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ON TRANSFORMATIONS OF MULTIVARIATE ARMA PROCESSES

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Transformations of multivariate ARMA processes are investigated such that they preserve the ARMA structure. A theorem is given that characterizes a multivariate ARMA process using a property of its covariance function. The theorem is applied to the linear transformation of a multivariate ARMA process and to the scalar product of two ARMA processes.

1. INTRODUCTION

Let $\boldsymbol{X}_t$ be an l-dimensional ARMA (p, q) process. Let $g: \mathbb{R}^l \to \mathbb{R}^s$ be a measurable function. Define a process $\boldsymbol{Z}_t$ by $\boldsymbol{Z}_t = g(\boldsymbol{X}_t)$. Generally, $\boldsymbol{Z}_t$ is not an ARMA process. However, we are interested in such functions g, for which $\boldsymbol{Z}_t$ is also an AR process. It is important to know the orders of the process $\boldsymbol{Z}_t$. In the most cases we find only some bounds for them. From the practical point of view, our considerations enable to express complicated ARMA processes in a form of some transformations of simpler ARMA processes.

The problem of transformations of one-dimensional ARMA processes was investigated in several papers. The results concern mainly sum, product and aggregation of ARMA processes. A unified approach was presented by Engel [2]. It is based on a theorem which characterizes a one-dimensional ARMA process using a property of its covariance function. In Engel's paper also all other references can be found.

There exist only few papers devoted to transformations of multidimensional ARMA processes. They concern mainly the sum of ARMA processes. Let us mention the paper by Lütkepohl [8], who considers a transformation $\boldsymbol{Y}_t = \boldsymbol{F}\boldsymbol{X}_t$ where $\boldsymbol{F}$ is a matrix with real elements.

In our paper we prove some assertions on the scalar product of two ARMA processes. The methods are based on Theorem 3.1, which characterizes a multivariate ARMA process by means of its covariance function. Theorem 3.1 is a generalization of the assertion for one-dimensional processes given in [2]. The problems concerning bounds for orders of the model are demonstrated on some examples.

2. PRELIMINARIES

Let $\varepsilon_t = (\varepsilon_t^1, \ldots, \varepsilon_t^m)'$ be an m-dimensional white noise with $\mathsf{E}_t\varepsilon_t = \mathbf{0}$ and $\operatorname{var} \varepsilon_t = \mathbf{I}_m$, where $\mathbf{I}_m$ is the unit matrix of the type $m \times m$. Let $\mathbf{A}_0, \ldots, \mathbf{A}_p$ and $\mathbf{D}_0, \ldots, \mathbf{D}_q$ be $m \times m$ matrices with real elements such that $\mathbf{A}_p \neq \mathbf{0}$, $\mathbf{D}_q \neq \mathbf{0}$ and that the following conditions are fulfilled:

a) $\mathbf{A}_0$ and $\mathbf{D}_0$ are regular.

b) We define $\mathbf{D}(z) = \sum_{k=0}^{q} \mathbf{D}_k z^k$.

c) If we define $\mathbf{A}(z) = \sum_{k=0}^{p} \mathbf{A}_k z^k$ then the equation $|\mathbf{A}(z)| = 0$ has all roots outside the unit circle.

Then there exist $m \times m$ matrices $\mathbf{C}_0, \mathbf{C}_1, \ldots$ such that

$$[\mathbf{A}(z)]^{-1} \mathbf{D}(z) = \sum_{k=0}^{\infty} \mathbf{C}_k z^k , \quad |z| < 1 ,$$

$$\sum_{k=0}^{\infty} |c_{jl}^k| < \infty \quad \text{for} \quad j, l = 1, \ldots, m$$

and the process $\mathbf{X}_t$ given by

$$\mathbf{X}_t = \sum_{k=0}^{\infty} \mathbf{C}_k \varepsilon_{t-k} \tag{2.1}$$

is the unique solution of

$$\mathbf{A}_0 \mathbf{X}_t + \mathbf{A}_1 \mathbf{X}_{t-1} + \ldots + \mathbf{A}_p \mathbf{X}_{t-p} = \mathbf{D}_0 \varepsilon_t + \mathbf{D}_1 \varepsilon_{t-1} + \ldots + \mathbf{D}_q \varepsilon_{t-q} \tag{2.2}$$

which has the form of a linear process. The process $\mathbf{X}_t$ is called an m-dimensional ARMA process of the order p and q. We denote it by $\mathrm{ARMA}_m(p, q)$. We shall use the symbol $\mathrm{ARMA}_m[p^*, q^*]$ for an $\mathrm{ARMA}_m(p, q)$ model such that its true order p, q satisfies $p \leqq p^*$, $q \leqq q^*$. If B is the back-shift operator, then (2.2) can be equivalently written in the form

$$\mathbf{A}(\mathsf{B})\, \mathbf{X}_t = \mathbf{D}(\mathsf{B})\, \varepsilon_t \tag{2.3}$$

where

$$\mathbf{A}(\mathsf{B}) = \sum_{k=0}^{p} \mathbf{A}_k \mathsf{B}^k = (A_{ij}(\mathsf{B}))_{i,j=1}^m = \Big(\sum_{k=0}^{p} a_{ij}^k \mathsf{B}^k\Big)_{i,j=1}^m$$

$$\mathbf{D}(\mathsf{B}) = \sum_{k=0}^{q} \mathbf{D}_k \mathsf{B}^k = (D_{ij}(\mathsf{B}))_{i,j=1}^m = \Big(\sum_{k=0}^{q} d_{ij}^k \mathsf{B}^k\Big)_{i,j=1}^m$$

Denote

$$p_{ij} = \deg [A_{ij}(\mathsf{B})] , \quad q_{ij} = \deg [D_{ij}(\mathsf{B})]$$

where $\deg [\cdot]$ denotes the degree of a polynomial. Then it holds

$$p = \max_{i,j} p_{ij} , \quad q = \max_{i,j} q_{ij} .$$

The process $\mathbf{X}_t$ is stationary and we get from (2.2) that its covariance function

$\mathbf{R_X}(k)$ satisfies

(2.4) $$\mathbf{A}(\mathrm{B})\,\mathbf{R_X}(k) = \mathbf{0}\,, \quad k > q\,,$$

(2.5) $$\mathbf{A}(\mathrm{B})\,\mathbf{R_X}(q) = \mathbf{D}_q\mathbf{C}_0 \neq \mathbf{0}\,.$$

Let $\mathbf{A}^*(\mathrm{B}) = (A^*_{ij}(\mathrm{B}))^m_{i,j=1}$ be the adjoint matrix to $\mathbf{A}(\mathrm{B})$. Multiplying (2.3) from the left by $\mathbf{A}^*(\mathrm{B})$ and using $|\mathbf{A}(\mathrm{B})|\,\mathbf{I}_m = \mathbf{A}^*(\mathrm{B})\,\mathbf{A}(\mathrm{B})$ we get

(2.6) $$|\mathbf{A}(\mathrm{B})|\,\mathbf{X}_t = \mathbf{A}^*(\mathrm{B})\,\mathbf{D}(\mathrm{B})\,\varepsilon_t\,.$$

This formula will be usefal in the next considerations. Let $H_{ij}(\mathrm{B})$ denotes the ijth cofactor of matrix $\mathbf{A}^*(\mathrm{B})\,\mathbf{D}(\mathrm{B})$. Then it holds $\max_{i,j}\{\deg[H_{ij}(\mathrm{B})]\} = \max\{\deg[A^*_{in}(\mathrm{B})]) + q_{nj};\ i, j, n = 1, \ldots, m\}$. Multiplying (2.6) from the right by $\mathbf{X}'_{t-k}$, taking expectations using (2.1) we obtain

(2.7) $$|\mathbf{A}(\mathrm{B})|\,\mathbf{R_X}(k) = \mathbf{0}$$
$$\text{if} \quad k > \max\{\deg[A^*_{ij}(\mathrm{B})] + q_{jn}\,;\ i, j, n = 1, \ldots, m\}\,.$$

It is known that the matrix $\mathbf{f}(\lambda) = (f_{ij}(\lambda))^m_{i,j=1}$ of spectral densities of the process $\mathbf{X}_t$ exists and equals

$$\mathbf{f}(\lambda) = \frac{1}{2\pi}\,[\mathbf{A}(\lambda)]^{-1}\,\mathbf{D}(\lambda)\,\bar{\mathbf{D}}(\lambda)\,[\bar{\mathbf{A}}(\lambda)]^{-1}$$

where

$$\mathbf{A}(\lambda) = \sum_{k=0}^{p} \mathbf{A}_k\,\mathrm{e}^{-\mathrm{i}k\lambda}\,, \quad \bar{\mathbf{A}}(\lambda) = \sum_{k=0}^{p} \mathbf{A}'_k\,\mathrm{e}^{\mathrm{i}k\lambda}\,,$$
$$\mathbf{D}(\lambda) = \sum_{k=0}^{q} \mathbf{D}_k\,\mathrm{e}^{-\mathrm{i}k\lambda}\,, \quad \bar{\mathbf{D}}(\lambda) = \sum_{k=0}^{q} \mathbf{D}'_k\,\mathrm{e}^{\mathrm{i}k\lambda}\,.$$

Since the matrix $\mathbf{f}(\lambda)$ is regular, the process $\mathbf{X}_t$ is also called regular (see [5] and [6]).

3. CHARACTERIZATION OF MULTIDIMENSIONAL ARMA PROCESS

The following assertion is a generalization of Theorem 1 from [2].

Theorem 3.1. Let $\mathbf{X}_t$ be an m-dimensional regular stationary process with vanishing expectation. Then $\mathbf{X}_t$ is an $\mathrm{ARMA}_m(p, q)$ process if and only if there exist matrices $\mathbf{A}_0, \ldots, \mathbf{A}_p$ such that $\mathbf{A}_p \neq \mathbf{0}$ and that the following conditions are fulfilled:

(3.1) If we define $\mathbf{A}(z) = \sum_{k=0}^{p} \mathbf{A}_k z^k$ then the equation $|\mathbf{A}(z)| = 0$ has all roots outside the unit circle.

(3.2) $$\mathbf{A}(\mathrm{B})\,\mathbf{R_X}(k) = \mathbf{0} \quad \text{for} \quad k > q\,.$$

(3.3) $$\mathbf{A}(\mathrm{B})\,\mathbf{R_X}(q) \neq \mathbf{0}\,.$$

Proof. If $\mathbf{X}_t$ is an $\mathrm{ARMA}_m\,(p, q)$, then the assertion clearly holds.

We prove that the conditions are sufficient. We see that

$$\mathbf{R}_{\mathbf{A}(\mathrm{B})\mathbf{X}}(k) = \sum_{i=0}^{p} \sum_{j=0}^{p} \mathbf{A}_j \mathbf{R}_{\mathbf{X}}(k + i - j)\, \mathbf{A}'_i = \mathbf{0} \quad \text{for} \quad k > q\,,$$

$$\mathbf{R}_{\mathbf{A}(\mathrm{B})\mathbf{X}}(q) \neq \mathbf{0}\,.$$

It remains to show that if the covariance function $\mathbf{R}_{\mathbf{Z}}(k)$ of an m-dimensional process $\mathbf{Z}_t$ satisfies

$$\mathbf{R}_{\mathbf{Z}}(k) = \mathbf{0} \quad \text{for} \quad k > q\,, \tag{3.4}$$

$$\mathbf{R}_{\mathbf{Z}}(q) \neq \mathbf{0}\,, \tag{3.5}$$

then $\mathbf{Z}_t$ is a moving average process. From the Wold decomposition we obtain

$$\mathbf{Z}_t = \sum_{j=0}^{\infty} \mathbf{C}_j \boldsymbol{\eta}_{t-j} \tag{3.6}$$

where $\mathbf{C}_0$ is a regular matrix, $\boldsymbol{\eta}_t = (\eta_t^1, \ldots, \eta_t^m)'$ is an m-dimensional white noise and

$$\mathscr{H}_t(\boldsymbol{\eta}) = \mathscr{H}_t(\mathbf{Z}) \tag{3.7}$$

where $\mathscr{H}_t(\mathbf{Z}) = \mathscr{H}(Z_s^i;\, i = 1, \ldots, m, s < t)$ is a subspace of the Hilbert space $\mathscr{H}(\mathbf{Z}) = \mathscr{H}(Z_t^i;\, i = 1, \ldots, m,\, t = 0, \mp 1, \ldots)$ (see [9]). From (3.6) we get

$$\mathsf{E}\mathbf{Z}_t \boldsymbol{\eta}'_{t-k} = \mathbf{C}_k\,. \tag{3.8}$$

On the other hand, (3.4) implies

$$\mathsf{E}\mathbf{Z}_t \mathbf{Z}'_{t-k} = \mathbf{0} \quad \text{for} \quad k > q\,.$$

In view of (3.7) we have also

$$\mathsf{E}\mathbf{Z}_t \boldsymbol{\eta}'_{t-k} = \mathbf{0} \quad \text{for} \quad k > q\,. \tag{3.9}$$

Comparing (3.8) and (3.9) we can see that $\mathbf{C}_k = 0$ for $k > q$.

Assume that $\mathbf{C}_q = \mathbf{0}$. Then $\mathsf{E}\mathbf{Z}_t \boldsymbol{\eta}'_{t-q} = \mathbf{0}$ but in view of (3.7) we have also $\mathsf{E}\mathbf{Z}_t \mathbf{Z}'_{t-q} = \mathbf{0}$. This contradicts (3.5). Thus $\mathbf{X}_t$ is an $\mathrm{ARMA}_m\,(p, q)$ process. □

We shall apply the existence theorem to special transformations of the ARMA processes. If we know a formula for the covariance function of the process arising from a transformation of ARMA processes, it suffices to find a matrix $\mathbf{A}(\mathrm{B})$ such that the conditions (3.1), (3.2) and (3.3) are fulfilled and to use Theorem 3.1.

4. LINEAR TRANSFORMATION

We start with a transformation of the process which is based on the multiplication by a matrix, the elements of which are real numbers. The next theorem is given in [8] without proof.

Theorem 4.1. Let $\mathbf{F}$ be an $l \times m$ matrix with real elements. Let the rank of $\mathbf{F}$ be l. Let $\mathbf{X}_t$ be an $\mathrm{ARMA}_m\,(p, q)$ process. Then

$$\mathbf{Z}_t = \mathbf{F}\mathbf{X}_t$$

is an $\text{ARMA}_l\,[p^*, q^*]$ process, where

$$p^* = \deg\left[|\mathbf{A}_{\mathbf{X}}(\mathsf{B})|\right],$$

$$q^* = \max\left\{\deg\left[A_{ij}^{\mathbf{X}*}(\mathsf{B})\right] + q_{jn}\,;\; i, j, n = 1, \ldots, m\right\}.$$

By $A_{ij}^{\mathbf{X}*}(\mathsf{B})$ we denote the ijth cofactor of the adjoint matrix to $\mathbf{A}_{\mathbf{X}}^*(\mathsf{B})$.

Proof. The covariance function $\mathbf{R}_{\mathbf{Z}}(k)$ of the process $\mathbf{Z}_t$ is given by

$$\mathbf{R}_{\mathbf{Z}}(k) = \mathsf{E}\mathbf{Z}_t\mathbf{Z}'_{t-k} = \mathbf{F}\mathbf{R}_{\mathbf{X}}(k)\,\mathbf{F}'.$$

Because $\mathsf{E}\mathbf{Z}_t = \mathbf{0}$ and the covariance function $\mathbf{R}_{\mathbf{Z}}(k)$ does not depend on t, the process $\mathbf{Z}_t$ is stationary. From Theorem XVII.1 in Anděl [1] it follows that the process $\mathbf{Z}_t$ is regular. If we express the process $\mathbf{X}_t$ in the form (2.6) and if we denote $\mathbf{A}_{\mathbf{Z}}(\mathsf{B}) = |\mathbf{A}_{\mathbf{X}}(\mathsf{B})|\,\mathbf{I}_l$, then it holds

$$\mathbf{A}_{\mathbf{Z}}(\mathsf{B})\,\mathbf{R}_{\mathbf{Z}}(k) = \mathbf{F}|\mathbf{A}_{\mathbf{X}}(\mathsf{B})|\,\mathbf{R}_{\mathbf{X}}(k)\,\mathbf{F}' = \mathbf{0}$$

$$\text{for}\quad k > \max\left\{\deg\left[A_{ij}^{\mathbf{X}*}(\mathsf{B})\right] + q_{jn};\; i, j, n = 1, \ldots, m\right\}.$$

Applying Theorem 3.1 we get our assertion. □

Theorem 4.1 is to be understood in such a way that one of the ARMA models of the transformed process $\mathbf{Z}_t$ has the order p, q bounded by p^*, q^*. The following example can serve as an illustration.

Example. Consider a two-dimensional AR(1) process $\mathbf{X}_t$ given by

$$\begin{pmatrix} 5, & 10 \\ -1, & 1 \end{pmatrix} \mathbf{X}_t + \begin{pmatrix} 1, & 2 \\ 1, & 1 \end{pmatrix} \mathbf{X}_{t-1} = \varepsilon_t$$

where ε_t is a white noise with $\mathsf{E}\varepsilon_t = \mathbf{0}$, $\operatorname{var}\varepsilon_t = \mathbf{I}_2$. Let $\mathbf{F} = (1, 1)$. Consider the process

$$Z_t = \mathbf{F}\mathbf{X}_t = X_t^1 + X_t^2.$$

Then the spectral density $f_Z(\lambda)$ of the process Z_t is

$$f_Z(\lambda) = f_{11}(\lambda) + f_{12}(\lambda) + f_{21}(\lambda) + f_{22}(\lambda)$$

where $f_{ij}(\lambda)$ are elements of the matrix $\mathbf{f}(\lambda)$ of the spectral densities of the process $\mathbf{X}_t$. After some computations we get

$$f_Z(\lambda) = \frac{5|1 + (3 - \sqrt{8})\,\mathrm{e}^{-\mathrm{i}\lambda}|^2}{(3 - \sqrt{8})\,2\pi|15 - 2\,\mathrm{e}^{-\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}2\lambda}|^2}.$$

This spectral density corresponds to a one-dimensional process ARMA (2,1). We know from Theorem 4.1 that Z_t is an ARMA [2, 1] process. In this case the bounds for the order are equal to the true order.

Choose a matrix $\mathbf{F} = (1, a)$, where a is a real number. Define again

$$Z_t = \mathbf{F}\mathbf{X}_t = X_t^1 + aX_t^2.$$

Analogously as in the previous case we derive the spectral density $f_Z(\lambda)$ of this process

Z_t in the form

$$f_Z(\lambda) = \frac{106 - 104a + 28a^2 + (21 - 20a + 4a^2)(e^{-i\lambda} + e^{i\lambda})}{2\pi|15 - 2\,e^{-i\lambda} - e^{-i2\lambda}|^2}.$$

If $a = \frac{3}{2}$ or $a = \frac{7}{2}$ we get $21 - 20a + 4a^2 = 0$ and then the spectral density corresponds to an AR (2) process. Theorem 4.1 still gives ARMA [2, 1]. In this case the true order of the model is lower than that insured by Theorem 4.1.

The example shows that for a small order of the model of the process $\mathbf{X}_t$ Theorem 4.1 gives good bounds for the order of the transformed process $\mathbf{Z}_t$.

The following theorem is a consequence of Theorem 4.1. It concerns the sum of independent ARMA processes.

Theorem 4.2. Let $\mathbf{X}_t$ and $\mathbf{W}_t$ be independent m-dimensional $\text{ARMA}_m(p_{\mathbf{X}}, q_{\mathbf{X}})$ and $\text{ARMA}_m(p_{\mathbf{W}}, q_{\mathbf{W}})$ processes, respectively. Define

$$\mathbf{Z}_t = \mathbf{X}_t + \mathbf{W}_t.$$

Then $\mathbf{Z}_t$ is an $\text{ARMA}_m[p^*, q^*]$ process, where

$$\begin{aligned} p^* &= \deg[|\mathbf{A}_{\mathbf{X}}(\mathrm{B})|] + \deg[|\mathbf{A}_{\mathbf{W}}(\mathrm{B})|], \\ q^* &= p^* + \max\{G^{\mathbf{X}}_{ijn}, G^{\mathbf{W}}_{ijn};\ i, j, n = 1, \ldots, m\}, \\ G^{\mathbf{X}}_{ijn} &= \deg[A^{\mathbf{X}*}_{ij}(\mathrm{B})] + q^{\mathbf{X}}_{jn} - \deg[|\mathbf{A}_{\mathbf{X}}(\mathrm{B})|], \\ G^{\mathbf{W}}_{ijn} &= \deg[A^{\mathbf{W}*}_{ij}(\mathrm{B})] + q^{\mathbf{W}}_{jn} - \deg[|\mathbf{A}_{\mathbf{W}}(\mathrm{B})|]. \end{aligned}$$

5. SCALAR PRODUCT

Let $\mathbf{X}_t$ and $\mathbf{W}_t$ be m-dimensional stationary processes with $\mathsf{E}\mathbf{X}_t = \mu_{\mathbf{X}}$, $\mathsf{E}\mathbf{W}_t = \mu_{\mathbf{W}}$. Let $\mathbf{X}_t$ and $\mathbf{W}_t$ be stationary cross-correlated, i.e. their cross-correlation functions

$$\begin{aligned} \mathbf{R}_{\mathbf{XW}}(k) &= \mathsf{E}(\mathbf{X}_t - \mu_{\mathbf{X}})(\mathbf{W}_{t-k} - \mu_{\mathbf{W}})', \\ \mathbf{R}_{\mathbf{WX}}(k) &= \mathsf{E}(\mathbf{W}_t - \mu_{\mathbf{W}})(\mathbf{X}_{t-k} - \mu_{\mathbf{X}})' \end{aligned}$$

depend only on k and not on t.

Let a process Z_t be defined by

$$Z_t = \mathbf{X}_t' \mathbf{W}_t = \sum_{i=1}^{m} X^i_t W^i_t.$$

Then for its expectation μ_Z and for its covariance function $\mathbf{R}_Z(t, t-k)$ it holds

$$\mu_Z = \mathrm{Tr}[\mathbf{R}_{\mathbf{XW}}(0)] + \mu'_{\mathbf{X}}\mu_{\mathbf{W}}, \tag{5.1}$$

$$\begin{aligned} \mathbf{R}_Z(t, t-k) = {} & \mu'_{\mathbf{X}}\mathbf{R}_{\mathbf{W}}(k)\,\mu_{\mathbf{X}} + \mu'_{\mathbf{W}}\mathbf{R}_{\mathbf{X}}(k)\,\mu_{\mathbf{W}} + \mu'_{\mathbf{X}}\mathbf{R}_{\mathbf{WX}}(k)\,\mu_{\mathbf{W}} + \\ & + \mu'_{\mathbf{W}}\mathbf{R}_{\mathbf{XW}}(k)\,\mu_{\mathbf{W}} - \{\mathrm{Tr}[\mathbf{R}_{\mathbf{XW}}(0)]\}^2 + \\ & + \mathsf{E}\{[(\mathbf{X}_t - \mu_{\mathbf{X}})'(\mathbf{W}_t - \mu_{\mathbf{W}})][(\mathbf{X}_{t-k} - \mu_{\mathbf{X}})' \times \\ & \times (\mathbf{W}_{t-k} - \mu_{\mathbf{W}})]\} + \end{aligned} \tag{5.2}$$

$$+\ \mathsf{E}[(\mathbf{X}_t - \mu_{\mathbf{X}})' (\mathbf{W}_t - \mu_{\mathbf{W}})] [(\mathbf{X}_{t-k} - \mu_{\mathbf{X}})' \mu_{\mathbf{W}}] +$$
$$+\ \mathsf{E}[(\mathbf{X}_t - \mu_{\mathbf{X}})' (\mathbf{W}_t - \mu_{\mathbf{W}})] [\mu'_{\mathbf{X}}(\mathbf{W}_{t-k} - \mu_{\mathbf{W}})] +$$
$$+\ \mathsf{E}[(\mathbf{X}_t - \mu_{\mathbf{X}})' \mu_{\mathbf{W}}] [(\mathbf{X}_{t-k} - \mu_{\mathbf{X}})' (\mathbf{W}_{t-k} - \mu_{\mathbf{W}})] +$$
$$+\ \mathsf{E}[\mu'_{\mathbf{X}}(\mathbf{W}_t - \mu_{\mathbf{W}})] [(\mathbf{X}_{t-k} - \mu_{\mathbf{X}})' (\mathbf{W}_{t-k} - \mu_{\mathbf{W}})] .$$

We shall consider special cases in which $\mathbf{R}_Z(t, t-k)$ depends only on k so that the process Z_t is stationary.

Theorem 5.1. Let $\mathbf{X}_t$ and $\mathbf{W}_t$ be independent m-dimensional $\mathrm{ARMA}_m(p_{\mathbf{X}}, q_{\mathbf{X}})$ and $\mathrm{ARMA}_m(p_{\mathbf{W}}, q_{\mathbf{W}})$ processes, respectively. Let

$$Z_t = \mathbf{X}'_t \mathbf{W}_t = \sum_{i=0}^{m} X^i_t W^i_t .$$

Then Z_t is an $\mathrm{ARMA}_1[p^*, q^*]$ process, where

$$p^* = \deg [|\mathbf{A}_{\mathbf{X}}(\mathsf{B})|] \deg [|\mathbf{A}_{\mathbf{W}}(\mathsf{B})|] ,$$
$$q^* = p^* + \max \{T^{\mathbf{X}}_{ijn}, T^{\mathbf{W}}_{ijn};\ i, j, n = 1, \ldots, m\} ,$$
$$T^{\mathbf{X}}_{ijn} = \deg [A^{\mathbf{X}*}_{ij}(\mathsf{B})] + q^{\mathbf{X}}_{jn} - \deg [|\mathbf{A}_{\mathbf{X}}(\mathsf{B})|] ,$$
$$T^{\mathbf{W}}_{ijn} = \deg [A^{\mathbf{W}*}_{ij}(\mathsf{B})] + q^{\mathbf{W}}_{jn} - \deg [|\mathbf{A}_{\mathbf{W}}(\mathsf{B})|] .$$

Proof. It follows from (5.1) and (5.2) that

$$\mathsf{E}Z_t = 0 ,$$

$$R_Z(k) = \mathrm{Tr} [\mathbf{R}_{\mathbf{X}}(k) \mathbf{R}_{\mathbf{W}}(k)] = \sum_{i=1}^{m} \sum_{j=1}^{m} R^{\mathbf{X}}_{ij}(k) R^{\mathbf{W}}_{ij}(k) . \tag{5.3}$$

If we use for $\mathbf{X}_t$ the model (2.6), then

$$|\mathbf{A}_{\mathbf{X}}(\mathsf{B})| \mathbf{R}_{\mathbf{X}}(k) = \begin{pmatrix} |\mathbf{A}_{\mathbf{X}}(\mathsf{B})| R^{\mathbf{X}}_{11}(k), \ldots, |\mathbf{A}_{\mathbf{X}}(\mathsf{B})| R^{\mathbf{X}}_{1m}(k) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ |\mathbf{A}_{\mathbf{X}}(\mathsf{B})| R^{\mathbf{X}}_{m1}(k), \ldots, |\mathbf{A}_{\mathbf{X}}(\mathsf{B})| R^{\mathbf{X}}_{mm}(k) \end{pmatrix} = \mathbf{0} .$$

$$\text{for} \quad k > \max \{\deg [A^{\mathbf{X}*}_{ij}(\mathsf{B})] + q^{\mathbf{X}}_{jn};\ i, j, n = 1, \ldots, m\} .$$

We have m^2 difference equations for $R^{\mathbf{X}}_{ij}(k)$. Let $(\alpha^{\mathbf{X}}_j)^{-1}$, $j = 1, \ldots, \deg [|\mathbf{A}_{\mathbf{X}}(\mathsf{B})|]$ be the roots of the polynomial $|\mathbf{A}_{\mathbf{X}}(z)|$. Then

$$R^{\mathbf{X}}_{ij}(k) = \sum_{l=1}^{\deg[|\mathbf{A}_{\mathbf{X}}(\mathsf{B})|]} c^{\mathbf{X}}_{ijl}(\alpha^{\mathbf{X}}_l)^k$$

$$\text{for} \quad k > \max \{T^{\mathbf{X}}_{ijn};\ i, j, n = 1, \ldots, m\} ,$$

where

$$T^{\mathbf{X}}_{ijn} = \deg [A^{\mathbf{X}*}_{ij}(\mathsf{B})] + q^{\mathbf{X}}_{in} - \deg [|\mathbf{A}_{\mathbf{X}}(\mathsf{B})|] .$$

Similar results we get for the process $\mathbf{W}_t$. Let $u = \deg [|\mathbf{A}_{\mathbf{X}}(\mathsf{B})|]$, $v = \deg [|\mathbf{A}_{\mathbf{W}}(\mathsf{B})|]$. Inserting into (5.3) we obtain

$$R_Z(k) = \sum_{l=1}^{u} \sum_{s=1}^{v} d^Z_{ls}(\alpha^{\mathbf{X}}_l \alpha^{\mathbf{W}}_s)^k$$

$$\text{for } k > \max \{T^{\mathbf{X}}_{ijn}, T^{\mathbf{W}}_{ijn};\ i, j, n = 1, \ldots, m\} .$$

Define

$$A_Z(z) = \prod_{l=1}^{u} \prod_{s=1}^{v} (1 - \alpha_l^{\mathbf{X}} \alpha_s^{\mathbf{W}} z) .$$

Then we have that

$$A_Z(z)\, R_Z(k) = 0$$

when

$$k > \max \{T_{ijn}^{\mathbf{X}}, T_{ijn}^{\mathbf{W}};\ i, j, n = 1, \ldots, m\} + \deg [|\mathbf{A_X}(\mathrm{B})|] \deg [|\mathbf{A_W}(\mathrm{B})|] = q^* .$$

Now, we apply Theorem 3.1. □

Theorem 5.2. Let $\mathbf{X}_t$ be an m-dimensional Gaussian $\mathrm{ARMA}_m(p, q)$ process. Let

$$Z_t = \mathbf{X}_t' \mathbf{X}_t = \sum_{i=1}^{m} (X_t^i)^2 ,$$

$$V_t = Z_t - \mu_Z .$$

Then V_t is an $\mathrm{ARMA}_1 [p^*, q^*]$ process, where

$$p^* = \tfrac{1}{2} \deg [|\mathbf{A_X}(\mathrm{B})|] \{\deg [|\mathbf{A_X}(\mathrm{B})|] + 1\} ,$$

$$q^* = p^* + \max \{\deg [A_{ij}^{\mathbf{X}*}(\mathrm{B})] + q_{jn};\ i, j, n = 1, \ldots, m\} - \deg [|\mathbf{A_X}(\mathrm{B})|] .$$

Proof. If we insert $\mathbf{Y}_t = \mathbf{X}_t$ into (5.1) and (5.2), then from Isserlis's relation in [7] we obtain

$$\mathrm{E} V_t = 0 ,$$

$$R_V(k) = 2 \operatorname{Tr} [\mathbf{R_X}(k)\, \mathbf{R_X}(k)] = 2 \sum_{i=1}^{m} \sum_{j=1}^{m} (R_{ij}^{\mathbf{X}})^2 .$$

Further we proceed similarly as in the proof of Theorem 5.1. □

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