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ON THE CONSISTENCY OF A LEAST SQUARES IDENTIFICATION PROCEDURE*

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Conditions for the convergence of parameter estimates to the true value applicable in self-tuning control models are presented. Persistent excitation property is proved by control theory methods.

1. INTRODUCTION

The paper deals with random processes the trajectory of which fulfills

$$\mathrm{d}X_t = f(\alpha)\,X_t\,\mathrm{d}t + U_t\,\mathrm{d}t + \mathrm{d}W_t\,, \quad t \geqq 0\,. \tag{1}$$

In (1) $W = \{W_t, t \geqq 0\}$ is the n-dimensional Wiener process with incremental variance matrix h,

$$\mathrm{d}W_t\,\mathrm{d}W_t' = h\,\mathrm{d}t\,.$$

Prime denotes the transposition. $U = \{U_t, t \geqq 0\}$ is a random process nonanticipative with respect to W. $f(\alpha)$ denotes an $n \times n$-matrix of the form

$$f(\alpha) = f_0 + \alpha^1 f_1 + \ldots + \alpha^m f_m\,, \quad \alpha = (\alpha^1, \ldots, \alpha^m)' \in \mathbb{R}^m\,.$$

$f_0, f_1, \ldots, f_m$ are given matrices, α is a parameter the true value α_0 of which is to be estimated from the observation of X and U.

The paper continues the research of parameter estimation in linear systems initiated in [2], [5], and shows that the applications of control theory methods to the consistency problems presented in [4] can be developed to obtain explicit results. The methods were extended in [1] to embrace the estimates of the drift parameters.

The least squares estimate of α_0 on the basis of $\{X_t, t \leqq T\}$, $\{U_t, t \leqq T\}$ is denoted by α_T^*. It is defined as follows. Let l be a nonnegative definite symmetric matrix. Heuristically α_T^* is the minimizer of the quadratic functional

$$\int_0^T (\dot{X}_t - f(\alpha)\,X_t - U_t)'\,l(\dot{X}_t - f(\alpha)\,X_t - U_t)\,\mathrm{d}t\,, \tag{2}$$

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where $\dot{X}_t$ denotes the derivative of X_t which in fact does not exist. To improve this we substract from (2)

$$\int_0^T \dot{X}_t' l \dot{X}_t \,\mathrm{d}t$$

which does not depend on α and rewrite the remaining terms as

$$(3) \qquad \int_0^T (f(\alpha) X_t + U_t)' \, l(f(\alpha) X_t + U_t) \,\mathrm{d}t - 2 \int_0^T (f(\alpha) X_t + U_t)' \, l \,\mathrm{d}X_t \,.$$

Equating the derivatives of (3) with respect to α^i to 0 one obtains the linear system of equations

$$(4) \qquad \sum_j \int_0^T X' f_i' l f_j X \,\mathrm{d}t \, \alpha_T^{*j} = \int_0^T X' f_i' l (\mathrm{d}X - f_0 X \,\mathrm{d}t - U \,\mathrm{d}t) \,, \quad i = 1, \ldots, m \,,$$

for $\alpha_T^{*1}, \ldots, \alpha_T^{*m}$. We remark that (4) is a recursive estimation procedure (see [1]).

The estimator α_T^* is *consistent* if $\alpha_T^* \to \alpha_0$ in probability. It is *strongly consistent* if $\alpha_T^* \to \alpha_0$ almost surely (abbreviated a.s.).

2. STATEMENT AND PROOF OF RESULTS

Lemma 1. Let g be an $n \times n$-matrix. If

$$(5) \qquad \frac{1}{T} \int_0^T |X_t|^2 \,\mathrm{d}t \,, \quad T > 0 \,,$$

is bounded in probability (respectively a.s.), then

$$(6) \qquad \lim_{T \to \infty} \frac{1}{T} \int_0^T X_t' g' \,\mathrm{d}W_t = 0 \quad \text{in prob. (respectively a.s.).}$$

Proof. Introduce

$$V_T = \int_0^T X' g' h g X \,\mathrm{d}t \,.$$

The following equation is satisfied

$$\int_0^T X' g' \,\mathrm{d}W = \mathscr{W}_{V_T} \,,$$

where $\{\mathscr{W}_s, s \geqq 0\}$ is a Wiener process. Let (5) be bounded in probability. Choose $\varepsilon > 0$ and find K_ε such that

$$\mathsf{P}(V_T / T \leqq K_\varepsilon) > 1 - \varepsilon \,, \quad T > 0 \,.$$

Then

$$(7) \quad \mathsf{P}\left(\left|\frac{1}{T} \mathscr{W}_{V_T}\right| > \varepsilon\right) \leqq \varepsilon + 2 \, \mathsf{P}\left(\sup_{s \leqq K_\varepsilon T} \mathscr{W}_s > \varepsilon T\right) = \varepsilon + 4 \Phi(-\varepsilon T / \sqrt{(K_\varepsilon T)}) \,,$$

where $\Phi(y)$ is the standardized normal distribution function. The last term in (7) tends to 0 as $T \to \infty$, which proves (6) in probability.

The alternative with a.s. converegence is proved directly using the strong law of large numbers for $\mathscr{W}$. □

Proposition 1. Let the matrices

(8) $$\sqrt{l}\, f_i \sqrt{h}\,, \quad i = 1, \ldots, m\,,$$

be linearly independent where $\sqrt{l}, \sqrt{h}$ is the symmetric square root of l and of h, respectively. If

(9) $$\frac{1}{T}\int_0^T (|X_t|^2 + |U_t|^2)\, \mathrm{d}t\,, \quad T > 0\,,$$

is bounded in probability (respectively a.s.), and

(10) $$\lim_{T\to\infty} |X_T|^2/T = 0 \quad \text{in prob. (respectively a.s.)}\,,$$

then

(11) $$\lim_{T\to\infty} \alpha_T^* = \alpha_0 \quad \text{in prob. (respectively a.s.).}$$

Proof. Inserting (1) with $\alpha = \alpha_0$ into (4) we get

$$\sum_j \int_0^T X' f_i' l f_j X\, \mathrm{d}t(\alpha_T^{*j} - \alpha_0^j) = \int_0^T X' f_i' l\, \mathrm{d}W\,,$$

and hence

(12) $$\sum_{ij} \frac{1}{T}\int_0^T X' f_i'\, l f_j X\, \mathrm{d}t(\alpha_T^{*i} - \alpha_0^i)(\alpha_T^{*j} - \alpha_0^j) = \sum_i \frac{1}{T}\int_0^T X' f_i' l\, \mathrm{d}W(\alpha_T^{*i} - \alpha_0^i)\,.$$

To investigate the left-hand side of (12) take $\mu \in \mathbb{R}^m$, $|\mu| = 1$, and denote

(13) $$p(\mu) = \sum_i \mu^i \sqrt{l}\, f_i\,, \quad q(\mu) = p(\mu)'\, p(\mu)\,.$$

Consequently,

$$\sum_{ij} \frac{1}{T}\int_0^T X' f_i' l f_j X\, \mathrm{d}t\, \mu^i \mu^j = \frac{1}{T}\int_0^T X'\, q(\mu)\, X\, \mathrm{d}t\,.$$

Set $f = f(\alpha_0)$. It can be assumed that f is a stable matrix because without loss of generality it can be replaced by $f - aI$ where I is the unit matrix. Introduce the quadratic functional

(14) $$Q_T(\mu) = \int_0^T X'\, q(\mu)\, X\, \mathrm{d}t + c\int_0^T |U|^2\, \mathrm{d}t$$

where $c > 0$. Consider U as a control process and $Q_T(\mu)$ as a cost functional. The minimum of $\mathsf{E}Q_T$ over all U nonanticipative is obtained by solving a Riccati equation whose limiting form as $T \to \infty$ is

(15) $$wf + f'w - c^{-1}w^2 + q(\mu) = 0$$

where w is nonnegative definite. It follows then

(16) $$\inf_u \{2x'w(fx + u) + x'\, q(\mu)\, x + c|u|^2\} = 0\,, \quad x \in \mathbb{R}^n\,.$$

From (1) and (16) applying the Itô formula to $\int_0^T \mathrm{d}(X'wX)$ it follows that

(17) $$Q_T(\mu) - T\,\mathrm{trace}\,(hw) + X_T' w X_T \geqq 2\int_0^T X'w\, \mathrm{d}W\,.$$

Setting $v = c^{-1}w$ we get from (15)

$$vf + f'v - v^2 + c^{-1}\,q(\mu) = 0\,.$$

From here it follows that

$$v \sim c^{-1/2}\sqrt{q(\mu)}\,, \quad c \to 0+\,. \tag{18}$$

Because of the linear independence of (8), $\sqrt{h}\,q(\mu)\sqrt{h}$ is nonzero, and hence $\sqrt{h}\,q(\mu)^{1/4}$ is nonzero. Consequently,

$$\inf_{|\mu|=1} \operatorname{trace}\,(h\sqrt{q(\mu)}) = \inf_{|\mu|=1} \operatorname{trace}\,(\sqrt{h}\sqrt{q(\mu)}\sqrt{h}) > 0\,.$$

From (18) we deduce that

$$\operatorname{trace}\,(hv) \geqq r/\sqrt{c}\,,$$

or

$$\operatorname{trace}\,(hw) \geqq r\sqrt{c}\,, \tag{19}$$

where $r > 0$ is independent of μ and c, $|\mu| = 1$, $c < 1$.

Let (9) be bounded in probability and let (10) hold in probability. Applying Lemma 1 to the integral in (17) we obtain using (10) and (19) that for $\delta > 0$

$$\lim_{T\to\infty} \mathsf{P}\left(\frac{1}{T}\,Q_T(\mu) \geqq r\sqrt{c} - \delta\right) = 1\,. \tag{20}$$

Using (20) we shall estimate the left-hand side of (12).

Let $\varepsilon > 0$. Find K_ε such that

$$\mathsf{P}\left(\frac{1}{T}\int_0^T (|X|^2 + |U|^2)\,\mathrm{d}t \leqq K_\varepsilon\right) \geqq 1 - \varepsilon\,, \quad T > 0\,. \tag{21}$$

Then

$$\mathsf{P}\left(\left|\frac{1}{T}\int_0^T X'(q(\mu) - q(v))\,X\,\mathrm{d}t\right| \leqq |q(\mu) - q(v)|\,K_\varepsilon,\ \mu, v \in \mathbb{R}^m\right) \geqq 1 - \varepsilon\,. \tag{22}$$

Further fix $c > 0$ such that

$$r\sqrt{c} - cK_\varepsilon - 3\delta > 0\,. \tag{23}$$

Next choose a finite set μ_k, $k = 1, \ldots, N$, $|\mu_k| = 1$, such that

$$\inf_k |q(\mu) - q(\mu_k)|\,K_\varepsilon \leqq \delta \quad \text{whenever} \quad |\mu| = 1\,. \tag{24}$$

By virtue of (20) for $T > T_0$

$$\mathsf{P}\left(\frac{1}{T}\,Q_T(\mu_i) \geqq r\sqrt{c} - \delta,\ i = 1, \ldots, N\right) \geqq 1 - \varepsilon\,,$$

and hence from (14), (21) (23)

$$\mathsf{P}\left(\frac{1}{T}\int_0^T X'\,q(\mu_i)\,X\,\mathrm{d}t \geqq 2\delta,\ i = 1, \ldots, N\right) \geqq 1 - 2\varepsilon\,.$$

(22) and (24) imply the persistent excitation condition (see [3])

$$\mathsf{P}\left(\frac{1}{T}\int_0^T X'\, q(\mu)\, X\, \mathrm{d}t \geqq \delta|\mu|^2,\ \mu \in \mathbb{R}^m\right) \geqq 1 - 2\varepsilon\,. \tag{25}$$

Consequently,

$$\mathsf{P}\left(\sum_{ij} \frac{1}{T}\int_0^T X' f_i' l f_j X\, \mathrm{d}t (\alpha_T^{*i} - \alpha_0^i)(\alpha_T^{*j} - \alpha_0^j) \geqq \delta|\alpha_T^* - \alpha_0|^2\right) \geqq 1 - 2\varepsilon\,. \tag{26}$$

Regarding the right-hand side of (12) we have by Lemma 1 for $T > T_0$

$$\mathsf{P}\left(\left(\sum_i \left(\frac{1}{T}\int_0^T X' f_i' l\, \mathrm{d}W\right)^2\right)^{1/2} \leqq \delta^2\right) \geqq 1 - \varepsilon\,,$$

and hence

$$\mathsf{P}\left(\sum_i \frac{1}{T}\int_0^T X' f_i' l\, \mathrm{d}W (\alpha_T^{*i} - \alpha_0^i) \leqq \delta^2 |\alpha_T^* - \alpha_0|\right) \geqq 1 - \varepsilon\,. \tag{27}$$

From (12), (26), (27) it follows that

$$\mathsf{P}(|\alpha_T^* - \alpha_0| < \delta) \geqq 1 - 3\varepsilon\,, \quad T > T_0\,.$$

Note that δ in (23) can be chosen arbitrarily small. The validity of (11) in probability is thus established.

The boundedness of (9) and the validity of (10) almost surely implies

$$\mathsf{P}\left(\liminf_{T\to\infty} \frac{1}{T}\, Q_T(\mu) \geqq r\sqrt{c}\right) = 1\,.$$

Moreover $T > T_0$ can be added to the events whose probabilities are computed starting with (21) and ending with

$$\mathsf{P}(|\alpha_T^* - \alpha_0| < \delta\,, \quad T > T_0) \geqq 1 - 3\varepsilon\,,$$

which proves the validity of (11) almost surely. □

Assume next that h is singular, $0 < \operatorname{rank} h = s < n$. Renumbering the coordinates if necessary h can be expressed as

$$h = \begin{pmatrix} h^{00}, & h^{01} \\ h^{10}, & h^{11} \end{pmatrix} = (h^0, h^1)\,,$$

where $\operatorname{rank} h^{00} = s$. The same partitioning will be used also for the blocks of other matrices. Recall the definition (13) of $p(\mu)$, $q(\mu)$.

Proposition 2. The implication of Proposition 1 remains valid if

$$\operatorname{rank} p^1(\mu) < \operatorname{rank} p(\mu)\,, \quad \mu \in \mathbb{R}^m\,. \tag{28}$$

Proof. Write $X_t^0 = (X_t^1, \ldots, X_t^s)'$, similarly for U_t^0, W_t^0, and set $V_t = (X_t^{s+1}, \ldots, X_t^n)'$. From (1) it follows

$$\mathrm{d}X_t^0 = f^{00} X_t^0\, \mathrm{d}t + f^{01} V_t\, \mathrm{d}t + U_t^0\, \mathrm{d}t + \mathrm{d}W_t^0\,, \quad t \geqq 0\,.$$

Consider U^0 and V as control processes and proceed as in the proof of Proposition 1. It holds

$$X' q(\mu) X = X^{0\prime} q^{00}(\mu) X^0 + 2V' q(\mu)^{10} X^0 + V' q^{11}(\mu) V .$$

Note that

$$q^{00} = (p^0)' p^0 , \quad q^{10} = (p^1)' p^0 , \quad q^{11} = (p^1)' p^1 .$$

Without loss of generality it can be assumed that q^{11} is nonsingular, i.e., p^1 has linearly independent columns. Otherwise the dimension of V could be reduced. Moreover, let $h^{00} = I$, and let f^{00} be a stable matrix.

Introduce the functional

$$Q_T = \int_0^T (X^{0\prime} q^{00} X^0 + 2V' q^{10} X^0 + V' q^{11} V)\, \mathrm{d}t + c \int_0^T (|U|^2 + |V|^2)\, \mathrm{d}t .$$

We shall demonstrate the analogues of (17) and (19). The rest of the proof follows that of Proposition 1. Writing x, u instead if x^0 u^0 we replace (16) by

$$\inf_{(u,v)} \{2x'w(f^{00}x + f^{01}v + u) + x'q^{00}x + \\ + 2v'q^{10}x + v'q^{11}v + c(|u|^2 + |v|^2)\} = 0 . \tag{29}$$

The minimum of the expression in braces is attained for

$$u = -c^{-1}wx , \quad v = -(cI + q^{11})^{-1} ((f^{01})' w + q^{10}) x .$$

Inserting these values into (29) we obtain

$$x'(2wf^{00} - c^{-1}w^2 + q^{00} - wf^{01}(cI + q^{11})^{-1} (f^{01})' w - \\ - 2wf^{01}(cI + q^{11})^{-1} q^{10} - q^{01}(cI + q^{11})^{-1} q^{10} - \\ - cq^{01}(cI + q^{11})^{-2} (q^{10} + 2(f^{01})' w)) x = 0 . \tag{30}$$

From here we conclude that the asymptotic behaviour of $c^{-1}w$ as $c \to 0+$ depends on the matrix

$$q^{00} - q^{01}(q^{11})^{-1} q^{10} . \tag{31}$$

From

$$\inf_v |p^0 x + p^1 v|^2 = x'(q^{00} - q^{01}(q^{11})^{-1} q^{10}) x$$

it is seen that (28) implies that (31) is a nonzero matrix. Consequently,

$$\inf_{|\mu|=1} \operatorname{trace} (q^{00}(\mu) - q^{01}(\mu) q^{11}(\mu)^{-1} q^{10}(\mu)) > 0 .$$

From this inequality and from (30) it follows that

$$\operatorname{trace}(w) \geqq r \sqrt{c}$$

with $r > 0$. This inequality with the inequality

$$Q_T(\mu) - T \operatorname{trace}(w) + X_T^{0\prime} w X_T^0 \geqq 2 \int_0^T X^{0\prime} w \, \mathrm{d}W$$

enables us to continue as in the proof of Proposition 1. □

Example. A self-tuning control model is described by the equation

$$dX_t = f(\alpha_0) X_t \, dt + k(\alpha_t^*) X_t \, dt + dW_t, \quad t \geqq 0,$$

where $k(\alpha)$ are given feedback gain matrices. Assume that

$$\mathscr{K} = \{k(\alpha), \alpha \in \mathbb{R}^m\}$$

is a bounded set and that the following Liapunov type assumption (see [5]) is fulfilled. There exists a symmetric matrix $z > 0$ such that

$$z(f + gk) + (f + gk)' z + I \leqq 0, \quad k \in \mathscr{K}. \tag{32}$$

The inequalities denote positive definiteness and negative semidefiniteness, respectively. (32) implies (9), (10) and Propositions 1, 2 give sufficient condition for the self-tuning property.

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