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POLE PLACEMENT FOR GENERALIZED MDF'S

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The pole placement via static state feedback for generalized MFD's (Matrix Fraction Descriptions) is presented. The difference between the present and existing works on the subject, is that they examine the time domain models. It is shown that realization procedures are simple and easy to compute. Here no decomposition algorithm is needed. Instead the division theorem for polynomial matrices is required.

1. INTRODUCTION

Consider the linear, time invariant system of n first order coupled differential equations

$$E\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx \tag{1b}$$

where E is a singular square matrix, x is an n-vector of state variables, u is an m-vector of inputs sufficiently differentiable and y is a p-vector of outputs. A, B and C are constant matrices of appropriate dimensions. Systems of the form (1) are termed *singular* or generalized *state-space* or semistate or *descriptor*.

The input output relationship, under zero initial conditions, for system (1) is given by its transfer functions $H(s)$ as follows [1]

$$Y(s) = H(s)\,U(s) \tag{2a}$$

where

$$H(s) = C(sE - A)^{-1}\,B\,. \tag{2b}$$

Singular systems of the form (1) appear in many practical applications. For example system (1) may be formed from interconnected systems [2]. Many applications are found in the area of electric networks [3], [4], in Leontieff models in multisector economy [5], in Leslie population models in biology [6], [7] etc.

Systems (1) were studied first in the frequency domain by Rosenbrock [8] and then in the time domain (discrete and continuous), in [6], [7], [9], [10], [11]. In the frequency domain Vextghese et al. [1], [12], [13] examined their impulsive behavior. Christodoulou and Mertzios [14] and Christodoulou [15] examined the realization problem. Problems concerning controllability and observability appeared in [11], [16]–[21]. Finally many synthesis problems, related to feedback contiols have been examined recently, as for example the pole placement problem [22]–[25], the linear quadratic regulator [26]–[29], the decoupling [30]–[33] etc.

In this paper, we examine the following problem. Assume that we are given a generalized right MDF of the form

(3) $$H(s) = N(s)\, D^{-1}(s)$$

where $H(s)$ denotes a rational in s matrix function of dimension $p \times m$. The $N(s)$ and $D(s)$ are polynomial in s matrix functions, of dimensions $p \times m$ and $m \times m$ respectively. It is assumed that $\det(D(s)) \neq 0$. Here $H(s)$ might not be strictly proper or even proper. It is known that $H(s)$ may give many state space realizations of the form (1), [11]. Assume that we pick a realization of (3). Then we apply a linear static state feedback control law of the form

(4) $$u = Fx + Gv$$

in order to relocate its poles. This constitutes the pole placement problem for a realization of a generalized MFD. A solution to this problem, for state space models is presented by Cobb [22]. Also Lewis et al. [24], [25] and Armentano [23] applied the relative eigenstructure and geometric theory to solve the same problem. Here we reexamine it, on a MDF setup, using frequency domain techniques [33].

Our method complements the work of others. Moreover realization procedures for the control laws are simple and easily computable.

It is also mentioned that this method suggests new problems for further research, such as the construction of state observers, design of compensators, etc.

2. GENERALIZED MDF's

It is known that the generalized MFD (3) can be written as a sum of a strictly proper and a polynomial part (using the division Theorem of polynomial matrices [33], p. 389) as follows

(5) $$H(s) = Q(s) + \bar{H}(s)$$

In other terms, (5) is written as

(6) $$H(s) = N(s)\, D^{-1}(s) = Q(s) + R(s)\, D^{-1}(s)$$

where $R(s)\, D^{-1}(s)$ is a strictly proper right MFD and $Q(s)$ and $R(s)$ are *unique*. Many realizations for strictly proper MFD's have been presented in the systems literature, as for example in [34]–[36].

Given $R(s)\,D^{-1}(s)$, we write for $D(s)$

(7) $$D(s) = D_{hc}\,S(s) + D_{lc}\,\Psi(s)$$

where

$$S(s) = \operatorname{diag}\left[s^{k_1} \dots s^{k_m}\right]$$

where k_i are the column degrees of $D(s)$, D_{hc} is the highest column degree coefficient matrix of $D(s)$ and $\Psi(s)$ is defined in [33].

The term $D_{lc}\,\Psi(s)$ accounts for the remaining lower column-degree terms of $D(s)$, with D_{lc} a matrix of coefficients.

We then write for $R(s)\,D^{-1}(s)$

(8a) $$D(s)\,\xi_1(s) = U(s)$$

(8b) $$Y_1(s) = R(s)\,\xi_1(s)$$

The output equation (8b) is written as

$$Y_1(s) = R(s)\,\xi_1(s) = N_{lc}\,\Psi(s)\,\xi_1(s)$$

where N_{lc} is an appropriate matrix of coefficients which shows clearly that Y_1 is obtained as weighted sums of the states.

The polynomial part $Q(s)$ is written as

(9) $$Q(s) = N_{hcp}\,\Psi_p(s)$$

where N_{hcp} is an appropriate matrix of coefficients and $\Psi_p(s)$ is the following matrix

(10) $$\Psi_p^{\mathrm{T}}(s) = \begin{bmatrix} 1\ s \dots s^{n_1} & 0 & \dots & 0 \\ 0 & 1\ s \dots s^{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & 1\ s \dots s^{n_m} \end{bmatrix}$$

Here the n_i's are the column degrees of $Q(s)$ (Highest degrees of the polynomial elements in each column.) We then write for $Q(s)$

(11a) $$I\,\xi_2(s) = U(s)$$

(11b) $$Y_2(s) = Q(s)\,\xi_2(s)$$

Since each u_i is available, differentiate each n_i times to obtain all the required higher order derivatives. There exist m *chains* with n_i differentiatiors in each chain. The outputs of the differentiators are given by the entries of $\Psi_p(s)\,U(s)$.

The output equation (11b) takes the form

(12) $$Y_2(s) = N_{hcp}\,\Psi_p(s)\,\xi_2(s)$$

The overall realization for $H(s)$ (assuming we use the known realization for $R(s)$ . . $D^{-1}(s)$ from [33], p. 406) is shown in Fig. 1.

A controller form realization for a strictly proper MFD $R(s)\,D^{-1}(s)$ is given in [33]. Assume that the triple (A_1, B_1, C_1) is such a realization.

To find an analogous one for the polynomial part, we first set m chains of n_i

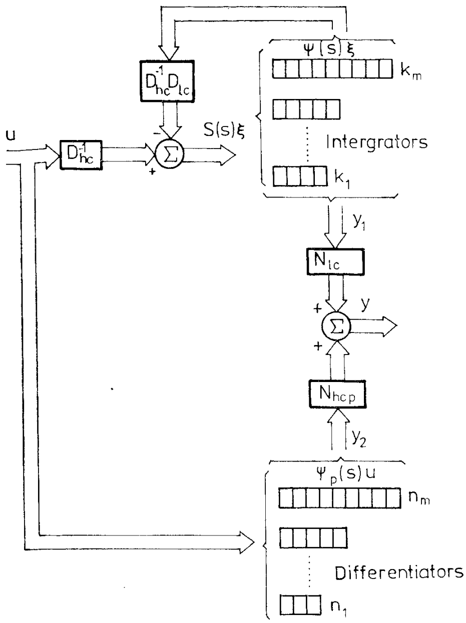

Fig. 1. A schematic of controller form realization.

differentiators each, with access to the input of the first differentiator of each chain and to the outputs of each differentiator of every chain. The corresponding system matrices for the realization of this core system are

$$\text{(13a)} \quad K = \text{block diag} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots 1 & 0 \end{bmatrix}, \; (n_i + 1) \times (n_i + 1), \quad i = 1, 2, \dots, m$$

$$\text{(13b)} \quad B_2^{\mathrm{T}} = \text{block diag} \{ [1 \;\; 0 \;\; \dots \;\; 0], \; 1 \times (n_i + 1), \quad i = 1, 2, \dots, m \}$$

$$\text{(13c)} \quad C_2 = -N_{hcp}$$

It can be checked by direct calculation that

$$\text{(14)} \quad C_2 (sK - I)^{-1} B_2 = Q(s)$$

This realization is controllable [17]. The final realization is

(15a) $$\begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

(15b) $$y = [c_1 \ c_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which has order $\sum_{i=1}^{m} k_i + \sum_{i=1}^{m} n_i + m$.

The beforementioned construction shows how to obtain from an MFD of the form $N(s)\, D^{-1}(s)$ a controllable realization. To obtain the minimal degree, it therefore seems reasonable to reduce MFD to "lowest" terms. This will be justified in the following theorem.

Theorem 2.1. Any realization of a generalized MFD with order equal to $\sum_{i=1}^{m} k_i + \sum_{i=1}^{m} n_i + m$ will be minimal (equivalently controllable and observable) iff the MFD is irreducible and the Q_{hc} (highest degree coefficient matrix in the polynomial $Q(s)$ *as in (6)) has rank m.*

Before proving Theorem 2.1, we explain what is the meaning of controllable and observable in singular systems.

Definition 2.1. The singular system (1) has all its finite modes controllable iff the matrix

$$[sE - AB]$$

has full rank for every finite s in $\mathscr{C}$. It has all infinite modes controllable iff

$$[E \ B]$$

is full rank. A singular system is called controllable iff all its modes (finite and infinite) are controllable.

Definition 2.2. The singular system (1) has all its finite modes observable iff the matrix

$$\begin{bmatrix} sE - A \\ C \end{bmatrix}$$

has full rank for every finite s in $\mathscr{C}$. It has all infinite modes observable iff

$$\begin{bmatrix} E \\ C \end{bmatrix}$$

is full rank. A singular system is called observable iff all its modes (finite and infinite) are observable.

The following lemmas will be needed for the proof of Theorem 2.1.

Lemma 2.1. If there exists one controllable and observable realization of

$N(s)\,D^{-1}(s)$, of order $(\sum_{i=1}^{m} k_i + \sum_{i=1}^{m} n_i + m)$ then all realizations of the same order will also be controllable and observable.

Proof. Use the definition of controllability and observability matrices for system (15) as follows [17]:

$$S_1 = [B_1 \;\; A_1B_1 \ldots A^{n_1-1}B_1] \tag{16a}$$

$$L_1^{\mathrm{T}} = [C_1^{\mathrm{T}} \;\; A_1^{\mathrm{T}} \ldots (A_1^{\mathrm{T}})^{n_1-1}\, C_1^{\mathrm{T}}] \tag{16b}$$

$$S_2 = [B_2 \;\; KB_2 \ldots K^{n_2-1}B_2] \tag{16c}$$

$$L_2^{\mathrm{T}} = [C_2^{\mathrm{T}} \;\; K^{\mathrm{T}}C_2^{\mathrm{T}} \ldots (K^{\mathrm{T}})^{n_2-1}C_2^{\mathrm{T}}] \tag{16d}$$

It is known that (1) is controllable iff S_1 and S_2 are full rank and observable iff L_1 and L_2 are full rank [17].

Assume now that we have the following two state space realizations

$$\begin{bmatrix} I & 0 \\ 0 & K_i \end{bmatrix}, \quad \begin{bmatrix} A_{1i} & 0 \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} B_{1i} \\ B_{2i} \end{bmatrix}, \quad [C_{1i} \; C_{2i}], \quad i = 1, 2\,.$$

Assume also that the realization which corresponds to $i = 1$ is minimal. Then since both are of the same order, it is true that

$$L_1(C_{11}, A_{11})\, S_1(A_{11}, B_{11}) = L_1(C_{12}, A_{12})\, S_1(A_{12}, B_{12}) \tag{17b}$$

$$L_2(C_{21}, K_1)\, S_2(K_1, B_{21}) = L_2(C_{22}, K_2)\, S_2(K_2, B_{22}) \tag{17b}$$

Using analogous reasonings, as for the strictly proper case ([33], p. 440), we easily conclude the proof. □

Lemma 2.2. Suppose $N(s)$ abd $D(s)$ are two polynomial matrices $p \times m$ and $m \times m$ respectively, with $\det(D(s)) \neq 0$. Then for the division algorithm

$$N(s)\, D^{-1}(s) = Q(s) + R(s)\, D^{-1}(s)$$

the following is true: $N(s)$ and $D(s)$ are right coprime iff $R(s)$ and $D(s)$ are right coprime.

Proof. Obvious, see [11]. □

Lemma 2.3. A controller form realization of $N(s)\, D^{-1}(s)$, of order equal to q, will have also its finite modes observable iff the MFD is irreducible. Moreover the infinite modes will be observable iff rank $(Q_{hc}) = m$, where Q_{hc} is defined in Theorem 2.1.

Proof. Use the division algorithm for $N(s)\, D^{-1}(s)$ as in (6). Then a well known lemma [33], for linear systems states that the strictly proper part $R(s)\, D^{-1}(s)$ will be observable iff $R(s)$ and $D(s)$ are right coprime. This together with Lemma 2.2 prove half of Lemma 2.3.

To prove the next half, it suffices to show that when $\begin{bmatrix} K \\ C \end{bmatrix}$ is full rank, then Q_{hc} has rank m, and vice versa.

To this and assume that the matrix $\begin{bmatrix} K \\ C \end{bmatrix}$ is not full rank. But then there exists a real vector $q \in \mathbb{R}^m$ which is different than zero and belongs to the null spaces of K and C simultaneously, i.e., $Kq = Cq = 0$. Assume that q is given by

$$q^{\mathrm{T}} = [q_1\ q_2 \ldots q_{n_1}\ q_{n_1+1} \ldots q_{n_1+n_2} \ldots \ldots q_{n_1+n_2+\ldots+n_m}]$$

By the construction of K in (13a), since $Kq = 0$, then $q_1 = \ldots = q_{n_1-1} = 0$, $q_{n_1+1} = \ldots = q_{n_1+n_2-1} = 0, \ldots, q_{n_m+n2+\ldots+n_m-1} = 0$. But $q_{n_1}, q_{n_1+n_2\ldots}$, $q_{n_1+n_2+\ldots+n_m} = 0$. Since $Cq = 0$, and the columns of C corresponding to $q_{n_1}, q_{n_1+n_2}, \ldots$ $\ldots, q_{n_1n_2+\ldots+n_m}$ are those of Q_{hc}, it holds

$$Q_{hc}\tilde{q} = 0$$

where $\tilde{q}^{\mathrm{T}} = [q_{n_1}, q_{n_1+n_2}, \ldots, q_{n_1+n_2+\ldots+n_m}]$. Hence Q_{hc} is not of rank m.

Conversely, assume there exists a $\tilde{q}$ such that $Q_{hc}\tilde{q} = 0$. Then expand $\tilde{q}$ to 0. by

$$\hat{q}^{\mathrm{T}} = [\underbrace{0\ 0 \ldots 0q_1}_{n_1} \mid \underbrace{0\ 0 \ldots 0q_2}_{n_2} \mid \ldots \mid \underbrace{0\ 0 \ldots 0q_m}_{n_m}]$$

Then for this $\hat{q}$ it is obvious that $C\hat{q} = 0$ and $K\hat{q} = 0$. This conclude the proof for the second part of Lemma 2.3.

Then the proof for Theorem 2.1 follows from the three lemmas above, and analogous reasonings as in the strictly proper, regular case. □

3. LINEAR FEEDBACK

Here we introduce the notion of generalized partial state which is defined via the vector

$$\xi(s) = \begin{bmatrix} \xi_1(s) \\ \xi_2(s) \end{bmatrix} \tag{18}$$

where $\xi_1(s)$ is the regular partial state for the strictly proper (8) and $\xi_2(s)$ the partial

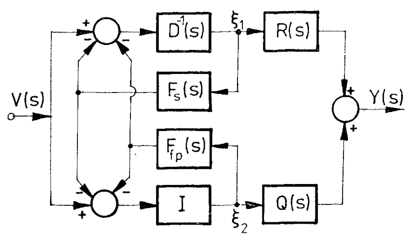

Fig. 2. A schematic of the closed loop system.

state for the polynomial part (11). Therefore, application of a constant state feedback of the form (4) to a state space realization of a generalized MFD (3), corresponds to polynomial feedback of the partial state as follows

(19) $$V(s) = [-F_r\, \Psi(s) \quad F_f\, \Psi_p(s)]\, \xi(s) + V(s)$$

where the matrix G multiplying the new input is set equal to unity $G = I$, and

(20) $$F = [F_s \quad F_f]$$

To this and the equations for the strictly proper and the polynomial part are written more compactly as

(21a) $$\begin{bmatrix} D(s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi_1(s) \\ \xi_2(s) \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix} V(s)$$

(21b) $$Y(s) = [R(s)\; Q(s)] \begin{bmatrix} \xi_1(s) \\ \xi_2(s) \end{bmatrix}$$

The closed loop MFD after the application of the feedback law (19) becomes

(22) $$\begin{bmatrix} D(s) + F_s\, \Psi(s) & F_f\, \Psi(s) \\ F_s\, \Psi(s) & I + F_f\, \Psi_p(s) \end{bmatrix} \xi(s) = \begin{bmatrix} I \\ I \end{bmatrix} V(s)$$

and (21b) remains the same. The matrix on the left of $\xi(s)$ is assumed invertible. Its inverse is given by

(23) $$\begin{bmatrix} D_{F_s}^{-1}(s)\, WL^{-1}T & WL^{-1} \\ L^{-1}T & L^{-1} \end{bmatrix}$$

where

$$\begin{aligned} D_F(s) &= D(s) + F_s\, \Psi(s) \\ W &= -D_{F_s}^{-1}(s)\, F_f\, \Psi_p(s) \\ T &= -F_s\, \Psi(s)\, D_{F_s}^{-1}(s) \\ L &= I + F_f\, \Psi_p(s) + F_s\, \Psi(s)\, W \end{aligned}$$

4. FEEDBACK FOR THE STRICTLY PROPER PART (SLOW FEEDBACK)

Let the feedback F in (20) satisfy

(24) $$F_f = 0$$

Then the control law becomes

(25) $$U(s) = -F_s\, \Psi(s)\, \xi_1(s) + V(s)$$

Using (24) in (23), we find the closed loop transfer function

(26) $$H_{F_s}(s) = (R(s) - Q(s)\, F_s\, \Psi(s))\, D_{F_s}^{-1}(s) + Q(s)$$

The closed loop transfer function (26), which is obtained after slow feedback application (to a controller realization of the MFD) consists of two parts, namely one generalized rational and one polynomial. Using the division Theorem we obtain for the left hand side term of (26):

$$(R(s) - Q(s) F_s \Psi(s)) D_{F_s}^{-1}(s) = Q_1(s) + R_1(s) D_{F_s}^{-1}(s) \tag{27}$$

where $R_1(s) D_{F_s}^{-1}(s)$ is strictly proper. Now $H_{F_s}(s)$ becomes

$$H_{F_s}(s) = [Q(s) + Q_1(s)] + R_1(s) D_{F_s}^{-1}(s) \tag{28}$$

The closed loop system with transfer function given by (28) has poles identical to the roots of the polynomial:

$$p_{F_s}(s) = \det (D_{F_s}(s)) \tag{29}$$

Based on the above results we may now state the following theorem:

Theorem 4.1. Assume we are given a realization of a generalized MFD, to which we apply slow feedback. The finite poles of this realization can be assigned arbitrarily, via this method, iff all of them are controllable.

Proof. Assume that the realization is controllable. Then by applying suitable transformation, we obtain the controller form. But relationships (28) and (29) show that in this case, the poles of the closed loop characteristic polynomial can be assigned arbitrarily under slow feedback.

Conversely, assume that the realization is not finitely controllable, and that all poles can be assigned by slow feedback. Use Gantmakher's [37] transformation to bring the realization, to a form which is composed of a slow and fast subsystems. Next, since the system is not finitely controllable, we can use another transformation to bring it to the form

$$\begin{bmatrix} \bar{A}_{1C} & \bar{A}_{12} \\ 0 & \bar{A}_{1\bar{C}} \end{bmatrix}, \quad \begin{bmatrix} \bar{B}_{1C} \\ 0 \end{bmatrix}$$

Then it is clear, that not all finite modes are assignable, which is a contradiction. This concludes the proof of our theorem. □

The above result is proved in the time domain by Cobb [22].

5. FEEDBACK FOR THE POLYNOMIAL PART (FAST FEEDBACK)

Let F satisfy

$$F_s = 0 \tag{30}$$

Then the control law becomes

$$U(s) = -F_f \Psi_p(s) \xi_2(s) + V(s) \tag{31}$$

Using (30) in (23), we find for the closed loop transfer function

$$H_{F_f}(s) = R(s) D^{-1}(s) - R(s) D^{-1}(s) F_f \Psi_p(s) (I + F_f \Psi_p(s))^{-1} + \\ + Q(s) (I + F_f \Psi_p(s))^{-1} . \tag{32}$$

Use the following notation

$$H_1(s) = R(s)\,D^{-1}(s)$$
$$H_2(s) = \Psi_p(s)\,(I + F_f\,\Psi_p(s))^{-1}$$

Then $H_{F_f}(s)$ becomes

$$H_{F_f}(s) = H_1(s) - H_1(s)\,F_f\,H_2(s) + N_{hcp}\,H_2(s) \tag{33}$$

where $H_1(s)$ is a strictly proper MFD for every F_f and $H_2(s)$ can become a proper MFD for some F_f. The poles of $H_2(s)$ are those of

$$\det(I + F_f\,\Psi'_p(s)) \tag{34}$$

and are *induced* to the controller form realization via F_f. In this form, the controller F_f can be always selected, so that $H_2(s)$ is *proper* (not strictly proper). This means. that the entire transfer function can be made proper by the selection of a suitable F_f. Thus impulses may be eliminated. Hence the following theorem is true.

Theorem 5.1. Impulses in a realization of $H(s)$ can be eliminated by fast feedback iff it is infinitely controllable.

Proof. If the realization is controllable, then apply eq. (34) to eliminate the impulses. Conversely assume that the realization is not infinitely controllable. Then use exactly the same reasoning as in the proof of Theorem 4.1. □

5. CONCLUSIONS

It has been shown that the pole placement in a realization of an MFD can be accomplished in two stages. First the division Theorem is applied to the MFD. Then if any impulsive behaviour is present to the controller form realization, it is eliminated under the procedure of Theorem 5.1.

In the second stage, again we used the division Theorem to the closed loop MFD, after fast feedback, and calculated slow feedback to place the finite eigenvalues in the new controller form realization.

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