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CONTROLLABILITY OF A CLASS OF PERTURBED NONLINEAR SYSTEMS

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Sufficient conditions are established for the controllability of general nonlinear system of the form

$$\dot{x} = g(t, x) + B(t, x)\, u + f(t, x, \dot{x}, u)$$

1. INTRODUCTION

The controllability of various nonlinear control systems has been studied by several authors [3]. One type of method used in many of these studies has been perturbation techniques (for references see [6]). In particular, the results of Dauer [7] give sufficient conditions for controllability of perturbed nonlinear systems. Dacka [5] introduced a new method of analysis to study the controllability of nonlinear systems with implicit derivatives, based on the measure of noncompactness of a set and Darbo's fixed point theorem. This method has been extended to a larger class of nonlinear dynamical systems by Balachandran [2]. The purpose of this paper is to study the problem of controllability of a class of perturbed nonlinear systems with implicit derivative by the method of Dacka [5] (see also [4]). The results generalise the results of Dacka [5] and Dauer [7].

2. PRELIMINARIES

Consider the following nonlinear control system

$$\dot{x}(t) = g(t, x(t)) + B(t, x(t))\, u(t) + f(t, x(t), \dot{x}(t), u(t)), \quad x(t_0) = x_0 \tag{1}$$

defined on $[t_0, t_1] = I$, where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Let $g: I \times \mathbb{R}^n \to \mathbb{R}^n$ have continuous second derivatives with respect to x and continuous first derivative with respect to t, let $\partial g / \partial x$ be bounded, let $B(t, x)$ be an $n \times m$ matrix whose elements are bounded and continuous in $I \times \mathbb{R}^n$ and continuously differentiable in x, let $f(t, x, \dot{x}, u)$ be continuous and bounded. Further of each $y, \bar{y} \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $t \in I$,

$$|f(t, x, y, u) - f(t, x, \bar{y}, u)| \leqq k|y - \bar{y}| \tag{2}$$

where k is a positive constant such that $k < 1$.

Then there exists a solution $y(t, s, x_0)$ [1, 8] of

$$\dot{y} = g(t, y)$$
$$y(s, s, x_0) = x_0$$

defined on I. It follows that the corresponding Jacobian matrix function

$$Z(t, s, x) = \frac{\partial y(t, s, x)}{\partial x}$$

is bounded on $I \times I \times \mathbb{R}^n$ and is the fundamental matrix solution of

$$\frac{\partial Z}{\partial t} = \left[\frac{\partial g(t, y(t, s, x))}{\partial y}\right] Z$$

such that $Z(t, t, x)$ is the identity matrix. By Alekseev's variation of parameters formula [1], for every continuous function $u(t)$ the solution of (1) is given by

$$x(t) = y(t, t_0, x_0) + \int_{t_0}^{t} Z(t, s, x)\, B(s, x)\, u(s)\, \mathrm{d}s + + \int_{t_0}^{t} Z(t, s, x) f(s, x, \dot{x}, u)\, \mathrm{d}s \tag{3}$$

We say that system (1) is completely controllable if for any $x_0, x_1 \in \mathbb{R}^n$ there exists continuous control function $u(t)$, defined on I such that a solution x of (1) satisfies $x(t_1) = x_1$.

Define the controllability matrix

$$W(t_0, t_1; x) = \int_{t_0}^{t_1} Z(t_1, t; x)\, B(t, x)\, B^*(t, x)\, Z^*(t_1, t, x)\, \mathrm{d}t \tag{4}$$

and

$$q(t_1, x, \dot{x}, u) = x_1 - y(t_1, t_0, x_0) - \int_{t_0}^{t_1} Z(t_1, s, x) f(s, x, \dot{x}, u)\, \mathrm{d}s \tag{5}$$

where the star denotes the matrix transpose.

3. MAIN RESULT

Theorem. If the assumptions about the system (1) given above are satisfied, and moreover

$$\inf_{x \in C_n^1[t_0, t_1]} \det W(t_0, t_1; x) > 0$$

then the system (1) is completely controllable.

Proof. The proof of the theorem is similar to the proofs given in [2, 4, 5] and hence it will be only sketched. Define the nonlinear transformation by

$$T\colon C_m(I) \times C_n^1(I) \to C_m(I) \times C_n^1(I)$$

by

$$T([u, x])(t) = [T_1([u, x])(t), T_2([u, x])(t)]$$

where the pair of operators T_1 and T_2 is defined as follows:

(6) $$T_1([u, x])(t) = B^*(t, x(t))\, Z^*(t_1, t, x(t))\, W^{-1} q(t, x, \dot{x}, u)$$

and

(7) $$T_2([u, x])(t) = y(t, t_0, x_0) + \int_{t_0}^{t} Z(t, s, x(s))\, B(s, x(s))\, T_1([u, x])(s)\, \mathrm{d}s + \int_{t_0}^{t} Z(t, s, x(s))\, f(s, x(s), \dot{x}(s), T_1([u, x])(s))\, \mathrm{d}s$$

Obviously the operator T is continuous and maps the space $C_m(I) \times C_n^1(I)$ into itself. Consider the closed convex subset of $C_m(I) \times C_n^1(I)$ by

$$H = \{[u, x] \in C_m(I) \times C_n^1(I) : \|x\| \leqq N_1,\ \|u\| \leqq N_2,\ \|\dot{x}\| \leqq N_3\}$$

where N_1, N_2, N_3 are positive constants depending on the bounds of g, B, f and Z. The mapping T transforms the set H into H. As in [2, 4, 5] the family of the functions $T_1([u, x])(t)$ with $[u, x] \in H$ is equicontinuous. Further the modulus of continuity of $DT_2([u, x])(t)$ (D denotes the derivative with respect to t) can be estimated by (see [4])

$$|DT_2([x, u])(t) - DT_2([x, u])(s)| \leqq k|\dot{x}(t) - \dot{x}(s)| + \beta(|t - s|), \quad \text{with} \quad \beta(h) = o(h),$$

and so, for any set $E \subset H$, it follows that

$$\mu(TE) \leqq k\mu(E),$$

where μ stands for the measure of noncompactness.

By the Darbo fixed point theorem the mapping T has at least one fixed point; therefore, there exists functions $u^* \in C_m(I)$ and $x^* \in C_n^1(I)$ such that

(8) $$u^*(t) = T_1([u^*, x^*])(t)$$

(9) $$x^*(t) = T_2([u^*, x^*])(t)$$

The functions (8) and (9) are the required solutions. Further it is easy to verify that the control $u(t)$ given in (8) steers the system (1) from x_0 to $x(t_1) = x_1$. Hence the system (1) is completely controllable.

Remark. If we assume that the nonlinear function in equation (1) also satisfies the Lipschitz condition, with respect to the state variable, then we can obtain the unique response determined by any control.

4. EXAMPLE

Consider the scalar system

(10) $$\dot{x} = -\tfrac{1}{2}x^3 + x^2 u + \sin\frac{\dot{x}}{2}$$

For any fixed $x \in C_1^1(I)$, the solution of (10) is given by [9]

$$x(t) = y(t, t_0; x_0) + \int_{t_0}^{t} [x^2(s)(t - s) + 1]^{-3/2} \left[x^2(s) u(s) + \sin \frac{\dot{x}(s)}{2} \right] ds$$

where $y(t, t_0; x_0)$ is the solution of $\dot{y} = -\frac{1}{2}y^3$ such that

$$y(t, t_0; x_0) = x_0[x_0^2(t - t_0) + 1]^{-1/2}$$

and

$$Z(t, t_0, x_0) = [x_0^2(t - t_0) + 1]^{-3/2}$$

The controllability matrix W is

$$W(t_0, t_1; x) = \int_{t_0}^{t_1} x^4(s) [x^2(s)(t_1 - s) + 1]^{-3} ds$$

If $t_1 > t_0$, then the infimum of det $W(t_0, t_1; x)$ is greater than zero. Further f satisfies Lipschitz condition with respect to the variable $\dot{x}$ with the constant $k = \frac{1}{2}$. Thus from the above theorem, the dynamical system (10) is completely controllable.

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