla V(y)) + y' q y + y' q_0 + c |s|^2 - \Theta_c \right] = 0 \, ,$$

tr denotes the trace operator. The solution of this equation is found in the form

$$V(y) = y'vy + y'v_0 \,. \tag{22}$$

Inserting (22) into (21) we get

$$\inf_s \left[(2y'v + v_0')\, s + \operatorname{tr}(hv) + y'qy + y'q_0 + cs's - \Theta_c\right] = 0 \,. \tag{23}$$

By differentiating the term in the square brackets with respect to s we obtain the optimal value of s

$$s = -(2vy + v_0)/2c \,,$$

which is substituted into (23) again. This yields the following equations for v, v_0, and $\Theta_c(\mu)$,

$$-\,v^2/c + q = 0 \,, \tag{24}$$

$$v_0'v/c + q_0' = 0 \,, \tag{25}$$

$$-\,v_0'v_0/4c + \operatorname{tr}(hv) - \Theta_c = 0 \,. \tag{26}$$

From (24) and (20) it follows that

$$v = \sqrt{(c)}\, |\mu^0|\, (f_1'lf_1)^{1/2} \,.$$

The symmetric matrix $f_1'lf_1$ can be expressed as

$$f_1'lf_1 = Y \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_p & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} Y' = Y\Lambda Y' \,,$$

where Y is an ortogonal matrix built from the characteristic vectors. It holds that

$$(f_1'lf_1)^{1/2} = Y \begin{bmatrix} \sqrt{\lambda_1} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\lambda_p} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} Y' \,.$$

First suppose that $f_1'lf_1$ is nonsingular, i.e. $p = n$. Then from (25)

$$v_0 = \pm\, 2 \sqrt{(c)}\, (f_1'lf_1)^{-1/2} f_1' \sqrt{(l)}\, \Big(\sqrt{(l)} \sum_{j=1}^{q} e_j \mu^j\Big) \,.$$

Hence,

$$v_0'v_0/(4c) = \Big(\sqrt{(l)} \sum_{j=1}^{q} e_j \mu^j\Big)' \sqrt{(l)}\, f_1 (f_1'lf_1)^{-1} f_1' \sqrt{(l)} \,.\, \Big(\sqrt{(l)} \sum_{j=1}^{q} e_j \mu^j\Big) = q_1(\mu) \,.$$

Let $f_1'lf_1$ be singular. To prove the solvability of (25) multiply vv_0 from the left by the ith column of Y,

$$y^{i\prime} v v_0 = \sqrt{(c)}\, |\mu^0|\, \sqrt{(\lambda_i)}\, y^{i\prime} v_0 .$$

From (25) and (20) it follows

$$\sqrt{(c)}\, |\mu^0|\, \sqrt{(\lambda_i)}\, y^{i\prime} v_0 = -2\mu^0 c b_i , \tag{27}$$

where

$$b_i = y^{i\prime} f_1' \sqrt{(l)} \left(\sqrt{(l)} \sum_{j=1}^{q} e_j \mu^j\right).$$

If $\lambda_i = 0$, then $b_i = 0$. Hence,

$$v_0 = \pm \sum_{i=1}^{p} 2 \sqrt{(c)} \frac{1}{\lambda_i} b_i y^i$$

is a solution of (25) and

$$v_0' v_0/(4c) = \sum_{i=1}^{p} \frac{b_i^2}{\lambda_i} .$$

Limit passage using nonsingular matrices enables us to deduce from (27) the inequality

$$v_0' v_0/(4c) \leqq q_1(\mu) .$$

Finally from (26) it follows that

$$\Theta_c(\mu) + q_1(\mu) \geqq \operatorname{tr}(hv) = \sqrt{(c)}\, |\mu^0|\, \operatorname{tr}\left(h(f_1' l f_1)^{1/2}\right). \tag{28}$$

Since $\sqrt{(l)} f_1' \sqrt{(h)} \neq 0$, the following matrices are nonzero

$$\sqrt{(h)}\,(f_1' l f_1)\sqrt{(h)}\,, \quad \sqrt{(h)}\,(f_1' l f_1)^{1/4}\,, \quad \sqrt{(h)}\,(f_1' l f_1)^{1/2}\sqrt{(h)} .$$

The trace of the last matrix equals $\operatorname{tr}\left(h(f_1' l f_1)^{1/2}\right)$, which consequently is positive.

Denote by $\varphi(y, s)$ the term in the square brackets in (21). According to (21) $\varphi(y, s) \geqq 0$ for $s \in \mathbb{R}^n$. The Itô formula gives

$$\int_0^T \mathrm{d}V(X_t) = \int_0^T \varphi(X_t, S_t)\,\mathrm{d}t + \int_0^T \nabla V(X_t)\,\mathrm{d}W_t - Q_T + \Theta_c T .$$

Hence,

$$\frac{1}{T} Q_T - \Theta_c \geqq \frac{1}{T}\left(V(X_0) - V(X_T) + \int_0^T \nabla V(X_t)\,\mathrm{d}W_t\right). \tag{29}$$

The right-hand side of (29) converges to zero a.s. in virtue of (17) and (18). It results

$$\varliminf_{T\to\infty} Q_T/T \geqq \Theta_c . \tag{30}$$

It follows from (30) that

$$\varliminf_{T\to\infty} L_T(\mu) \geqq \Theta_c(\mu) + q_1(\mu) - c \varlimsup_{T\to\infty} \frac{1}{T}\int_0^T |S_t|^2\,\mathrm{d}t .$$

Since (17) holds,

$$\varlimsup_{T\to\infty} \frac{1}{T}\int_0^T |S_t|^2\,\mathrm{d}t \leqq C_1$$

for some constant C_1. Hence,

$$\varliminf_{T\to\infty} L_T(\mu) \geqq \Theta_c(\mu) + q_1(\mu) - cC_1 . \tag{31}$$

We aim to fulfil

$$\varliminf_{T\to\infty} L_T(\mu) \geqq \delta\,, \quad \mu \in \mathbb{R}^{q+1}\,, \quad |\mu| = 1\,, \quad \text{a.s.}\,, \tag{32}$$

for a $\delta > 0$.

If $|\mu^0| = 0$, then $q(\mu) = q_0(\mu) = 0$ and

$$L_T(\mu) = q_1(\mu) \geqq \delta_1\,, \quad |\mu| = 1\,, \quad \text{with} \quad \delta_1 > 0$$

as follows from (20) and from the assumption that $\sqrt{(l)}\, e_i$, $i = 1, \ldots, q$, are linearly independent. Relation (17) implies the uniform continuity of $L_T(\mu)$ in μ, $|\mu| = 1$. This gives the validity of $L_T(\mu) \geqq \frac{1}{2}\delta_1$, $T > 0$, for μ, $|\mu| = 1$, with $|\mu^0|$ sufficiently small, i.e. $|\mu^0| \leqq \gamma$, $\gamma > 0$. From (31) and (28)

$$\varliminf_{T\to\infty} L_T(\mu) \geqq \sqrt{(c)}\, |\mu^0| \operatorname{tr}\left(h(f_1' l f_1)^{1/2}\right) - cC_1\,. \tag{33}$$

For $|\mu^0| > \gamma$ there exists $c > 0$ such that the right-hand side of (33) is positive. Using the uniform continuity of $L_T(\mu)$ in μ, $|\mu| = 1$, we get (32). Hence,

$$\varliminf_{T\to\infty} (\alpha_T^* - \alpha_0)' \frac{1}{T}\int_0^T Z_t' l Z_t \,\mathrm{d}t\, (\alpha_T^* - \alpha_0) \geqq \delta |\alpha_T^* - \alpha_0|^2 \text{ a.s.} \tag{34}$$

From (19) it follows

$$\varliminf_{T\to\infty} \left|\frac{1}{T}\int_0^T Z_t' l \,\mathrm{d}W_t\right| \geqq \delta |\alpha_T^* - \alpha_0|\,.$$

On the other hand

$$\frac{1}{T}\int_0^T Z_t' l \,\mathrm{d}W_t$$

converges to zero a.s., as $T \to \infty$, provided (17) holds, as it is seen by expressing the integral by means of a random time change in a Wiener process. Hence,

$$\alpha_t^* \to \alpha_0 \quad \text{a.s.}, \quad t \to \infty\,,$$

is a consequence of (34). □

4. ASYMPTOTIC RESULTS

In this section the limit distribution of the estimate and of quadratic functionals will be derived.

Assume first that the true value α_0 is known and consider the control in the form

$$U_t = k(\alpha_0)\, X_t + k_0(\alpha_0) = kX_t + k_0\,. \tag{35}$$

Then (5) becomes

$$\mathrm{d}X_t = (f + gk)\, X_t \,\mathrm{d}t + (e + gk_0)\,\mathrm{d}t + \mathrm{d}W_t\,. \tag{36}$$

Provided that the matrix $f + gk$ is stable, X_t has as $t \to \infty$ asymptotically normal distribution $N(m, v)$, where m and v fulfil the following equations

(37) $$(f + gk)\, m + (e + gk_0) = 0\,,$$

(38) $$(f + gk)\, v + v(f + gk)' + h = 0\,.$$

Set

$$\bar{X}_t = X_t - m\,.$$

From (36) it follows

(39) $$\mathrm{d}\bar{X}_t = (f + gk)\, \bar{X}_t\, \mathrm{d}t + \mathrm{d}W_t\,.$$

In the case that the true value α_0 is unknown, (12) is used, i.e.

(40) $$U_t = k(\alpha_t^*)\, X_t + k_0(\alpha_t^*) = k_t^* X_t + k_{0t}^*\,.$$

(5) can be then rewritten as

(41) $$\mathrm{d}\bar{X}_t = (f + gk_t)\, \bar{X}_t\, \mathrm{d}t + g((k_t^* - k)\, m + (k_{0t}^* - k_0))\, \mathrm{d}t + \mathrm{d}W_t\,.$$

Note that (39) is the limit case of (41) for $k_t^* \to k$ and $k_{0t}^* \to k_0$ as $t \to \infty$.

Next we shall study the asymptotic behaviour of quadratic cost

(42) $$C_T = \int_0^T (\bar{X}_t' c \bar{X}_t + \bar{X}_t' c_0)\, \mathrm{d}t$$

as $T \to \infty$. When X_t is used instead of $\bar{X}_t$, the cost can be transformed to the form (42) up to an additive constant. The mean of the integrand in (42) with respect to the limit distribution $N(0, v)$ is

$$\mathsf{E}_\infty(\bar{X}_t' c \bar{X}_t + \bar{X}_t' c_0) = \operatorname{tr}(vc)\,.$$

Denote by $\Theta = \operatorname{tr}(vc)$ the limiting average cost.

The cost potential for initial state $X_0 = x$ and control (35) has the expression

$$P_x = \int_0^\infty \mathsf{E}_x(\bar{X}_t' c \bar{X}_t + \bar{X}_t' c_0)\, \mathrm{d}t\,.$$

It can be proved that

$$P_x = \bar{x}' w \bar{x} + \bar{x}' w_0 + \text{const.}\,,$$

where $\bar{x} = x - m$, and w and w_0 fulfil the equations

(43) $$w(f + gk) + (f + gk)'\, w + c = 0\,,$$

(44) $$(f + gk)'\, w_0 + c_0 = 0\,.$$

We shall need the following equation for investigating the asymptotic behaviour of C_T as $T \to \infty$.

Lemma 1. For any nonanticipative control of the form $U_t = K_t X_t + K_{0t}$ it holds

(45) $$C_T - T\Theta + \bar{X}_T' w \bar{X}_T + \bar{X}_T' w_0 - \bar{x}' w \bar{x} - \bar{x}' w_0 - \int_0^T (2w\bar{X}_t + w_0)'\, g((K_t - k)\, \bar{X}_t + (K_t - k)\, m + (K_{0t} - k_0))\, \mathrm{d}t = \int_0^T (2w\bar{X}_t + w_0)'\, \mathrm{d}W_t\,.$$

Proof. Using the Itô formula and the relation (41) for $U_t = K_t X_t + K_{0t}$ we get

$$\overline{X}'_T w \overline{X}_T - \bar{x}' w \bar{x} = \int_0^T \mathrm{d}\overline{X}'_t w \overline{X}_t =$$
$$= 2 \int_0^T \overline{X}'_t w((f + gK_t)\, \overline{X}_t + g(K_t - k)\, m + g(K_{0t} - k_0))\, \mathrm{d}t +$$
$$+ 2 \int_0^T \overline{X}'_t w \,\mathrm{d}W_t + T \operatorname{tr}(wh)\,.$$

Analogously

$$\overline{X}'_T w_0 - \bar{x}' w_0 = w'_0 \int_0^T \mathrm{d}\overline{X}_t =$$
$$= w'_0 \int_0^T ((f + gK_t)\, \overline{X}_t + g(K_t - k)\, m + g(K_{0t} - k_0))\, \mathrm{d}t + w'_0 \int_0^T \mathrm{d}W_t\,.$$

Further, it holds

$$\operatorname{tr}(wh) = \operatorname{tr}(vc) = \Theta\,.$$

To obtain this result equation (38) is multiplied from left by the matrix w and equation (43) from right by the matrix v and the trace operator is applied to both equations. Hence,

$$C_T - T\Theta + \overline{X}'_T w \overline{X}_T + \overline{X}'_T w_0 - \bar{x}' w \bar{x} - \bar{x}' w_0 = \tag{46}$$
$$= \int_0^T (\overline{X}' c \overline{X} + \overline{X}' c_0)\, \mathrm{d}t + \int_0^T (2\overline{X}' w + w'_0)\, \mathrm{d}W +$$
$$+ \int_0^T (2\overline{X}' w + w'_0)\, ((f + gK)\, \overline{X} + g(K - k)\, m + g(K_0 - k_0))\, \mathrm{d}t =$$
$$= \int_0^T (\overline{X}' c \overline{X} + 2\overline{X}' w (f + gk)\, \overline{X})\, \mathrm{d}t + \int_0^T (\overline{X}' c_0 + w'_0 (f + gk)\, \overline{X})\, \mathrm{d}t +$$
$$+ \int_0^T (2\overline{X}' w + w'_0)\, g((K - k)\, \overline{X} + (K - k)\, m + (K_0 - k_0))\, \mathrm{d}t +$$
$$+ \int_0^T (2\overline{X}' w + w'_0)\, \mathrm{d}W\,.$$

The first two integrals on the right-hand side of (46) are equal to zero in consequence of relations (43) and (44). This imply the validity of (45). □

Next we return to the system of equations (7) for the estimate α_t^*. Assume the strong consistency of α_t^*, i.e.

$$\alpha_t^* \to \alpha_0 \quad \text{a.s.} \quad \text{as} \quad t \to \infty\,, \tag{47}$$

and make the following hypothesis.

Assumption 2. The matrix $f + gk$ is stable, and $k(\alpha)$ and $k_0(\alpha)$ are continuous at α_0.

Since the Liapunov condition is fulfilled in a neighbourhood of α_0, the results from previous section can be used in the proof that the law of large numbers holds for quadratic functionals (see [2]). Hence, by the law of large numbers A_T/T converges as $T \to \infty$ to the matrix $a = (a_{ij})_{i,j=0,\ldots,q}$ fulfilling

$$a_{00} = \operatorname{tr}(f'_1 l f_1 v) + m' f'_1 l f_1 m\,,$$
$$a_{0i} = a_{i0} = m' f'_1 l e_i\,, \quad i = 1, \ldots, q\,,$$
$$a_{ij} = a_{ji} = e'_i l e_j\,, \qquad i, j = 1, \ldots, q\,.$$

The matrix a is supposed to be nonsingular. Then, according to (9),

$$(\alpha_T^* - \alpha_0) \sim a^{-1} \frac{1}{T} \int_0^T Z_t' l \, \mathrm{d} W_t . \tag{48}$$

Denote for $z \in [0, 1]$

$$ {}^zY_T = ({}^zY_T^0, {}^zY_T^1, \ldots, {}^zY_T^q, {}^zY_T^{q+1})' =$$
$$= \frac{1}{\sqrt{T}} (\int_0^{Tz} X_t' f_1' l \, \mathrm{d}W_t , \int_0^{Tz} e_1' l \, \mathrm{d}W_t, \ldots, \int_0^{Tz} e_q' l \, \mathrm{d}W_t, \int_0^{Tz} (2w\bar{X}_t + w_0)' \, \mathrm{d}W_t)' .$$

We shall study the limit distribution of the process $\{{}^zY_T, z \in [0, 1]\}$ as $T \to \infty$. Consider, e.g. the element

$$ {}^zY_T^0 = \frac{1}{\sqrt{T}} \int_0^{Tz} X_t' f_1' l \, \mathrm{d} W_t = \frac{1}{\sqrt{T}} \mathscr{W}_{T(V_{Tz})/T} = {}^T\mathscr{W}_{V_{Tz}/T} , \tag{49}$$

where ${}^T\mathscr{W}_u$ is a Wiener process and

$$V_{Tz} = \int_0^{Tz} X_t' f_1' l h l f_1 X_t \, \mathrm{d}t$$

by the known representation of Wiener integrals.

V_T/T approaches as $T \to \infty$ the value

$$b = \operatorname{tr}(f_1' l h l f_1 v) + m' f_1' l h l f_1 m$$

according to the law of large numbers. This consideration indicates that the process $\{{}^zY_T^0, z \in [0, 1]\}$ converges weakly as $T \to \infty$ to the Wiener process fulfilling $(\mathrm{d}Z)^2 = b \, \mathrm{d}t$. When we investigate all the vector ${}^zY_T$, $z \in [0, 1]$, we take linear combination of its elements and prove using the same consideration that the process $\{{}^zY_T, z \in [0,1]\}$ converges weakly as $T \to \infty$ to the $(q + 2)$-dimensional Wiener process

$$(\mathscr{W}_z^0, \mathscr{W}_z^1, \ldots, \mathscr{W}_z^q, \mathscr{W}_z^{q+1})' = (\tilde{\mathscr{W}}_z', \mathscr{W}_z^{q+1})', z \in [0, 1] , \tag{50}$$

with incremental variance matrix

$$\begin{pmatrix} b & p \\ p & d \end{pmatrix},$$

where

$$d = 4 \operatorname{tr}(w h w' v) + w_0' h w ,$$

p is the $(1 + q)$-dimensional vector with elements

$$p_0 = 2 \operatorname{tr}(v f_1' l h w) + m' f_1' l h w_0 ,$$
$$p_i = e_i' l h w_0 , \quad i = 1, \ldots, q ,$$

and b is the $((1 + q) \times (1 + q))$-matrix, the elements of which are

$$b_{00} = \operatorname{tr}(f_1' l h l f_1 v) + m' f_1' l h l f_1 m ,$$
$$b_{0i} = b_{i0} = m' f_1' l h l e_i , \quad i = 1, \ldots, q ,$$
$$b_{ij} = b_{ji} = e_i' l h l e_j , \qquad i, j = 1, \ldots, q .$$

Using this limit relation for ${}^zY_T$ and using (48) we get the following proposition.

Proposition 2. Let Assumption 2 and (47) hold, and let the matrix a be nonsingular. Then $\sqrt{(T)}\,(\alpha_T^* - \alpha_0)$ has asymptotic distribution $\mathrm{N}(0, a^{-1}ba^{-1})$ as $T \to \infty$.

Next the equation (45) is used for C_{Tz}, i.e.

$$C_{Tz} - \Theta Tz = \int_0^{Tz} (2w\bar{X}_t + w_0)'\, g((k_t^* - k)\,\bar{X}_t + (k_t^* - k)\,m + (k_{0t}^* - k_0))\,\mathrm{d}t + \int_0^{Tz} (2w\bar{X}_t + w_0)'\,\mathrm{d}W_t + o_p(\sqrt{T})\,.$$

The integrand of the first integral on the right-hand side is denoted by I_1. Provided that the functions $k(\alpha)$, $k_0(\alpha)$ are twice continuously differentiable at α_0, we can use their Taylor development at α_0. Set

$$\frac{\partial}{\partial\alpha^i}\,k(\alpha_0) = k^i\,,\quad \frac{\partial}{\partial\alpha^i}\,k_0(\alpha_0) = k_0^i\,,\quad i = 0, \ldots, q\,,$$

and

(51) $$u_i(\bar{X}) = (2w\bar{X} + w_0)'\,g(k^i\bar{X} + k^i m + k_0^i)\,,\quad i = 0, \ldots, q\,.$$

Then

$$I_1 = \sum_{i=0}^{q} u_i(\bar{X}_t)\,(\alpha_t^{i*} - \alpha_0^i) + o_p((|\bar{X}_t|^2 + 1)\,|\alpha_t^* - \alpha_0|^2)\,,$$

and hence,

$$C_{Tz} - \Theta Tz = \sum_{i=0}^{q} \int_0^{Tz} \frac{1}{t} \int_0^t u_i(\bar{X}_s)\,\mathrm{d}s\,(\alpha_t^{i*} - \alpha_0^i)\,\mathrm{d}t + \sqrt{(T)}\,{}^zY_T^{q+1} + o_p(\sqrt{T})\,.$$

Using the substitution $t = yT$ we get after rearrangements

$$C_{Tz} - \Theta Tz = \sum_{i=0}^{q} \int_0^{z} \frac{1}{y} \int_0^{yT} u_i(\bar{X}_s)\,\mathrm{d}s\,(\alpha_{yT}^{i*} - \alpha_0^i)\,\mathrm{d}y + \sqrt{(T)}\,{}^zY_T^{q+1} + o_p(\sqrt{T})\,.$$

From (48) it follows

$$\sqrt{(T)}\,(\alpha_{Tz}^{*i} - \alpha_0^i) \sim j_i\,a^{-1}\frac{1}{z}\,{}^z\tilde{Y}_T\,,$$

where ${}^z\tilde{Y}_T = ({}^zY_T^0, \ldots, {}^zY_T^q)'$ and j_i is the row vector having 1 at ith position and 0 elsewhere. Hence,

$$\frac{1}{\sqrt{T}}(C_{Tz} - \Theta Tz) = \sum_{i=0}^{q} \int_0^{z} \frac{1}{Tt} \int_0^{tT} u_i(\bar{X}_s)\,\mathrm{d}s\,\sqrt{(T)}\,(\alpha_{tT}^{i*} - \alpha_0^i)\,\mathrm{d}t + {}^zY_T^{q+1} + o_p(1) =$$

$$= \sum_{i=0}^{q} \int_0^{z} \frac{1}{Tt} \int_0^{tT} u_i(\bar{X}_s)\,\mathrm{d}s\,\frac{1}{t}\,j_i\,a^{-1}\,{}^t\tilde{Y}_T\,\mathrm{d}t + {}^zY_T^{q+1} + o_p(1)\,.$$

From (51) applying the law of large numbers it can be established that

$$\frac{1}{T}\int_0^T u_i(\bar{X}_s)\,\mathrm{d}s\,,\quad i = 0, \ldots, q\,,$$

approaches as $T \to \infty$ the value

$$r^i = 2\,\mathrm{tr}\,(wgk^iv) + w_0'gk^im + w_0'gk_0^i\,,\quad i = 0, \ldots, q\,.$$

This yields using (49) and (50) that $(C_{Tz} - \Theta Tz)/\sqrt{T}$ converges weakly as $T \to \infty$ to

$$\int_0^z \frac{1}{t} \sum_{i=0}^{q} r^i j_i a^{-1} \widetilde{\mathscr{W}}_t \, \mathrm{d}t + \mathscr{W}_z^{q+1} .$$

Set $r = (r^0, r^1, \ldots, r^q)'$. From the above consideration the following proposition can be formulated.

Proposition 3. Let the matrix a be nonsingular and let the functions $k(\alpha)$, $k_0(\alpha)$ be twice continuously differentiable at α_0. Assume

$$U_t = k(\alpha_t^*) X_t + k_0(\alpha_t^*) , \quad t \geqq 0 ,$$

where α_t^* is the least squares estimate of α_0 satisfying

$$\lim_{t \to \infty} \alpha_t^* = \alpha_0 \quad \text{a.s.}$$

Then the distribution of the process $\{(C_{Tz} - \Theta Tz)/\sqrt{T}, z \in [0, 1]\}$ converges weakly as $T \to \infty$ to the distribution of

$$\int_0^z \frac{1}{t} Z_t^1 \, \mathrm{d}t + Z_z^2 , \quad z \in [0, 1] ,$$

where $Z = \{(Z_t^1, Z_t^2), t \in [0,1]\}$ is the two-dimensional Wiener process with incremental variance matrix

$$\begin{pmatrix} r'a^{-1}ba^{-1}r & p'a^{-1}r \\ p'a^{-1}r & d \end{pmatrix} .$$

5. EXAMPLES

5.1. Elimination of the drift

We shall consider the model of linear controlled system

$$\mathrm{d}X_t = fX_t \, \mathrm{d}t + e(\alpha) \, \mathrm{d}t + U_t \, \mathrm{d}t + \mathrm{d}W_t , \quad t \geqq 0 ,$$

where

$$e(\alpha) = e_0 + e_1\alpha^1 + \ldots + e_q\alpha^q .$$

Assume that α_0, the true value of parameter $\alpha = (\alpha^1, \ldots, \alpha^q)'$, is unknown. The least squares estimate α_t^* satisfies the following system of equations

$$(52) \qquad \sum_{j=1}^{q} \int_0^T e_i' l e_j \, \mathrm{d}t \, \alpha_T^{j*} = \int_0^T e_i' l(\mathrm{d}X_t - fX_t \, \mathrm{d}t - e_0 \, \mathrm{d}t - U_t \, \mathrm{d}t) , \; i = 1, \ldots, q.$$

Denote by a a $(q \times q)$-matrix with elements

$$a_{ij} = e_i' l e_j , \quad i, j = 1, \ldots, q ,$$

and suppose that a is nonsingular. From (52) we obtain

$$(53) \qquad \sqrt{(T)}(\alpha_T^* - \alpha_0) = a^{-1}(e_1, \ldots, e_q)' \, lW_T/\sqrt{(T)} = a^{-1}e'lW_T/\sqrt{(T)} .$$

Since $W_T/\sqrt{T}$ has distribution $\mathrm{N}(0, h)$, it holds

$$\sqrt{(T)}\,(\alpha_T^* - \alpha_0) \sim \mathrm{N}(0, a^{-1}e'lhlea^{-1})\,.$$

To offset the drift $e(\alpha)$ we introduce the control in the form $U_t = -e(\alpha_t^*)$. Hence,

$$\mathrm{d}X_t = fX_t\,\mathrm{d}t + e(\alpha_0)\,\mathrm{d}t - e(\alpha_t^*)\,\mathrm{d}t + \mathrm{d}W_t\,.$$

Using the matrix $F(t) = \exp(tf)$ and the relation (53) for $(\alpha_T^* - \alpha_0)$ the expression for X_t is obtained in the form

$$X_t = F(t)\,(X_0 + \textstyle\int_0^t F(s)^{-1}\,(f + bs^{-1})\,W_s\,\mathrm{d}s) + W_t\,,$$

where

$$b = -ea^{-1}e'l\,.$$

Computation of the variance matrix

$$q(t) = \mathsf{E}(X_t - \mathsf{E}X_t)\,(X_t - \mathsf{E}X_t)'$$

yields

$$q(t) = \tilde{q}(t) - q_1/t + o(t^{\delta-2})\,, \quad t \to \infty\,, \quad \delta > 0\,,$$

where $\tilde{q}(t)$ denotes the variance matrix of $\tilde{X}_t$ fulfilling

$$\mathrm{d}\tilde{X}_t = f\tilde{X}_t\,\mathrm{d}t + \mathrm{d}W_t\,,$$

and q_1 satisfies the equation

$$fq_1 + q_1f' + (bhb' + bh)\,f^{-1} + f^{-1}(bhb' + bh)' = 0\,.$$

5.2. Recursive model of self-tuning control

Let

$$\mathrm{d}X_t = \alpha fX_t\,\mathrm{d}t + gU_t\,\mathrm{d}t + \mathrm{d}W_t\,, \tag{54}$$

where U_t is one-dimensional. We look for k such that for $U_t = -k'X_t$ the system (54) has a transfer function with beforehand selected poles, i.e.

$$0 = \det(zI - \alpha f + gk') = D(z) = z^n + d_1\,z^{n-1} + \ldots + d_{n-1}z + d_n\,,$$

where $d_1, \ldots, d_n$ are fixed.

According to the Ackermann formula (see [1]) this k has the following expression

$$k' = (0, \ldots, 0, 1)\,(g, \alpha fg, \alpha^2f^2g, \ldots, \alpha^{n-1}f^{n-1}g)^{-1}\,D(\alpha f)\,.$$

After rearrangements we get

$$\begin{aligned} k' &= (0, \ldots, 0, 1)\,(g, fg, f^2g, \ldots, f^{n-1}g)^{-1}\,. \\ &\quad .\,(f^n\alpha + d_1f^{n-1} + \sum_{i=1}^{n-1} d_{i+1}f^{n-1-i}\alpha^{-i})\,. \end{aligned} \tag{55}$$

In the case that the parameter α is unknown, we use the least squares estimate α_t^* fulfilling the recursive relations

$$\begin{aligned} \mathrm{d}\alpha_t^* &= P_tX_t'f'l(\mathrm{d}X_t - \alpha_t^*fX_t - gk_t'X_t\,\mathrm{d}t)\,, \\ \mathrm{d}P &= -P_t^2X_t'f'lfX_t\,\mathrm{d}t\,, \end{aligned}$$

as follows from (11). Applying the Itô formula to $d(\alpha_t^*)^j$ we obtain from (55) the recursive expression for the estimate of the control k

$$dk_t = (0, \ldots, 0, 1)(g, fg, \ldots, f^{n-1}g)^{-1} \,.$$
$$. \left[f^n \, d\alpha_t^* + \sum_{i=1}^{n-1} d_{i+1} f^{n-i-1}\left(i(\alpha_t^*)^{-i-1} \, d\alpha_t^* + \tfrac{1}{2} i(i+1)(\alpha_t^*)^{-i-2} (d\alpha_t^*)^2\right)\right].$$

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