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ASYMPTOTIC THEORY OF PARAMETER ESTIMATION FOR GAUSS-MARKOV RANDOM FIELDS

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The simultaneous estimation of the mean value and the spectral density of a Gauss-Markov random field is understood as a parameter estimation problem with a regular parameter family of probability distributions. A method for estimating is proposed and its asymptotic properties are investigated. Namely, the consistency, the asymptotic normality, and the asymptotic efficiency are proved.

0. INTRODUCTION

For the statistical analysis of spatial data on a regular lattice we use the Gauss-Markov random fields to serve as the probability models. Due to Rozanov's [7] result, this natural assumption yields a quite easy form of the spectral densities with a finite number of unknown parameters. Since there is a one-to-one relation between these parameters and a corresponding vector of covariances, we can estimate the unknown parameters via estimating the covariances. The present paper is mainly devoted to deriving the asymptotic properties of the estimate as it is usual in the general theory of statistical estimation. We prove the consistency, the asymptotic normality, and the asymptotic efficiency, which are the typical properties of a maximum likelihood type estimator (though our estimator is not exactly the maximum likelihood one).

The proposed method of estimation does not generalize the techniques of time series analysis, which is quite uneasy in case of higher dimension random fields. Rather, it employs the ideas of the theory of Gibbs random fields as it was developed in frame of statistical mechanics. In spite of the fact that we do not explicitly need any result of this theory and all the steps could be performed without mentioning this connection, many of the results here are obtained by re-formulating the general results concerning statistical analysis of Gibbs random fields (cf. [4]).

For the approach to the Gaussian random fields from the Gibbsian point of view see Dobrushin [1], and Künsch [5] who has already proved some of the results with the aid of a bit different methods. The main improvement of the present paper consists in the assumption of unknown mean value and in the treating the problem of the LAN (locally asymptotic normality) condition and the efficiency.

1. GAUSS-MARKOV RANDOM FIELDS

By random field we mean a stochastic process on a d-dimensional lattice $\mathscr{Z}^d$, its distribution is a probability measure defined on the space $\mathscr{R}^{\mathscr{Z}^d}$ with the appropriate product σ-algebra. If the distribution is translation invariant we call the random field stationary.

Let us denote by $\mathfrak{K} = \{\mathscr{V} \subset \mathscr{Z}^d;\ 0 < |\mathscr{V}| < \infty\}$ the system of all non-void finite subsets of the lattice $\mathscr{Z}^d$ (the symbol $|\mathscr{V}|$ is used for the cardinality of the set $\mathscr{V}$).

A random field is Gaussian if the projections $\mathbf{P}_{\mathscr{V}}$, $\mathscr{V} \in \mathfrak{K}$, of its distribution $\mathbf{P}$ are given by the appropriate finite dimensional normal distributions.

A random field $\{X_t\}_{t\in\mathscr{Z}^d}$ is $\mathscr{L}$-Markov, $0 \notin \mathscr{L} \in \mathfrak{K}$, if for every $\mathscr{V} \in \mathfrak{K}$ the conditional distribution of $X_{\mathscr{V}} = \{X_t\}_{t\in\mathscr{V}}$, given $X_j = x_j$ for $j \notin \mathscr{V}$, depends only on the values x_j for $j \in \partial_{\mathscr{L}}\mathscr{V} = \{k \notin \mathscr{V};\ (k - \mathscr{L}) \cap \mathscr{V} \neq \emptyset\}$, i.e.

$$\mathbf{P}_{\mathscr{V}}(\cdot \mid x_j, j \notin \mathscr{V}) = \mathbf{P}_{\mathscr{V}}(\cdot \mid x_j, j \in \partial_{\mathscr{L}}\mathscr{V}) \quad \text{a.s. } [\mathbf{P}] .$$

A random field is regular if its distribution $\mathbf{P}$, restricted to the tail σ-algebra $\bigcap_{\mathscr{V}\in\mathfrak{K}} \sigma\{X_j, j \notin \mathscr{V}\}$, is a trivial zero-one probability measure.

Rozanov ([7], Theorem 3) proved that a stationary Gaussian random field is $\mathscr{L}$-Markov regular iff its spectral density is given by the expression

$$f(\lambda) = c(1 - \sum_{t\in\mathscr{L}} \alpha(t)\, e^{it\lambda})^{-1}$$

for $\lambda \in [-\pi, \pi]^d = \mathscr{I}_d$ (the scalar product is meant by $t\lambda$).

In this paper we shall deal only with the random fields of such a type.

Since the spectral density is a non-negative real-valued function, we may also write

$$f(\lambda) = c(1 - \sum_{t\in\mathscr{L}} \alpha(t) \cos t\lambda)^{-1} ,$$

and the symmetry $\alpha(t) = \alpha(-t)$ must hold for every $t \in \mathscr{L}$, which implicitly means that the set $\mathscr{L}$ must be symmetric, i.e. $\mathscr{L} = -\mathscr{L}$.

Let us denote $\boldsymbol{a}(0) = c^{-1}$, $\boldsymbol{a}(t) = -c^{-1}\alpha(t)$ for $t \in \mathscr{L}$, $\boldsymbol{a}(t) = 0$ otherwise. Then we can easily see that

$$\boldsymbol{a}(t) = (2\pi)^{-d} \int_{\mathscr{I}_d} e^{-it\lambda} f(\lambda)^{-1}\, d\lambda \quad \text{for every} \quad t \in \mathscr{Z}^d ,$$

i.e. $\{\boldsymbol{a}(t)\}_{t\in\mathscr{Z}^d}$ are the Fourier coefficients of the function reciprocal to the spectral density.

Simultaneously, according to the well-known one-to-one correspondence, the Fourier coefficients of the spectral density f are given by the covariance function, i.e.

$$R(k) = (2\pi)^{-d} \int_{\mathscr{I}_d} e^{ik\lambda} f(\lambda)\, d\lambda = \mathsf{E}[X_{t+k} - \mu][X_t - \mu]$$

for every $k, t \in \mathscr{Z}^d$, where $\mu = \mathsf{E} X_t$ for every $t \in \mathscr{Z}^d$ is the constant mean value.

A direct computation shows that

$$\sum_{t \in \mathscr{Z}^d} \boldsymbol{a}(t)\, \boldsymbol{R}(t + k) = \delta_{0k}$$

holds for every $k \in \mathscr{Z}^d$ (here δ_{0k} is the known Kronecker's symbol), wherefrom we conclude that the infinite matrix $\mathbf{A} = (\boldsymbol{a}(t - s))_{t,s \in \mathscr{Z}^d}$ is inverse to the infinite covariance matrix $\mathbf{R} = (\boldsymbol{R}(t - s))_{t,s \in \mathscr{Z}^d}$.

Using the above results we may express the conditional expectation

$$\mathsf{E}[X_t - \mu \mid X_s, s \neq t] = \sum_{s \in \mathscr{L} + t} \alpha(s - t)[X_s - \mu],$$

the conditional variance

$$\mathsf{E}[X_t - \mu - \sum_{s \in \mathscr{L} + t} \alpha(s - t)[X_s - \mu]]^2 = c,$$

and, finally, the conditional density

$$d\mathbf{P}_t(x_t \mid x_s, s \neq t) =$$

$$= \exp\{-\tfrac{1}{2}\boldsymbol{a}(0)\, x_t^2 - h x_t - x_t \sum_{s \in \mathscr{L} + t} \boldsymbol{a}(t - s)\, x_s\}\, [\mathbf{Z}_t(x_s, s \neq t)]^{-1}\, dx_t,$$

where $h = -\mu \sum_{t \in \mathscr{L}} \boldsymbol{a}(t)$, and $\mathbf{Z}_t(x_s, s \neq t)$ is the appropriate normalizing constant.

The formula above indicates that a stationary Gaussian random field may be understood as a Gibbs field with a finite range pair potential $\boldsymbol{U}$, given by

$$U_{\{t\}}(x_t) = \tfrac{1}{2}\boldsymbol{a}(0)\, x_t^2 + h x_t, \quad t \in \mathscr{Z}^d$$

$$\boldsymbol{U}_{\{t,s\}}(x_t, x_s) = \boldsymbol{a}(t - s)\, x_t x_s, \quad t, s \in \mathscr{Z}^d, \quad t \neq s.$$

This approach to the Gaussian random fields, developed by Dobrushin [1] and partly by Künsch [5], will not be explicitly followed in this paper, nevertheless it seems useful to realize this connection.

2. PARAMETER FAMILY OF GAUSS-MARKOV RANDOM FIELDS

Let us suppose the set $\mathscr{L}$ to be fixed. We denote $\mathscr{M} = \{0\} \cup \{k \in \mathscr{L}; k \succ 0\} \subset \mathscr{Z}^d$, where "$\succ$" is some linear, e.g. the lexicographical ordering.

Then the spectral density may be written in the form

$$f_U(\lambda) = [2 \,.\, \sum_{k \in \mathscr{M}} U(k) \cos k\lambda]^{-1} \quad \text{for every} \quad \lambda \in \mathscr{I}_d,$$

where $U(0) = \frac{1}{2}\boldsymbol{a}(0)$, $U(k) = \boldsymbol{a}(k)$ for $k \in \mathscr{M} \setminus \{0\}$, and for the mean value we have

$$\mu_{h,U} = -[2 \,.\, \sum_{k \in \mathscr{M}} U(k)]^{-1} \,.\, h = -f_U(0) \,.\, h\,.$$

From now, we shall not distinguish between a random field and its distribution and we shall use the term "random field" rather for the distribution.

Thus, every random field under consideration is determined by the $(1 + |\mathcal{M}|)$-dimensional vector parameter $\boldsymbol{\theta} = (h, \boldsymbol{U}) = (h, \{U(k)\}_{k \in \mathcal{M}})$, where $h \in \mathcal{R}$ may be arbitrary and the vector $\boldsymbol{U} \in \mathcal{R}^{|\mathcal{M}|}$ should guarantee a correct definition of the spectral density. In order not to complicate further steps, we assume

$$\boldsymbol{U} \in \mathfrak{D}_{\mathcal{M}} = \{\boldsymbol{U} = \{U(k)\}_{k \in \mathcal{M}} \in \mathcal{R}^{|\mathcal{M}|};\ f_{\boldsymbol{U}}(\lambda)^{-1} = 2 \,.\sum_{k \in \mathcal{M}} U(k) \cos k\lambda > 0 \text{ for every } \lambda \in \mathcal{I}_d\} .$$

The set of considered parameters is therefore given by the Cartesian product $\mathcal{R} \otimes \mathfrak{D}_{\mathcal{M}} = \boldsymbol{\Theta}$ which is an open subset of $\mathcal{R}^{1+|\mathcal{M}|}$ thanks to the strict positivity which is required in the definition of the set $\mathfrak{D}_{\mathcal{M}}$.

For every vector parameter $\boldsymbol{\theta} = (h, \boldsymbol{U}) \in \boldsymbol{\Theta}$ there exists exactly one stationary Gaussian $\mathcal{L}$-Markov random field $\mathbf{P}^{\boldsymbol{\theta}}$ with the mean value $\mu_{h,\boldsymbol{U}}$ and the spectral density $f_{\boldsymbol{U}}$. Thus, within the estimation problem we shall deal with the parameter family

$$\mathbb{P} = \{\mathbf{P}^{\boldsymbol{\theta}}\}_{\boldsymbol{\theta} \in \boldsymbol{\Theta}}$$

of probability distributions.

First of all we shall show that the parameter family $\mathbb{P}$ satisfies a basic regularity condition which makes possible to construct asymptotically optimal statistical procedures.

For fixed $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\boldsymbol{\beta} \in \mathcal{R}^{1+|\mathcal{M}|}$ we denote

$$\mathfrak{K}_{\boldsymbol{\theta},\boldsymbol{\beta}} = \{\mathcal{V} \in \mathfrak{K};\ \boldsymbol{\theta} + |\mathcal{V}|^{-1/2}\, \boldsymbol{\beta} \in \boldsymbol{\Theta}\} .$$

(Note that $\mathfrak{K}_{\boldsymbol{\theta},\boldsymbol{\beta}} \neq \emptyset$ since $\boldsymbol{\Theta}$ being open.) For the simplicity of the notation we shall write

$$\mathbf{P}_{\mathcal{V}}^{\boldsymbol{\theta},\boldsymbol{\beta}} \quad \text{instead of} \quad \mathbf{P}_{\mathcal{V}}^{\boldsymbol{\theta}+|\mathcal{V}|^{-\frac{1}{2}}\boldsymbol{\beta}} , \quad \mathcal{V} \in \mathfrak{K}_{\boldsymbol{\theta},\boldsymbol{\beta}} ,$$

for the corresponding finite dimensional projection.

The parameter family $\mathbb{P}$ is said to be locally asymptotically normal (LAN) if for every fixed $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $\boldsymbol{\beta} \in \mathcal{R}^{1+|\mathcal{M}|}$ it holds

$$\frac{\mathrm{d}\mathbf{P}_{\mathcal{V}}^{\boldsymbol{\theta},\boldsymbol{\beta}}}{\mathrm{d}\mathbf{P}_{\mathcal{V}}^{\boldsymbol{\theta}}} = \exp\{\boldsymbol{\beta}^{\mathrm{T}}\, \Delta_{\mathcal{V}}^{\boldsymbol{\theta}} - \tfrac{1}{2}\, \boldsymbol{\beta}^{\mathrm{T}}\, \boldsymbol{\Gamma}_{\boldsymbol{\theta}} \boldsymbol{\beta} + G_{\mathcal{V}}^{\boldsymbol{\theta}}\} \quad \text{for every } \mathcal{V} \in \mathfrak{K}_{\boldsymbol{\theta},\boldsymbol{\beta}} ,$$

where

$$G_{\mathcal{V}}^{\boldsymbol{\theta}} \to 0 \quad \text{in probability} \quad [\mathbf{P}^{\boldsymbol{\theta}}]$$

and

$$\mathfrak{L}_{\mathbf{P}^{\boldsymbol{\theta}}}(\Delta_{\mathcal{V}}^{\boldsymbol{\theta}}) \Rightarrow \mathbb{N}_{1+|\mathcal{M}|}(0, \boldsymbol{\Gamma}_{\boldsymbol{\theta}}) \quad \text{for} \quad \mathcal{V} \nearrow \mathcal{Z}^d ,$$

i.e. $\Delta_{\mathcal{V}}^{\boldsymbol{\theta}}$ tends in distribution $[\mathbf{P}^{\boldsymbol{\theta}}]$ to the $(1 + |\mathcal{M}|)$-dimensional normal distribution with zero mean and the covariance matrix $\boldsymbol{\Gamma}_{\boldsymbol{\theta}}$.

The convergence $\mathcal{V} \nearrow \mathcal{Z}^d$ is defined to satisfy the condition $|\mathcal{V}|^{-1} .\ |\mathcal{V}_k| \to 1$ for every $k \in \mathcal{Z}^d$, where $\mathcal{V}_k = \mathcal{V} \cap (\mathcal{V} - k)$.

For every $\mathscr{V} \in \mathfrak{K}$ with $\mathscr{V}_k \neq \emptyset$, $k \in \mathscr{M}$, let us define

$$\hat{\mu}_{\mathscr{V}}(X_{\mathscr{V}}) = |\mathscr{V}|^{-1} \sum_{t \in \mathscr{V}} X_t$$

and

$$\hat{M}_{\mathscr{V},k}(X_{\mathscr{V}}) = |\mathscr{V}_k|^{-1} \sum_{t \in \mathscr{V}_k} X_t X_{t+k}\,, \quad k \in \mathscr{M}\,.$$

Obviously

$$\mathsf{E}_{\mathbf{P}^{\boldsymbol{\theta}}}[\hat{\mu}_{\mathscr{V}}] = \mu_{h,\boldsymbol{U}} \quad \text{and} \quad \mathsf{E}_{\mathbf{P}^{\boldsymbol{\theta}}}[\hat{M}_{\mathscr{V},k}] = \boldsymbol{M}_{h,\boldsymbol{U}}(k) = \boldsymbol{R}_{\boldsymbol{U}}(k) + \mu^2_{h,\boldsymbol{U}}\,.$$

Further, for every $k, l \in \mathscr{Z}^d$ let us define

$$\mathbf{W}^{\boldsymbol{U}}(k, l) = (2\pi)^{-d} \int_{\mathscr{I}_d} 2 \,.\, \cos k\lambda \,.\, \cos l\lambda \,.\, [f_{\boldsymbol{U}}(\lambda)]^2 \,\mathrm{d}\lambda = $$
$$= \sum_{t \in \mathscr{Z}^d} \boldsymbol{R}_{\boldsymbol{U}}(t + k)\,[\boldsymbol{R}_{\boldsymbol{U}}(t + l) + \boldsymbol{R}_{\boldsymbol{U}}(t - l)]\,.$$

The latter equality holds due to the absolute convergence of the covariance function: $\sum_{t \in \mathscr{Z}^d} |\boldsymbol{R}_{\boldsymbol{U}}(t)| < \infty$ (cf. Corollary VII.1.9 in [8]) and the known relation $f_{\boldsymbol{U}}(\lambda) = \sum_{t \in \mathscr{Z}^d} \boldsymbol{R}_{\boldsymbol{U}}(t)\, \mathrm{e}^{\mathrm{i}t\lambda}$.

Now, we may introduce the $(1 + |\mathscr{M}|) \times (1 + |\mathscr{M}|)$-matrix

$$\boldsymbol{\Psi}_{\boldsymbol{\theta}} = \begin{pmatrix} f_{\boldsymbol{U}}(0) & -2h\, f_{\boldsymbol{U}}(0)^2 \,.\, \mathbf{1}_{\mathscr{M}} \\ -2h\, f_{\boldsymbol{U}}(0)^2\, \mathbf{1}^{\mathrm{T}}_{\mathscr{M}} & \mathbf{W}^{\boldsymbol{U}}_{\mathscr{M},\mathscr{M}} + (2h)^2 f_{\boldsymbol{U}}(0)^3 \,.\, \mathbf{1}_{\mathscr{M},\mathscr{M}} \end{pmatrix}$$

which will play an important role in what follows. By $\mathbf{1}_{\mathscr{M}}$ and $\mathbf{1}_{\mathscr{M},\mathscr{M}}$ we mean the vector and the matrix, respectively, with constant elements equal to one, while the unit matrix will be denoted by $\mathbf{I}_{\mathscr{M},\mathscr{M}}$.

Lemma 2.1. The matrix $\boldsymbol{\Psi}_{\boldsymbol{\theta}}$ is positive definite for every $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Proof. For every $c \in \mathscr{R}$, $\boldsymbol{b} \in \mathscr{R}^{|\mathscr{M}|}$ we may write

$$(c, \boldsymbol{b})^{\mathrm{T}}\, \boldsymbol{\Psi}_{\boldsymbol{\theta}}(c, \boldsymbol{b}) = f_{\boldsymbol{U}}(0)\,(c - 2h\, f_{\boldsymbol{U}}(0) \,.\, \mathbf{1}^{\mathrm{T}}_{\mathscr{M}}\boldsymbol{b})^2 +$$
$$+ 2 \,.\, (2\pi)^{-d} \int_{\mathscr{I}_d} (\sum_{k \in \mathscr{M}} \boldsymbol{b}_k \cos k\lambda)^2 f_{\boldsymbol{U}}(\lambda)^2 \,\mathrm{d}\lambda \geqq 0$$

while the equality occurs only for $\boldsymbol{b} = 0$, $c = 0$. □

Theorem 2.2. The parameter family $\mathbb{P}$ is LAN with

$$\Delta^{\boldsymbol{\theta}}_{\mathscr{V}} = -|\mathscr{V}|^{1/2}(\hat{\mu}_{\mathscr{V}} - \mu_{h,\boldsymbol{U}}, \{\hat{\boldsymbol{M}}_{\mathscr{V},k} - \boldsymbol{M}_{h,\boldsymbol{U}}(k)\}_{k \in \mathscr{M}})$$

and the positive definite $\boldsymbol{\Gamma}_{\boldsymbol{\theta}} = \boldsymbol{\Psi}_{\boldsymbol{\theta}}$.

The proof will be given in Section 5. □

Now, let us introduce the transform

$$\Phi\colon \boldsymbol{\Theta} \to \mathscr{R}^{1 + |\mathscr{M}|}$$

defined by the formula

$$\Phi(\boldsymbol{\theta}) = -(\mu_{h,\boldsymbol{U}}, \{\boldsymbol{M}_{h,\boldsymbol{U}}(k)\}_{k \in \mathscr{M}}) \quad \text{for every} \quad \boldsymbol{\theta} = (h, U) \in \boldsymbol{\Theta}\,.$$

Lemma 2.3. The transform Φ is one-to-one.

Proof. Let $\Phi(\boldsymbol{\theta}) = \Phi(\tilde{\boldsymbol{\theta}})$ for $\boldsymbol{\theta}, \tilde{\boldsymbol{\theta}} \in \boldsymbol{\Theta}$. Then $\mu_{h,U} = \mu_{\tilde{h},\tilde{U}}$ and $\boldsymbol{R}_U(k) = \boldsymbol{R}_{\tilde{U}}(k)$ for $k \in \mathscr{M}$.

Let us denote $\boldsymbol{h}_{U,\tilde{U}}(\lambda) = [f_{\tilde{U}}(\lambda)]^{-1} . f_U(\lambda) - 1 - \log\{[f_{\tilde{U}}(\lambda)]^{-1} f_U(\lambda)\}$. Since $\boldsymbol{h}_{U,\tilde{U}} \geqq 0$ it must be $\int_{\mathscr{I}_d} \boldsymbol{h}_{U,\tilde{U}}(\lambda)\, \mathrm{d}\lambda \geqq 0$ and by symmetry also $\int_{\mathscr{I}_d} \boldsymbol{h}_{\tilde{U},U}(\lambda)\, \mathrm{d}\lambda \geqq 0$. But $\int_{\mathscr{I}_d} (\boldsymbol{h}_{U,\tilde{U}}(\lambda) + \boldsymbol{h}_{\tilde{U},U}(\lambda))\, \mathrm{d}\lambda = \int_{\mathscr{I}_d} (f_U(\lambda) - f_{\tilde{U}}(\lambda))\,([f_{\tilde{U}}(\lambda)]^{-1} - [f_U(\lambda)]^{-1})\, \mathrm{d}\lambda = 0$ by assumption and therefore $\int_{\mathscr{I}_d} \boldsymbol{h}_{U,\tilde{U}}(\lambda)\, \mathrm{d}\lambda = \int_{\mathscr{I}_d} \boldsymbol{h}_{\tilde{U},U}(\lambda)\, \mathrm{d}\lambda = 0$. Since $\boldsymbol{h}_{U,\tilde{U}} \geqq 0$ it must hold $\boldsymbol{h}_{U,\tilde{U}} = 0$ a.s. $[d\lambda]$. And it is possible only if $f_U = f_{\tilde{U}}$ a.s. $[\mathrm{d}\lambda]$, wherefrom we directly conclude that $U = \tilde{U}$ and therefore also $h = \tilde{h}$. □

Let us denote by

$$\mathbf{J}(\boldsymbol{\theta}) = \left(\frac{\partial \Phi_k}{\partial \theta_j}(\boldsymbol{\theta})\right)_{k,j=0,\dots,|\mathscr{M}|}$$

the Jacobi matrix of the transform Φ at the point $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Lemma 2.4. For every $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ the matrix $\mathbf{J}(\boldsymbol{\theta})$ is equal to $\boldsymbol{\Psi}_0$, and all its elements are continuous on $\boldsymbol{\Theta}$.

Proof. The identity $\mathbf{J}(\boldsymbol{\theta}) = \boldsymbol{\Psi}_0$ can be obtained by direct differentiation, while the continuity of the matrix elements is straightforward. □

Corollary 2.5. The transform Φ is regular on the open set $\boldsymbol{\Theta}$ which namely means that the open subsets of $\boldsymbol{\Theta}$ are mapped on the open subsets of $\Phi(\boldsymbol{\Theta}) \subset \mathscr{R}^{1+|\mathscr{M}|}$.

3. PARAMETER ESTIMATION

Suppose we are given a collection of observations $\bar{x}_{\mathscr{V}} = \{\bar{x}_t\}_{t\in\mathscr{V}} \in \mathscr{R}^{|\mathscr{V}|}$, $\mathscr{V} \in \mathfrak{R}$, generated by a random field $\mathbf{P}^{\boldsymbol{\theta}^0} \in \mathbb{P}$ with an unknown parameter $\boldsymbol{\theta}^0 \in \boldsymbol{\Theta}$. We shall treat the problem how to estimate $\boldsymbol{\theta}^0$ from the given sample.

The transform Φ has been introduced in order to estimate the parameter $\boldsymbol{\theta}^0$ via estimating the transformed parameter

$$\Phi^0 = \Phi(\boldsymbol{\theta}^0) = -(\mu_{h^0,U^0}, \{\boldsymbol{M}_{h^0,U^0}(k)\}_{k\in\mathscr{M}}) .$$

The latter one can be easily estimated, namely we set

$$\hat{\Phi}_{\mathscr{V}}(\bar{x}_{\mathscr{V}}) = -(\hat{\mu}_{\mathscr{V}}(\bar{x}_{\mathscr{V}}) , \quad \{\hat{\boldsymbol{M}}_{\mathscr{V},k}(\bar{x}_{\mathscr{V}})\}_{k\in\mathscr{M}})$$

(see Section 2 for the definition of the terms). Then $\hat{\Phi}_{\mathscr{V}}$ is an unbiased estimate of Φ^0, and its asymptotic properties are investigated in the following theorem.

Theorem 3.1. The estimate $\hat{\Phi}_{\mathscr{V}}$ is consistent, i.e.

$$\hat{\Phi}_{\mathscr{V}} \to \Phi^0 \quad \text{a.s.} \quad [\mathbf{P}_{\boldsymbol{\theta}^0}] ,$$

and asymptotically normal, i.e.

$$\mathfrak{L}_{\mathbf{P}^{\boldsymbol{\theta}^0}}(|\mathscr{V}|^{1/2}\,(\hat{\Phi}_{\mathscr{V}} - \Phi^0)) \Rightarrow \mathbb{N}_{1+|\mathscr{M}|}(0, \boldsymbol{\Psi}_{\boldsymbol{\theta}^0}) \quad \text{for} \quad \mathscr{V} \nearrow \mathscr{Z}^d .$$

Proof. The consistency follows from the d-dimensional ergodic theorem (Theorem VIII.6.9 in [2]) and the asymptotic normality is proved in Lemma 5.5 and Theorem 5.4. □

Remark 3.2. For the asymptotic normality in the preceding theorem it is important to use the unbiased estimates $\hat{M}_{\mathscr{V},k}$, $k \in \mathscr{M}$, because the bias of the estimate $M^*_{\mathscr{V},k}(\bar{x}_{\mathscr{V}}) = |\mathscr{V}|^{-1} \sum_{t \in \mathscr{V}_k} \bar{x}_t \bar{x}_{t+k}$ could be bigger then the random fluctuations (cf. Remark 3.14 in [5]).

Now we may define the estimate of the original parameter $\boldsymbol{\theta}^0$ as the inverse transform of the estimate of the transformed parameter, i.e.

$$\hat{\boldsymbol{\theta}}_{\mathscr{V}} = \Phi^{-1}(\hat{\Phi}_{\mathscr{V}}) = \Phi^{-1}(-\hat{\mu}_{\mathscr{V}}, \{-\hat{M}_{\mathscr{V},k}\}_{k \in \mathscr{M}}) .$$

Unfortunately, the estimate $\hat{\boldsymbol{\theta}}_{\mathscr{V}}$ is correctly defined only for

$$\hat{\Phi}_{\mathscr{V}} \in \Phi(\boldsymbol{\Theta}) ,$$

and the estimate $\{\hat{M}_{\mathscr{V},k}\}_{k \in \mathscr{M}}$ of the second moments does not in general ensure the positive definiteness of the empirical covariance function, which is a necessary condition here. Nevertheless, the following theorem holds.

Theorem 3.3. The estimate $\hat{\boldsymbol{\theta}}_{\mathscr{V}} = \Phi^{-1}(\hat{\Phi}_{\mathscr{V}})$ is defined with probability tending to one for $\mathscr{V} \nearrow \mathscr{Z}^d$.

Proof. Thanks to the regularity of the transform Φ on the open set $\boldsymbol{\Theta}$, the image $\Phi(\boldsymbol{\theta}^0)$ of $\boldsymbol{\theta}^0$ is an inner point of $\Phi(\boldsymbol{\Theta})$.

Therefore with some $\varepsilon > 0$ we have

$$\mathbf{P}^{\boldsymbol{\theta}^0}(\hat{\Phi}_{\mathscr{V}} \in \Phi(\boldsymbol{\Theta})) \geqq \mathbf{P}^{\boldsymbol{\theta}^0}(\|\hat{\Phi}_{\mathscr{V}} - \Phi^0\| < \varepsilon) \to 1 \quad \text{for} \quad \mathscr{V} \nearrow \mathscr{Z}^d$$

due to the consistency of the estimate $\hat{\Phi}_{\mathscr{V}}$. □

Thus, we could define the estimate $\hat{\boldsymbol{\theta}}_{\mathscr{V}}$ for $\hat{\Phi}_{\mathscr{V}} \notin \Phi(\boldsymbol{\Theta})$ arbitrarily, but it seems convenient to define the estimate in a way which at least partially respects the given estimates of the moments, e.g.

$$\hat{\boldsymbol{\theta}}_{\mathscr{V}} = \Phi^{-1}(-\hat{\mu}_{\mathscr{V}}, \{-\tilde{M}_{\mathscr{V},k}\}_{k \in \mathscr{M}}) ,$$

where $\tilde{M}_{\mathscr{V},0} = \hat{M}_{\mathscr{V},0}$ and $\tilde{M}_{\mathscr{V},k} = 0$ for $k \neq 0$.

The estimate $\hat{\boldsymbol{\theta}}_{\mathscr{V}}$, obtained in such a way, need not be unbiased but its asymptotic properties may be easily derived.

Theorem 3.4. The estimate $\hat{\boldsymbol{\theta}}_{\mathscr{V}}$ of the parameter $\boldsymbol{\theta}^0$ is consistent and asymptotically normal with the covariance matrix $\boldsymbol{\Psi}^{-1}_{\boldsymbol{\theta}^0}$.

Proof. The consistency follows immediately from the continuity of the transform Φ^{-1}. Proving the asymptotic normality, we may proceed similarly as Rao [5] in Theorem 6a.2(II) and Theorem 6a.2(III). □

4. EFFICIENCY

In this section we turn our attention to the question of efficiency of the estimate. Usually, an estimate is said to be (asymptotically) efficient if its (asymptotic) variance assumes minimal available value. But, this definition is meaningful only with some classes of problems and therefore we shall follow the general theory of efficiency based on Rao's definition of the notion.

The estimate $\hat{\boldsymbol{\theta}}_{\mathscr{V}}$ of a parameter $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is said to be asymptotically efficient (in sense of Rao) if for some non-random $(1 + |\mathscr{M}|) \times (1 + |\mathscr{M}|)$-matrix $\mathbf{C}_{\boldsymbol{\theta}}$

$$|\mathscr{V}|^{1/2} (\hat{\boldsymbol{\theta}}_{\mathscr{V}} - \boldsymbol{\theta} - |\mathscr{V}|^{-1} \mathbf{C}_{\boldsymbol{\theta}} \, \boldsymbol{l}^{\boldsymbol{\theta}}_{\mathscr{V}}) \to 0 \quad \text{in probability} \quad [\mathbf{P}^{\boldsymbol{\theta}}]$$

for $\mathscr{V} \nearrow \mathscr{Z}^d$, where

$$\boldsymbol{l}^{\boldsymbol{\theta}}_{\mathscr{V}}(X_{\mathscr{V}}) = \left(\frac{\mathrm{d} \log \mathbf{P}^{\boldsymbol{\theta}}_{\mathscr{V}}(X_{\mathscr{V}})}{\mathrm{d}\theta_j} \right)_{j=0,\ldots,|\mathscr{M}|}$$

is the score vector.

The definition states that $\hat{\boldsymbol{\theta}}_{\mathscr{V}}$, appropriately centered and scaled, is asymptotically linearly related to the score vector $\boldsymbol{l}^{\boldsymbol{\theta}}_{\mathscr{V}}$.

It can be shown (cf. e.g. [9]), under the general LAN model, that, roughly speaking, if $\tilde{\boldsymbol{\theta}}_{\mathscr{V}}$ is any other estimator satisfying some regularity condition,

$$\lim_{\mathscr{V} \nearrow \mathscr{Z}^d} \mathbf{P}^{\boldsymbol{\theta}}\{|\mathscr{V}|^{1/2} (\tilde{\boldsymbol{\theta}}_{\mathscr{V}} - \boldsymbol{\theta}) \in \mathscr{K}\} \leqq \lim_{\mathscr{V} \nearrow \mathscr{Z}^d} \mathbf{P}^{\boldsymbol{\theta}}\{|\mathscr{V}|^{1/2} (\hat{\boldsymbol{\theta}}_{\mathscr{V}} - \boldsymbol{\theta}) \in \mathscr{K}\} ,$$

where $\mathscr{K}$ is a $(1 + |\mathscr{M}|)$-dimensional bounded convex set symmetric about the origin, i.e. $\hat{\boldsymbol{\theta}}_{\mathscr{V}}$ attains the maximal possible concentration about the true value of the parameter. For further discussion of this rather complicated problem see e.g. [3].

Here, we shall show that the estimate $\hat{\boldsymbol{\theta}}_{\mathscr{V}}$ of a parameter $\boldsymbol{\theta}^0 \in \boldsymbol{\Theta}$, introduced in the preceding section, satisfies the Rao's definition of asymptotic efficiency. And again, we shall at first prove the property for the transformed parameter $\Phi^0 = \Phi(\boldsymbol{\theta}^0)$.

Lemma 4.1. The estimate $\hat{\Phi}_{\mathscr{V}}$ is asymptotically efficient with $\mathbf{C}_{\Phi^0} = \boldsymbol{\Psi}_{\boldsymbol{\theta}^0}$, i.e.

$$|\mathscr{V}|^{1/2}(\hat{\Phi}_{\mathscr{V}} - \Phi^0 - |\mathscr{V}|^{-1} \boldsymbol{\Psi}_{\boldsymbol{\theta}^0} \boldsymbol{l}^{\Phi^0}_{\mathscr{V}}) \to 0 \quad \text{in probability} \quad [\mathbf{P}^{\boldsymbol{\theta}^0}] .$$

Proof. The statement follows immediately from the rules for differentiation, namely

$$\frac{\mathrm{d} \log \mathbf{P}^{\boldsymbol{\theta}}}{\mathrm{d}\theta_j} = \sum_{k=0}^{|\mathscr{M}|} \frac{\mathrm{d} \log \mathbf{P}^{\boldsymbol{\theta}}}{\mathrm{d}\Phi_k} \frac{\mathrm{d}\Phi_k}{\mathrm{d}\theta_j} = \sum_{k=0}^{|\mathscr{M}|} \boldsymbol{\Psi}_{\boldsymbol{\theta}}(j, k) \, . \, \boldsymbol{l}^{\Phi}_{\mathscr{V},k} ,$$

and Lemma 5.10 together with Lemma 5.5. □

Now, we can easily obtain the result for the original parameter.

Theorem 4.2. The estimate $\hat{\boldsymbol{\theta}}_{\mathscr{V}}$ of the parameter $\boldsymbol{\theta}^0$ is asymptotically efficient with $\mathbf{C}_{\boldsymbol{\theta}^0} = \boldsymbol{\Psi}^{-1}_{\boldsymbol{\theta}^0}$, i.e.

$$|\mathscr{V}|^{1/2} (\hat{\boldsymbol{\theta}}_{\mathscr{V}} - \boldsymbol{\theta}^0 - |\mathscr{V}|^{-1} \boldsymbol{\Psi}^{-1}_{\boldsymbol{\theta}^0} \boldsymbol{l}^{\boldsymbol{\theta}^0}_{\mathscr{V}}) \to 0 \quad \text{in probability} \quad [\mathbf{P}^{\boldsymbol{\theta}^0}]$$

Proof. Using the idea of the proof of Theorem 6a.2(III), [6], the statement follows from the preceding lemma. □

Since the LAN condition is satisfied in our case we may conclude that the proposed estimator is asymptotically optimal in the sense described above.

5. AUXILIARY RESULTS AND LIMIT THEOREMS

In this section we fix $\boldsymbol{\theta} = (h, U) \in \boldsymbol{\Theta}$ and $\boldsymbol{\beta} = (\tilde{h}, U^{\sim}) \in \mathscr{R}^{1+|\mathscr{M}|}$. Omitting the indices, we shall write $\boldsymbol{R}$ and $\boldsymbol{R}^{\sim}$ for the covariance functions, f and $\tilde{f}$ for the spectral densities, μ and $\tilde{\mu}$ for the mean values corresponding to the random fields $\mathbf{P}^{\mathbf{0}}$ and $\mathbf{P}^{\mathbf{0}+|\mathscr{V}|^{-1/2}\boldsymbol{\beta}}$, respectively.

In Section 1 we have denoted $\mathbf{A} = (a(t-s))_{t,s\in\mathscr{Z}^d}$ the infinite matrix inverse to the infinite covariance matrix $\mathbf{R}$. Similarly, we introduce the infinite matrix $\mathbf{B} = (b(t-s))_{t,s\in\mathscr{Z}^d}$ defined by

$$\begin{aligned} b(0) &= 2U^{\sim}(0) \\ b(t) &= b(-t) = U^{\sim}(t) \quad \text{for} \quad t \in \mathscr{M} \setminus \{0\} \\ b(t) &= 0 \qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

Obviously, $\mathbf{A} + |\mathscr{V}|^{-1/2}\,\mathbf{B}$ is inverse to $\boldsymbol{R}^{\sim}$.

All expectations, variances etc. are meant with respect to the random field $\mathbf{P}^{\mathbf{0}}$ in this section.

For the simplicity let us denote

$$\mathbf{g}_{\mathscr{V}} = \mu\,.\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\,\mathbf{1}_{\mathscr{V}} + \tilde{h}\,\mathbf{1}_{\mathscr{V}} \quad \text{for every} \quad \mathscr{V} \in \mathfrak{K}\,.$$

Lemma 5.1. It holds

(i) $$\tfrac{1}{2}\lim_{\mathscr{V}\nearrow\mathscr{Z}^d} |\mathscr{V}|^{-1}\,\mathrm{Tr}((\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^2) = U^{\sim\mathrm{T}}\mathbf{W}_{\mathscr{M},\mathscr{M}}U^{\sim}\,;$$

(ii) $$\lim_{\mathscr{V}\nearrow\mathscr{Z}^d} |\mathscr{V}|^{-1}\,\mathbf{g}_{\mathscr{V}}^{\mathrm{T}}\mathbf{R}_{\mathscr{V}\mathscr{V}}\mathbf{g}_{\mathscr{V}} = f(0)\,(\tilde{h} - h\,f(0)\,2\,\mathbf{1}_{\mathscr{M}}^{\mathrm{T}}U^{\sim})^2\,.$$

Proof. We may write

$$\tfrac{1}{2}|\mathscr{V}|^{-1}\mathrm{Tr}((\mathbf{B}_{\mathscr{V}\mathscr{V}}\boldsymbol{R}_{\mathscr{V}\mathscr{V}})^2) = \tfrac{1}{2}\sum_{k,l\in\mathscr{Z}^d}\sum b(k)\,b(l)\,.$$
$$.\sum_{t\in\mathscr{Z}^d}|\mathscr{V}|^{-1}\,|\mathscr{V}\cap(\mathscr{V}-l)\cap(\mathscr{V}-t)\cap(\mathscr{V}-k-t)|\,\boldsymbol{R}(t-l)\,\boldsymbol{R}(t+k)\,.$$

But $|\mathscr{V}|^{-1}\,|\mathscr{V}\cap(\mathscr{V}-l)\cap(\mathscr{V}-t)\cap(\mathscr{V}-k-t)| \to 1$ for $\mathscr{V}\nearrow\mathscr{Z}^d$ for every fixed $k, l, t \in \mathscr{Z}^d$, and therefore, due to the dominated convergence theorem, the limit is

$$\tfrac{1}{2}\sum_{k,l\in\mathscr{Z}^d}\sum b(k)\,b(l)\sum_{t\in\mathscr{Z}^d}\boldsymbol{R}(t-l)\,\boldsymbol{R}(t+k) =$$
$$= \tfrac{1}{2}\sum_{k,l\in\mathscr{M}}\sum U^{\sim}(k)\,U^{\sim}(l)\,.\sum_{t\in\mathscr{Z}^d}[\boldsymbol{R}(t-l)+\boldsymbol{R}(t+l)]\,[\boldsymbol{R}(t+k)+\boldsymbol{R}(t-k)] =$$
$$= U^{\sim\mathrm{T}}\,\mathbf{W}_{\mathscr{M},\mathscr{M}}U^{\sim}\,.$$

Similarly, we can show

$$|\mathscr{V}|^{-1}\,\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\,\mathbf{1}_{\mathscr{V}}\to\sum_{t\in\mathscr{Z}^d}\boldsymbol{R}(t)=f(0)\;;$$

$$|\mathscr{V}|^{-1}\,\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}\,\mathbf{1}_{\mathscr{V}}\to f(0)\,.\,2\,.\,\mathbf{1}_{\mathscr{M}}^{\mathrm{T}}\boldsymbol{U}^{\sim}\;;$$

$$|\mathscr{V}|^{-1}\,\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\,\mathbf{1}_{\mathscr{V}}\to f(0)\,.\,(2\,.\,\mathbf{1}_{\mathscr{M}}^{\mathrm{T}}\boldsymbol{U}^{\sim})^2\,,$$

which (if we realize $\mu = -h\,f(0)$) proves (ii). □

For any $N \times N$-matrix $\boldsymbol{D}$ we denote

$$\varrho(\boldsymbol{D}) = \max_{i=1,\dots,N} |c_j|$$

where c_j, $j = 1, \dots, N$ are the eigenvalues of the matrix $\boldsymbol{D}$.

Lemma 5.2. For every fixed $\lambda \in \mathscr{R}$ it holds

(i) $$\lim_{\mathscr{V}\nearrow\mathscr{Z}^d}\exp\{-\tfrac{1}{2}\,\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathrm{Tr}(\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})\}\,[\mathrm{Det}\,(\mathbf{I}_{\mathscr{V}\mathscr{V}}-\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})]^{-1/2} = \\ = \exp\{-\tfrac{1}{2}\lambda^2\boldsymbol{U}^{\sim\mathrm{T}}\,\mathbf{W}_{\mathscr{M},\mathscr{M}}\boldsymbol{U}^{\sim}\}\;;$$

(ii) $$\lim_{\mathscr{V}\nearrow\mathscr{Z}^d}\exp\{-\tfrac{1}{2}\lambda^2|\mathscr{V}|^{-1}\,\mathbf{g}_{\mathscr{V}}^{\mathrm{T}}\mathbf{R}_{\mathscr{V}\mathscr{V}}(\mathbf{I}_{\mathscr{V}\mathscr{V}}-\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^{-1}\,\mathbf{g}_{\mathscr{V}}\} = \\ = \exp\{-\tfrac{1}{2}\lambda^2\,.\,f(0)\,.\,(\tilde{h}-h\,.\,f(0)\,.\,2\,.\,\mathbf{1}_{\mathscr{M}}^{\mathrm{T}}\boldsymbol{U}^{\sim})^2\}\,.$$

Proof. Denoting $K_b = \sum_{t\in\mathscr{Z}^d}|\boldsymbol{b}(t)|$, $K_R = \sum_{t\in\mathscr{Z}^d}|\boldsymbol{R}(t)|$ we have $\varrho(\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}) \leqq$ $\leqq |\mathscr{V}|^{-1/2}\,|\lambda|\,K_b\,K_R < 1$ for sufficiently large $|\mathscr{V}|$.

Thus, we may write

$$[\mathrm{Det}\,(\mathbf{I}_{\mathscr{V}\mathscr{V}}-\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})]^{-1/2} = \exp\{\tfrac{1}{2}\sum_{n=1}^{\infty}n^{-1}\,\mathrm{Tr}\,[(\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^n]\}\,.$$

Since

$$|\mathrm{Tr}\,[(\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^n]| \leqq |\mathscr{V}|\,[\varrho(\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})]^n \leqq \\ \leqq |\mathscr{V}|^{1-n/2}\,(|\lambda|\,K_bK_R)^n$$

it holds

$$|\sum_{n=3}^{\infty}n^{-1}\,\mathrm{Tr}\,[(\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^n]| \leqq \sum_{n=1}^{\infty}(n+2)^{-1}\,|\mathscr{V}|^{-n/2}\,(|\lambda|K_bK_R)^{n+2} \leqq \\ \leqq (|\lambda|\,K_bK_R)^2\,|\log\,(1-|\mathscr{V}|^{-1/2}\,|\lambda|\,K_bK_R)| \to 0 \quad \text{for} \quad \mathscr{V}\nearrow\mathscr{Z}^d\,.$$

Wherefrom we directly obtain

$$\lim_{\mathscr{V}\nearrow\mathscr{Z}^d}\exp\{-\tfrac{1}{2}\,\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathrm{Tr}\,(\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})\}\,[\mathrm{Det}\,(\mathbf{I}_{\mathscr{V}\mathscr{V}}-i\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})]^{-1/2} = \\ = \lim_{\mathscr{V}\nearrow\mathscr{Z}^d}\exp\{-\tfrac{1}{2}\lambda^2|\mathscr{V}|^{-1}\,\tfrac{1}{2}\,\mathrm{Tr}\,[(\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^2]\} = \exp\{-\tfrac{1}{2}\lambda^2\boldsymbol{U}^{\sim\mathrm{T}}\,\mathbf{W}_{\mathscr{M}\mathscr{M}}\boldsymbol{U}^{\sim}\}$$

according to the statement (i) of Lemma 3.1.

Further, due to the given bound for the maximum eigenvalue we may also write

$$(\mathbf{I}_{\mathscr{V}\mathscr{V}}-\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^{-1} = \sum_{n=0}^{\infty}(\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^n\,,$$

and therefore

$$\varrho(\mathbf{R}_{\mathscr{V}\mathscr{V}}(\mathbf{I}_{\mathscr{V}\mathscr{V}} - i\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^{-1} - \mathbf{R}_{\mathscr{V}\mathscr{V}}) = \varrho(\mathbf{R}_{\mathscr{V}\mathscr{V}}\sum_{n=1}^{\infty}(\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^n)$$

$$\leqq \sum_{n=1}^{\infty}|\lambda|^n\,|\mathscr{V}|^{-n/2}\,K_b^nK_R^{n+1} = |\mathscr{V}|^{-1/2}\,|\lambda|\,K_bK_R^2[1 - |\mathscr{V}|^{-1/2}\,|\lambda|\,K_bK_R]^{-1} \to 0 \quad \text{for} \quad \mathscr{V} \nearrow \mathscr{Z}^d\,.$$

Since

$$|\mathscr{V}|^{-1}\,\mathbf{g}_{\mathscr{V}}^{\mathrm{T}}\mathbf{g}_{\mathscr{V}} \leqq \tilde{h}^2 + 2|h\mu|\,K_b + \mu^2K_b^2 = K_1$$

we conclude

$$|(\mathscr{V}|^{-1}\,\mathbf{g}_{\mathscr{V}}^{\mathrm{T}}(\mathbf{R}_{\mathscr{V}\mathscr{V}}(\mathbf{I}_{\mathscr{V}\mathscr{V}} - \mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^{-1} - \mathbf{R}_{\mathscr{V}\mathscr{V}})\,\mathbf{g}_{\mathscr{V}}| \leqq$$

$$\leqq |\mathscr{V}|^{-1/2}\,K_1\,\frac{K_R^2K_b|\lambda|}{1 - |\lambda|\,K_bK_R|\mathscr{V}|^{-1/2}} \to 0 \quad \text{for} \quad \mathscr{V} \nearrow \mathscr{Z}^d$$

which, together with (ii) of Lemma 5.1, yields the second statement of the present lemma. □

Now, let us define the linear-quadratic form

$$\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}}) = \tfrac{1}{2}X_{\mathscr{V}}^{\mathrm{T}}\mathbf{B}_{\mathscr{V}\mathscr{V}}X_{\mathscr{V}} + \tilde{h}\,\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}X_{\mathscr{V}} \quad \text{for every} \quad \mathscr{V} \in \mathfrak{K}\,.$$

Lemma 5.3. It holds

(i) $\mathsf{E}[\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}})] = \tfrac{1}{2}(\mathsf{Tr}\,(\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}) + \mu^2\,\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\,\mathbf{1}_{\mathscr{V}}) + \mu\tilde{h}\,\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\,\mathbf{1}_{\mathscr{V}}$;

(ii) $\mathsf{var}\,[\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}})] = \tfrac{1}{2}\,\mathsf{Tr}\,[(\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^2] + \mathbf{g}_{\mathscr{V}}^{\mathrm{T}}\mathbf{R}_{\mathscr{V}\mathscr{V}}\mathbf{g}_{\mathscr{V}}$;

(iii) $\lim_{\mathscr{V}\nearrow\mathscr{Z}^d}|\mathscr{V}|^{-1}\,\mathsf{var}\,[\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}})] = \boldsymbol{\beta}^{\mathrm{T}}\,\Psi_0\boldsymbol{\beta}$.

Proof. By direct calculation, properly substituting for the moments of the normal distribution, we obtain the statements (i) and (ii), while the statement (iii) is proved in Lemma 5.1. □

Theorem 5.4. (CLT). The form $\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}})$ is asymptotically normally distributed, i.e.

$$\mathfrak{L}(|\mathscr{V}|^{-1/2}\,(\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}}) - \mathsf{E}\,[\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}})])) \Rightarrow \mathbb{N}(0, \boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\Psi}_0\boldsymbol{\beta})\,.$$

Proof. We express the characteristic function

$$\varphi_{\mathscr{V}}(\lambda) = \mathsf{E}\exp\{\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,(\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}}) - \mathsf{E}\,[\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}})])\} =$$

$$= \exp\{-\tfrac{1}{2}\,\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathsf{Tr}\,(\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})\}\,.\,(2\pi)^{-|\mathscr{V}|/2}\,[\mathrm{Det}\,(\mathbf{R}_{\mathscr{V}\mathscr{V}})]^{-1/2}\,.$$

$$.\int\exp\{-\tfrac{1}{2}(x_{\mathscr{V}} - \mu\,\mathbf{1}_{\mathscr{V}})^{\mathrm{T}}\,(\mathbf{R}_{\mathscr{V}\mathscr{V}}^{-1} - \mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}})\,(x_{\mathscr{V}} - \mu\,\mathbf{1}_{\mathscr{V}}) +$$

$$+ \mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{g}_{\mathscr{V}}^{\mathrm{T}}(x_{\mathscr{V}} - \mu\,\mathbf{1}_{\mathscr{V}})\}\,\mathrm{d}x_{\mathscr{V}} = \exp\{-\tfrac{1}{2}\,\mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathsf{Tr}\,(\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})\}\,.$$

$$.\,[\mathrm{Det}\,(\mathbf{I}_{\mathscr{V}\mathscr{V}} - \mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})]^{-1/2}\,.$$

$$.\,\exp\{-\tfrac{1}{2}\lambda^2|\mathscr{V}|^{-1}\,\mathbf{g}_{\mathscr{V}}^{\mathrm{T}}\mathbf{R}_{\mathscr{V}\mathscr{V}}(\mathbf{I}_{\mathscr{V}\mathscr{V}} - \mathrm{i}\lambda|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^{-1}\,\mathbf{g}_{\mathscr{V}}\}\,.$$

Thus, according to Lemma 5.2,

$$\lim_{\mathscr{V}\nearrow\mathscr{Z}^d}\varphi_{\mathscr{V}}(\lambda) = \exp\{-\tfrac{1}{2}\lambda^2\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\Psi}_0\boldsymbol{\beta}\}$$

which proves the theorem. □

Lemma 5.5. It holds

$$\operatorname{var}\left(|\mathscr{V}|^{-1/2}\,\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}}) + \boldsymbol{\beta}^{\mathrm{T}}\,\Delta^{0}_{\mathscr{V}}\right) \to 0 \quad \text{for} \quad \mathscr{V} \nearrow \mathscr{Z}^d .$$

Proof. Since

$$\mathbf{F}_{\mathscr{V}} = |\mathscr{V}|\,\tilde{h}\hat{\mu}_{\mathscr{V}} + \sum_{k\in\mathscr{M}} |\mathscr{V}_k|\, U^{\sim}(k)\,\hat{M}_{\mathscr{V},k}$$

we have

$$\operatorname{var}\left(|\mathscr{V}|^{-1/2}\,\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}}) + \boldsymbol{\beta}^{\mathrm{T}}\,\Delta^{0}_{\mathscr{V}}\right) = |\mathscr{V}|^{-1} \operatorname{var}\left(\sum_{k\in\mathscr{M}} (|\mathscr{V}|_k - |\mathscr{V}|)\, U^{\sim}(k)\,\hat{M}_{\mathscr{V},k}\right) \leqq$$

$$\leqq |\mathscr{M}|\,|\mathscr{V}|^{-1} \sum_{k\in\mathscr{M}} (|\mathscr{V}_k| - |\mathscr{V}|)^2\,(U^{\sim}(k))^2 \operatorname{var}(\hat{M}_{\mathscr{V},k}) \leqq 2R(0)\,|\mathscr{M}|\,.$$

$$.\sum_{k\in\mathscr{M}} (U^{\sim}(k))^2\,|\mathscr{V}|^{-1}\,|\mathscr{V}_k|^{-1}\,(|\mathscr{V}_k| - |\mathscr{V}|)^2 \to 0 \quad \text{for} \quad \mathscr{V} \nearrow \mathscr{Z}^d . \qquad \square$$

From now, the further steps are directed to the proof of Theorem 2.2.

Lemma 5.6. It holds

$$\mathbf{R}^{\sim} = \mathbf{R} - |\mathscr{V}|^{-1/2}\,\mathbf{R}^{\sim}\mathbf{B}\mathbf{R} = \mathbf{R}\sum_{n=0}^{\infty}\left(-|\mathscr{V}|^{-1/2}\,\mathbf{B}\mathbf{R}\right)^n .$$

Proof. The first equality is straightforward if we realize $\mathbf{R}^{\sim -1} = \mathbf{A} + |\mathscr{V}|^{-1/2}\,\mathbf{B} = \mathbf{R}^{-1} + |\mathscr{V}|^{-1/2}\,\mathbf{B}$. The second equality follows from the first one since $\varrho(\mathbf{BR}) \leqq K < \infty$. $\square$

As a corollary to the preceding lemma we have

$$\sum_{t\in\mathscr{Z}^d} |R^{\sim}(t) - R(t)| < \infty ,$$

or, more generally, the convergence $\sum_{t\in\mathscr{Z}^d} |R_{\bar{U}}(t)|$ is uniform for $\bar{U}$ from some neighborhood of U.

Lemma 5.7. It holds

(i) $\lim\limits_{\mathscr{V}\nearrow\mathscr{Z}^d} \operatorname{Tr}\left\{\left[\left(\mathbf{R}^{\sim -1}_{\mathscr{V}\mathscr{V}} - \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}} - |\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\right)\mathbf{R}_{\mathscr{V}\mathscr{V}}\right]^2\right\} = 0;$

(ii) $\lim\limits_{\mathscr{V}\nearrow\mathscr{Z}^d} \left[(\mu - \tilde{\mu})\,\mathbf{R}^{\sim -1}_{\mathscr{V}\mathscr{V}}\,\mathbf{1}_{\mathscr{V}} - |\mathscr{V}|^{-1/2}\,\mathbf{g}_{\mathscr{V}}\right]^{\mathrm{T}} \mathbf{R}_{\mathscr{V}\mathscr{V}}\left[(\mu - \tilde{\mu})\,\mathbf{R}^{\sim -1}_{\mathscr{V}\mathscr{V}}\,\mathbf{1}_{\mathscr{V}} - |\mathscr{V}|^{-1/2}\,\mathbf{g}_{\mathscr{V}}\right] = 0.$

Proof. Let us denote $\mathbf{D}_{\mathscr{V}\mathscr{V}} = \mathbf{R}_{\mathscr{V}\mathscr{V}} - \mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}} - |\mathscr{V}|^{-1/2}\,\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}$. Then

$$\operatorname{Tr}\left\{\left[\left(\mathbf{R}^{\sim -1}_{\mathscr{V}\mathscr{V}} - \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}} - |\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\right)\mathbf{R}_{\mathscr{V}\mathscr{V}}\right]^2\right\} =$$

$$= \operatorname{Tr}\left\{\left[\mathbf{R}^{1/2}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim -1}_{\mathscr{V}\mathscr{V}}\mathbf{D}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{-1/2}_{\mathscr{V}\mathscr{V}}\right]\left[\mathbf{R}^{1/2}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim -1}_{\mathscr{V}\mathscr{V}}\mathbf{D}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{-1/2}_{\mathscr{V}\mathscr{V}}\right]^{\mathrm{T}}\right\} \leqq$$

$$\leqq \left[\varrho(\mathbf{R}^{1/2}_{\mathscr{V}\mathscr{V}})\,\varrho(\mathbf{R}^{\sim -1}_{\mathscr{V}\mathscr{V}})\,\varrho(\mathbf{R}^{-1/2}_{\mathscr{V}\mathscr{V}})\right]^2 . \operatorname{Tr}\left\{\mathbf{D}_{\mathscr{V}\mathscr{V}}\mathbf{D}^{\mathrm{T}}_{\mathscr{V}\mathscr{V}}\right\}$$

due to a well-known result of the matrix calculus.

From Lemma 5.6 it follows that

$$\mathbf{D}_{\mathscr{V}\mathscr{V}} = |\mathscr{V}|^{-1/2}\left[\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{B}_{\mathscr{V}\mathscr{V}^c}\mathbf{R}_{\mathscr{V}^c\mathscr{V}} + \mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}^c}\mathbf{B}_{\mathscr{V}^c\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}} + \mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}^c}\mathbf{B}_{\mathscr{V}^c\mathscr{V}^c}\mathbf{R}_{\mathscr{V}^c\mathscr{V}}\right].$$

Hence, if we substitute this expression into $\operatorname{Tr}(\mathbf{D}_{\mathscr{V}\mathscr{V}}\mathbf{D}^{\mathrm{T}}_{\mathscr{V}\mathscr{V}})$ we can easily find that all the terms tend to zero and therefore $\operatorname{Tr}(\mathbf{D}_{\mathscr{V}\mathscr{V}}\mathbf{D}^{\mathrm{T}}_{\mathscr{V}\mathscr{V}})$ tends to zero for $\mathscr{V} \nearrow \mathscr{Z}^d$.

Since

$$[\varrho(\mathbf{R}^{1/2}_{\mathscr{V}\mathscr{V}})\,\varrho(\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}})\,\varrho(\mathbf{R}^{-1/2}_{\mathscr{V}\mathscr{V}})]^2 \leqq \max f(\lambda)\,[\min f(\lambda)]^{-1}\,[\min \tilde{f}(\lambda)]^{-2} \leqq C < \infty\,,$$

the proof of (i) is completed.

In order to prove (ii) let us denote

$$\mathbf{q}_{\mathscr{V}} = [(\mu - \tilde{\mu})\,\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\,\mathbf{1}_{\mathscr{V}} - |\mathscr{V}|^{-1/2}\,\mathbf{g}_{\mathscr{V}}]\,.$$

Then we may write $\mathbf{q}^{\mathrm{T}}_{\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}\mathbf{q}_{\mathscr{V}} \leqq \varrho(\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}})\,(\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{q}_{\mathscr{V}})^{\mathrm{T}}\,(\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{q}_{\mathscr{V}})$. Obviously, it holds

$$|\mathscr{V}|^{-1}\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\,.\,\mathbf{1}_{\mathscr{V}} \to f(0)$$
$$|\mathscr{V}|^{-1}\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\,.\,\mathbf{1}_{\mathscr{V}} \to f(0)\,.\,2\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{M}}U^{\sim}$$
$$|\mathscr{V}|^{-1}\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\,.\,\mathbf{1}_{\mathscr{V}} \to f(0)^2$$
$$|\mathscr{V}|^{-1}\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\,.\,\mathbf{1}_{\mathscr{V}} \to f(0)^2\,.\,2\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{M}}U^{\sim}$$
$$|\mathscr{V}|^{-1}\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{V}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\,.\,\mathbf{1}_{\mathscr{V}} \to f(0)^2\,.\,(2\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{M}}U^{\sim})^2\,.$$

Further, since $f(0) - \tilde{f}(0) = |\mathscr{V}|^{-1/2}\,.\,2\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{M}}U^{\sim}\,f(0)\tilde{f}(0)$ and $\mu = -h\,f(0)$, we have

$$|\mathscr{V}|^{1/2}\,(\mu - \tilde{\mu}) = |\mathscr{V}|^{1/2}\,(-h\,f(0) + (h + |V|^{-1/2}\tilde{h})\,\tilde{f}(0)) =$$
$$= \tilde{h}\tilde{f}(0) - h\,\tilde{f}(0)\,f(0)\,[2\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{M}}U^{\sim}] \to f(0)\,[\tilde{h} + \mu\,.\,2\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{M}}U^{\sim}]\,.$$

Using all these results we obtain

$$(\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{q}_{\mathscr{V}})^{\mathrm{T}}\,(\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{V}}\mathbf{q}_{\mathscr{V}}) \to 0\,.$$

And again $\varrho(\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}) \leqq \max f(\lambda)\,[\min \tilde{f}(\lambda)]^{-2} \leqq C_1 < \infty$ and the proof is finished. □

Lemma 5.8. It holds

(i) $\lim\limits_{\mathscr{V}\nearrow\mathscr{Z}^d} \{\frac{1}{2}\log \mathrm{Det}\,(\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}) + \frac{1}{2}\,\mathrm{Tr}\,(\mathbf{I}_{\mathscr{V}\mathscr{V}} - \mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})\} = -\frac{1}{2}U^{\sim\mathrm{T}}\,\mathbf{W}_{\mathscr{M}\mathscr{M}}U^{\sim}$

(ii) $\lim\limits_{\mathscr{V}\nearrow\mathscr{Z}^d} |\mathscr{V}|^{-1}\,.\,\mathbf{1}^{\mathrm{T}}_{\mathscr{V}}\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\,.\,\mathbf{1}_{\mathscr{V}} = f(0)^{-1}\,.$

Proof. Since, according to Lemma 5.6

$$\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}} = \mathbf{I}_{\mathscr{V}\mathscr{V}} + |\mathscr{V}|^{-1/2}\,\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{Z}^d}\mathbf{B}\mathbf{R}_{\mathscr{Z}^d\mathscr{V}}$$

and

$$\varrho(\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}^{\sim}_{\mathscr{V}\mathscr{Z}^d}\mathbf{B}\mathbf{R}_{\mathscr{Z}^d\mathscr{V}}) \leqq C_3 < \infty\,,$$

we obtain by the same arguments as in Lemma 5.2 that

$$\lim_{\mathscr{V}\nearrow\mathscr{Z}^d} \{\tfrac{1}{2}\log \mathrm{Det}\,(\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}}) + \tfrac{1}{2}\,\mathrm{Tr}\,(\mathbf{I}_{\mathscr{V}\mathscr{V}} - \mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})\} =$$
$$= -\tfrac{1}{2}\lim_{\mathscr{V}\nearrow\mathscr{Z}^d} \tfrac{1}{2}\,\mathrm{Tr}\,[(\mathbf{I}_{\mathscr{V}\mathscr{V}} - \mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^2]\,.$$

Therefore by the obvious inequality

$$|\tfrac{1}{2}\,\mathrm{Tr}\,[(\mathbf{I}_{\mathscr{V}\mathscr{V}} - \mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^2] - \tfrac{1}{2}\,\mathrm{Tr}\,[(|\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{R}_{\mathscr{V}\mathscr{V}})^2]| \leqq$$
$$\leqq \tfrac{1}{2}\,\mathrm{Tr}\,[((\mathbf{R}^{\sim\,-1}_{\mathscr{V}\mathscr{V}} - \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}} - |\mathscr{V}|^{-1/2}\,\mathbf{B}_{\mathscr{V}\mathscr{V}})\,\mathbf{R}_{\mathscr{V}\mathscr{V}})^2]\,,$$

together with Lemma 5.1 (i) and Lemma 5.7 (i) we finish the proof of the first statement.

In order to prove the second one we observe that

$$|\mathscr{V}|^{-1}\left|\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\mathbf{1}_{\mathscr{V}}-\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\mathbf{A}_{\mathscr{V}\mathscr{V}}\mathbf{1}_{\mathscr{V}}\right|\leqq|\mathscr{V}|^{-1}\left(|\mathscr{V}|^{-1/2}\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{1}_{\mathscr{V}}+\right.$$

$$\left.+\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}(\mathbf{A}_{\mathscr{V}\mathscr{V}^c}+|\mathscr{V}|^{-1/2}\mathbf{B}_{\mathscr{V}\mathscr{V}^c})\tilde{\mathbf{R}}_{\mathscr{V}^c\mathscr{V}^c}(\mathbf{A}_{\mathscr{V}^c\mathscr{V}}+|\mathscr{V}|^{-1/2}\mathbf{B}_{\mathscr{V}^c\mathscr{V}})\mathbf{1}_{\mathscr{V}}\right)\to 0\quad\text{for}\quad\mathscr{V}\nearrow\mathscr{Z}^d$$

and

$$|\mathscr{V}|^{-1}\,.\,\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\mathbf{A}_{\mathscr{V}\mathscr{V}}\mathbf{1}_{\mathscr{V}}\to 2\sum_{k\in\mathscr{M}}U(k)=f(0)^{-1}\,.$$

□

5.9. Proof of Theorem 2.2.

We may write

$$\log\frac{\mathrm{d}\mathbf{P}_{\mathscr{V}}^{\mathbf{0},\boldsymbol{\beta}}}{\mathrm{d}\mathbf{P}_{\mathscr{V}}^{\mathbf{0}}}(X_{\mathscr{V}})=-\tfrac{1}{2}(X_{\mathscr{V}}-\tilde{\mu}\,\mathbf{1}_{\mathscr{V}})^{\mathrm{T}}\,\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}(X_{\mathscr{V}}-\tilde{\mu}\,\mathbf{1}_{\mathscr{V}})+$$

$$+\tfrac{1}{2}(X_{\mathscr{V}}-\mu\,\mathbf{1}_{\mathscr{V}})\,\mathbf{R}_{\mathscr{V}\mathscr{V}}^{-1}(X_{\mathscr{V}}-\mu\,\mathbf{1}_{\mathscr{V}})+\tfrac{1}{2}\log\operatorname{Det}(\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\mathbf{R}_{\mathscr{V}\mathscr{V}})=\mathbf{Q}_{\mathscr{V}}^{1}(X_{\mathscr{V}})+\mathbf{Q}_{\mathscr{V}}^{2}\,,$$

where

$$\mathbf{Q}_{\mathscr{V}}^{1}(X_{\mathscr{V}})=\tfrac{1}{2}(X_{\mathscr{V}}-\mu\,\mathbf{1}_{\mathscr{V}})^{\mathrm{T}}\left[\mathbf{R}_{\mathscr{V}\mathscr{V}}^{-1}-\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\right](X_{\mathscr{V}}-\mu\,\mathbf{1}_{\mathscr{V}})-$$

$$-(\mu-\tilde{\mu})\,.\,\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}(X_{\mathscr{V}}-\mu\,.\,\mathbf{1}_{\mathscr{V}})-\tfrac{1}{2}\operatorname{Tr}(\mathbf{I}_{\mathscr{V}\mathscr{V}}-\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\mathbf{R}_{\mathscr{V}\mathscr{V}})\,,$$

and

$$\mathbf{Q}_{\mathscr{V}}^{2}=\tfrac{1}{2}\left[\log\operatorname{Det}(\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\mathbf{R}_{\mathscr{V}\mathscr{V}})+\operatorname{Tr}(\mathbf{I}_{\mathscr{V}\mathscr{V}}-\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\mathbf{R}_{\mathscr{V}\mathscr{V}})\right]-\tfrac{1}{2}(\mu-\tilde{\mu})^2\,.\,\mathbf{1}_{\mathscr{V}}^{\mathrm{T}}\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\,\mathbf{1}_{\mathscr{V}}.$$

Here, we have

$$\operatorname{var}\left(\mathbf{Q}_{\mathscr{V}}^{1}(X_{\mathscr{V}})+|\mathscr{V}|^{-1/2}\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}})\right]=$$

$$=\operatorname{var}\left\{\tfrac{1}{2}(X_{\mathscr{V}}-\mu\,.\,\mathbf{1}_{\mathscr{V}})^{\mathrm{T}}\left[\mathbf{R}_{\mathscr{V}\mathscr{V}}^{-1}-\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}+|\mathscr{V}|^{-1/2}\mathbf{B}_{\mathscr{V}\mathscr{V}}\right](X_{\mathscr{V}}-\mu\,.\,\mathbf{1}_{\mathscr{V}})-\right.$$

$$-\tfrac{1}{2}\operatorname{Tr}(\mathbf{I}_{\mathscr{V}\mathscr{V}}-\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\mathbf{R}_{\mathscr{V}\mathscr{V}}+|\mathscr{V}|^{-1/2}\mathbf{B}_{\mathscr{V}\mathscr{V}}\mathbf{B}_{\mathscr{V}\mathscr{V}})+$$

$$\left.+\left[|\mathscr{V}|^{-1/2}\mathbf{g}_{\mathscr{V}}-(\mu-\tilde{\mu})\,\mathbf{R}_{\mathscr{V}\mathscr{V}}^{-1}\mathbf{1}_{\mathscr{V}}\right]^{\mathrm{T}}(X_{\mathscr{V}}-\mu\,.\,\mathbf{1}_{\mathscr{V}})\right\}=$$

$$=\tfrac{1}{2}\operatorname{Tr}\left[((\mathbf{R}_{\mathscr{V}\mathscr{V}}^{-1}-\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}+|\mathscr{V}|^{-1/2}\mathbf{B}_{\mathscr{V}\mathscr{V}})\,\mathbf{R}_{\mathscr{V}\mathscr{V}})^2\right]+$$

$$+\left[|\mathscr{V}|^{-1/2}\mathbf{g}_{\mathscr{V}}-(\mu-\tilde{\mu})\,\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\,.\,\mathbf{1}_{\mathscr{V}}\right]^{\mathrm{T}}\mathbf{R}_{\mathscr{V}\mathscr{V}}\left[|\mathscr{V}|^{-1/2}\mathbf{g}_{\mathscr{V}}-(\mu-\tilde{\mu})\,\tilde{\mathbf{R}}_{\mathscr{V}\mathscr{V}}^{-1}\,.\,\mathbf{1}_{\mathscr{V}}\right]\to 0$$

for $\mathscr{V}\nearrow\mathscr{Z}^d$ due to Lemma 5.7. And by Lemma 5.5 we obtain

$$\operatorname{var}\left[\mathbf{Q}_{\mathscr{V}}^{1}(X_{\mathscr{V}})-\boldsymbol{\beta}^{\mathrm{T}}\Delta_{\mathscr{V}}^{\mathbf{0}}\right]\to 0\quad\text{for}\quad\mathscr{V}\nearrow\mathscr{Z}^d\,.$$

Finally, in the proof of Lemma 3.7 we have shown that

$$|\mathscr{V}|^{1/2}(\mu-\tilde{\mu})\to f(0)\,\tilde{h}-f(0)^2\,h\,2\,.\,\mathbf{1}_{\mathscr{M}}^{\mathrm{T}}U^{\sim}\,,$$

and therefore $\mathbf{Q}_{\mathscr{V}}^{2}\to-\frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\Psi}_{\mathbf{0}}\boldsymbol{\beta}$ for $\mathscr{V}\nearrow\mathscr{Z}^d$ by Lemma 5.8. □

Now, let us denote

$$l_{\mathscr{V}}^{\mathbf{0}}(X_{\mathscr{V}})=\left(l_{\mathscr{V},h}^{\mathbf{0}}(X_{\mathscr{V}})=\frac{\partial\log\mathbf{P}^{\mathbf{0}}(X_{\mathscr{V}})}{\partial h},\quad\left\{l_{\mathscr{V},k}^{\mathbf{0}}(X_{\mathscr{V}})=\frac{\partial\log\mathbf{P}^{\mathbf{0}}(X_{\mathscr{V}})}{\partial U(k)}\right\}_{k\in\mathscr{M}}\right).$$

Lemma 5.10. It holds

$$|\mathscr{V}|^{-1} \operatorname{var}[\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}}) + \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{l}^{\mathbf{0}}_{\mathscr{V}}(X_{\mathscr{V}})] \to 0 \quad \text{for} \quad \mathscr{V} \nearrow \mathscr{Z}^d .$$

Proof. By direct differentiation we obtain

$$l^{\mathbf{0}}_{\mathscr{V},h}(X_{\mathscr{V}}) = -f(0) \,.\, \mathbf{1}^{\mathrm{T}}_{\mathscr{V}} \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}}(X_{\mathscr{V}} - \mu \,.\, \mathbf{1}_{\mathscr{V}})$$

and

$$l^{\mathbf{0}}_{\mathscr{V},k}(X_{\mathscr{V}}) = -2 f(0) \,.\, \mathbf{1}^{\mathrm{T}}_{\mathscr{V}} \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}}(X_{\mathscr{V}} - \mu \,.\, \mathbf{1}_{\mathscr{V}}) -$$
$$- \tfrac{1}{2}[(X_{\mathscr{V}} - \mu \,.\, \mathbf{1}_{\mathscr{V}})^{\mathrm{T}} \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}} \mathbf{R}^{(k)}_{\mathscr{V}\mathscr{V}} \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}}(X_{\mathscr{V}} - \mu \,.\, \mathbf{1}_{\mathscr{V}}) - \operatorname{Tr}(\mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}} \mathbf{R}^{(k)}_{\mathscr{V}\mathscr{V}})] ,$$

for $k \in \mathscr{M}$, where

$$\mathbf{R}^{(k)}(j, l) = -\frac{\partial \mathbf{R}(j, l)}{\partial U(k)} = (2\pi)^{-d} \int_{\mathscr{I}_d} 2 \cos (j - l) \lambda \cos (k\lambda) [f(\lambda)]^2 \, \mathrm{d}\lambda =$$
$$= \mathbf{W}(j - l, k) \quad \text{for every} \quad j, l \in \mathscr{Z}^d , \quad k \in \mathscr{M} .$$

Thus,

$$\boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{l}^{\mathbf{0}}_{\mathscr{V}}(X_{\mathscr{V}}) = -[\tilde{h} + \mu 2 \,.\, \mathbf{1}^{\mathrm{T}}_{\mathscr{M}} U^{\sim}] f(0) \, \mathbf{1}^{\mathrm{T}}_{\mathscr{V}} \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}}(X_{\mathscr{V}} - \mu \,.\, \mathbf{1}_{\mathscr{V}}) -$$
$$- \tfrac{1}{2}[(X_{\mathscr{V}} - \mu \,.\, \mathbf{1}_{\mathscr{V}})^{\mathrm{T}} \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}} \mathbf{R}_{\mathscr{V}\mathscr{Z}^d} \mathbf{B} \mathbf{R}_{\mathscr{Z}^d\mathscr{V}} \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}}(X_{\mathscr{V}} - \mu \,.\, \mathbf{1}_{\mathscr{V}}) - \operatorname{Tr}(\mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}} \mathbf{R}_{\mathscr{V}\mathscr{Z}^d} \mathbf{B} \mathbf{R}_{\mathscr{Z}^d\mathscr{V}})]$$

since

$$\sum_{k \in \mathscr{M}} U^{\sim}(k) \, \mathbf{R}^{(k)}(j, l) = \tfrac{1}{2} \sum_{k \in \mathscr{Z}^d} \mathbf{B}(k, 0) (2\pi)^{-d} \int_{\mathscr{I}_d} 2 \cos (j - l) \lambda \cos \lambda k \,.$$
$$. \sum_{t \in \mathscr{Z}^d} R(t) \, \mathrm{e}^{\mathrm{i}t\lambda} f(\lambda) \, \mathrm{d}\lambda = \sum_{t,k \in \mathscr{Z}^d} \sum R(t - j) \, B(t - k) \, R(k - l)$$
$$\text{for every} \quad j, l \in \mathscr{Z}^d .$$

Thus

$$|\mathscr{V}|^{-1} \operatorname{var}[\mathbf{F}_{\mathscr{V}}(X_{\mathscr{V}}) + \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{l}^{\mathbf{0}}_{\mathscr{V}}(X_{\mathscr{V}})] = |\mathscr{V}|^{-1} \tfrac{1}{2} \operatorname{Tr}[(\mathbf{B}^{-1}_{\mathscr{V}\mathscr{V}} \mathbf{R}_{\mathscr{V}\mathscr{Z}^d} \mathbf{B} \mathbf{R}_{\mathscr{Z}^d\mathscr{V}} - \mathbf{B}_{\mathscr{V}\mathscr{V}} \mathbf{R}_{\mathscr{V}\mathscr{V}})^2] +$$
$$+ |\mathscr{V}|^{-1} [\mathbf{g}_{\mathscr{V}} - (\tilde{h} + \mu 2 \,.\, \mathbf{1}^{\mathrm{T}}_{\mathscr{M}} U^{\sim}) f(0) \, \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}} \,.\, \mathbf{1}_{\mathscr{V}}]^{\mathrm{T}} \, \mathbf{R}_{\mathscr{V}\mathscr{V}} [\mathbf{g}_{\mathscr{V}} - (\tilde{h} + \mu 2 \,.\, \mathbf{1}^{\mathrm{T}}_{\mathscr{M}} U^{\sim}) \,.$$
$$. f(0) \, \mathbf{R}^{-1}_{\mathscr{V}\mathscr{V}} \,.\, \mathbf{1}_{\mathscr{V}}]$$

$\to 0$ for $\mathscr{V} \nearrow \mathscr{Z}^d$, similarly as in Lemma 5.7. □

6. CONCLUDING REMARKS

I. The method for estimating could be used in a non-Gaussian case as well. We may just consider random fields with spectral densities of the given form. Nevertheless, the deriving asymptotic properties would be much more difficult (if not impossible without any additional assumptions).

II. The implementation of the method is connected with the "thermodynamical" properties of the random fields. We postpone this and some related questions for a forthcoming paper.

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