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DISCRETE MARGINAL PROBLEM FOR COMPLEX MEASURES

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This paper contains complete analysis of the problem to find all complex measures (on the Cartesian product of a finite system of finite nonempty sets) for every from which a family of its marginal measures is identical with a given family of complex measures. The transform mapping every density of the complex measure into the family of its marginal densities, called here the discrete Radon transform, is considered together with its dual transform and their inversions are given in effective forms. Illustrating examples are added.

1. FORMULATION OF THE PROBLEM

Let $n \geqq 2$ be an integer, $N = \{1, 2, \ldots, n\}$, and for $j \in N$ let $G_j = \{0, 1, \ldots, m_j - 1\}$, where $m_j \geqq 2$ is an integer too. For every subset $I \subset N$ we define $G_I = \bigtimes_{j \in I} G_j$ being the Cartesian product of the sets G_j, $j \in I$; $G_N = G$, $G_0 = \{0\}$ (0 is used instead of $\emptyset$ in all indices). Elements of G_I will be denoted a_I, b_I, c_I and for $a_I = (a_j)_{j \in I} \in G_I$, $a_j \in G_j$, the symbol a_I^J $(J \subset I)$ will denote an element of G_J formed from a_I by the projection on G_J, namely $a_I^J = (a_j)_{j \in J}$; $a_I^I = a_I$, $a_I^0 = 0$ (index N will be omitted by every type of notation).

Assuming that μ is a complex measure on G (we consider the σ-algebra on every finite set is simply the system of all its subsets), let μ^I $(I \subset N)$ be its marginal measure, i.e. the measure on G_I:

$$\forall A_I \subset G_I:\ \mu^I(A_I) = \mu(A_I \times G_{N-I}) = \sum_{b \in G,\, b^I \in A_I} \mu(\{b\}).$$

Let a non-empty system $\mathscr{A}$ of subsets of N and a family $(\mu_I)_{I \in \mathscr{A}}$ of complex measures (μ_I on G_I) be given. The discrete marginal problem for complex measures may be now formulated precisely; we look for all complex measures μ on G holding $\forall I \in \mathscr{A}$ $\mu^I = \mu_I$, i.e. having the given family of complex measures as the family of its marginal measures.

This problem was solved (probably first) by Kellerer in [1], where a necessary and sufficient condition for solvability and a formula for finding one of solutions are presented. Up to this time the interest has concentrated to more general marginal problems; arbitrary measure spaces are considered and constraints for measures are added (see Kellerer [1], [2] and Studený [3]). Especially, the marginal problem for probability measures becomes very important in view of applications to intensional expert systems (suggested by Perez and Jiroušek in [5]). In this case special approximative methods were found an developed (see [5] and [4]).

Before outlining the contents of the present paper we reformulate the problem in view of the theory of the Radon transform and in view of the mathematical background to the "image reconstruction from projections" problems; see Helgason [6] and Herman [7] respectively. We conserve existing connections in the terminology and notation.

As all measures considered here are naturally represented by their densities (with respect to counting measures) we shall deal with them only. Let C^{G_I} be the linear space of all complex functions on G_I (over the complex numbers) and let $C^{G_{\mathscr{A}}}$ be the linear space of all complex functions on $G_{\mathscr{A}} = \bigcup_{I \in \mathscr{A}} G_I$, i.e. every element g of $C^{G_{\mathscr{A}}}$ has the form $g = (g_I)_{I \in \mathscr{A}}$, where $g_I \in C^{G_I}$.

Let the discrete Radon transform be a linear mapping from C^G into $C^{G_{\mathscr{A}}}$ assigning to every $f \in C^G$ the function $\hat{f} = (\hat{f}_I)_{I \in \mathscr{A}} \in C^{G_{\mathscr{A}}}$:

$$\forall I \in \mathscr{A} \quad \forall a_I \in G_I\colon\ \hat{f}_I(a_I) = \sum_{b \in G,\, b^I = a_I} f(b)$$

and similarly let the dual transform to the Radon transform maps $g \in C^{G_{\mathscr{A}}}$ into $\check{g} \in C^G$.

Evidently, if $f \in C^G$ represents a complex measure μ on G then $\hat{f} = (\hat{f}_I)_{I \in \mathscr{A}}$ represents the family $(\mu^I)_{I \in \mathscr{A}}$ and the discrete marginal problem is equivalent to the solution of the equation $\hat{f} = g$ for a known $g \in C^{G_{\mathscr{A}}}$ representing the family $(\mu_I)_{I \in \mathscr{A}}$. The dual problem: the solution of the equation $\check{g} = f$ for a known $f \in C^G$ is closely connected with the preceding problem and will be interesting too.

In this paper these two dual problems are completely analysed by use of the concepts of the discrete convolutions and the discrete Fourier transform on C^G and $C^{G_{\mathscr{A}}}$. We present bases of the kernels and ranges of Radon transform and its dual, new conditions for solvability of the both problems and the sets of all solutions with effective formulae for finding the solutions with the smallest L^2-norm on C^G and $C^{G_{\mathscr{A}}}$.

2. DISCRETE CONVOLUTION, PRODUCT AND FOURIER TRANSFORM

Every set $G_j, j \in N$, will be understood as the cyclic group with the operation mod (m_j), every G_I, $I \subset N$, as the direct product of the groups G_j, $j \in I$.

We define on C^{G_I} two binary operations: the discrete convolution and the discrete product of a pair of functions:

$$\forall f_I, f'_I \in C^{G_I}: \ (f_I * f'_I)(a_I) = \sum_{b_I \in G_I} f_I(b_I) f'_I(a_I - b_I)$$
$$(f_I \,.\, f'_I)(a_I) = f_I(a_I) \,.\, f'_I(a_I)$$

both these operations are commutative, associative and linear in both arguments.

Let m_I denote the number of all elements of G_I: $m_I = \prod_{j \in I} m_j$, $m_0 = 1$ and let $m_{\mathscr{A}} = \sum_{I \in \mathscr{A}} m_I$. If we write:

$$[a_I, b_I] = 2\pi i \sum_{j \in I} \frac{a_j b_j}{m_j}, \quad [a_0, b_0] = 0, \quad \text{where} \quad a_I = (a_j)_{j \in I}, \quad b_I = (b_j)_{j \in I},$$

(we consider real operations here) then the discrete Fourier transform on G_I has the form:

$$\forall f_I \in C^{G_I}: \ \boldsymbol{F}_I f_I(a_I) = \sum_{b_I \in G_I} f_I(b_I) \exp [a_I, b_I]$$

and its inversion:

$$\boldsymbol{F}_I^{-1} f_I(a_I) = \frac{1}{m_I} \sum_{b_I \in G_I} f_I(b_I) \exp - [a_I, b_I] \,.$$

The trivial identity:

$$\sum_{b \in G} \exp \pm [a, b] = m_N \quad a = 0$$
$$= 0 \quad a \neq 0$$

will be used in the paper without references; e.g. it allows to verify these identities:

$$\forall f_I \in C^{G_I}: \quad \boldsymbol{F}_I^{-1} \boldsymbol{F}_I f_I = f_I \ \text{ and } \ \|\boldsymbol{F}_I f_I\|^2 = m_I \|f_I\|^2 \,,$$
$$\forall f_I, f'_I \in C^{G_I}: \ \boldsymbol{F}_I(f_I * f'_I) = \boldsymbol{F}_I f_I \,.\, \boldsymbol{F}_I f'_I \ \text{ and } \ (\boldsymbol{F}_I f_I, \boldsymbol{F}_I f'_I) = m_I (f_I, f'_I) \,,$$

assuming the scalar product and the norm on C^{G_I} are natural:

$$(f_I, f'_I) = \sum_{a_I \in G_I} f_I(a_I) \,.\, \overline{f'_I(a_I)} \qquad \|f_I\|^2 = \sum_{a_I \in G_I} |f_I(a_I)|^2 \,.$$

We shall need analogical operations on $C^{G_{\mathscr{A}}}$, we define them coordinatewisely: $\forall g, g' \in C^{G_{\mathscr{A}}}$; $g * g' = (g_I * g'_I)_{I \in \mathscr{A}}$ etc. The scalar product and the norm on $C^{G_{\mathscr{A}}}$ are defined by:

$$(g, g') = \sum_{I \in \mathscr{A}} (g_I, g'_I) \qquad \|g\|^2 = \sum_{I \in \mathscr{A}} \|g_I\|^2 \,.$$

The notion of δ_I^J-function on G_I $(J \subset I \subset N)$ will be used with a lot of advantages:

$$\delta_I(a_I) = 1 \quad a_I = 0$$
$$= 0 \quad \text{otherwise}$$

and

$$\delta_I^J(a_I) = \delta_J(a_I^J) \,;$$
$$\delta_I^0 = \delta_0 = 1 \,.$$

3. BASIC IDENTITIES

According to the introduced notation the Radon transform takes the form:

(1) $$\forall I \in \mathscr{A}, \quad \forall a_I \in G_I: \; \hat{f}_I(a_I) = \sum_{b \in G} f(b)\, \delta_I(a_I - b^I).$$

Then for $f \in C^G$, $g \in C^{G_{\mathscr{A}}}$:

$$(\hat{f}, g) = \sum_{I \in \mathscr{A}} (\hat{f}_I, g_I) = \sum_{I \in \mathscr{A}} \sum_{a_I \in G_I} \sum_{b \in G} f(b)\, \delta_I(a_I - b^I)\, \overline{g_I(a_I)} =$$
$$= \sum_{I \in \mathscr{A}} \sum_{b \in G} f(b)\, \overline{g_I(b^I)} = \sum_{b \in G} f(b) \sum_{I \in \mathscr{A}} \overline{g_I(b^I)} = (f, \check{g}),$$

what gives the form of the dual transform:

(2) $$\forall g \in C^{G_{\mathscr{A}}}, \quad \forall b \in G: \; \check{g}(b) = \sum_{I \in \mathscr{A}} g_I(b^I),$$

it may be called the backprojection (analogical transforms are in [7]).

Lemma 1.

(3) $$\forall f, f' \in C^G: \; (f * f')^\wedge = \hat{f} * \hat{f}',$$

(4) $$\forall f \in C^G, \quad \forall g \in C^{G_{\mathscr{A}}}: \; f * \check{g} = (\hat{f} * g)^\vee.$$

Proof. The proof is based on the following computations:

$$\forall I \in \mathscr{A}, \quad \forall a_I \in G_I: \; (f * f')^\wedge_I (a_I) = \sum_{b \in G} \sum_{c \in G} f(c) f'(b - c)\, \delta_I(a_I - b^I) =$$
$$= \sum_{c \in G} f(c) \sum_{b \in G} f'(b) \,.\, \delta_I(a_I - c^I - b^I) = \sum_{c \in G} f(c) \,.\, \hat{f}'(a_I - c^I) =$$
$$= \sum_{b_I \in G_I} \sum_{c \in G} f(c) \,.\, \hat{f}'(a_I - b_I) \,.\, \delta_I(c^I - b_I) = \sum_{b_I \in G_I} \hat{f}'(a_I - b_I) \hat{f}(b_I) =$$
$$= (\hat{f} * \hat{f}')_I (a_I),$$

$$\forall a \in G: \; (f * \check{g})(a) = \sum_{b \in G} f(b) \,.\, \sum_{I \in \mathscr{A}} g_I(a^I - b^I) =$$
$$= \sum_{I \in \mathscr{A}} \sum_{b \in G} f(b) \sum_{c_I \in G_I} g_I(a^I - c_I)\, \delta_I(c_I - b^I) = \sum_{I \in \mathscr{A}} \sum_{c_I \in G_I} g_I(a^I - c_I) \hat{f}_I(c_I) =$$
$$= \sum_{I \in \mathscr{A}} (\hat{f}_I * g_I)(a^I) = (\hat{f} * g)^\vee (a).$$

□

Lemma 2.

(5) $$\forall f \in C^G: \; (\hat{f})^\vee = f * \varphi,$$

(6) $$\forall g \in \mathscr{R}_\wedge: \; (\check{g})^\wedge = g * \hat{\varphi}$$

where $\varphi = \sum_{I \in \mathscr{A}} \delta^I$ ($\mathscr{R}_\vee$ is the range of the Radon transform).

Note. Analogical identities are known in the theory of the Radon transform (see Helgason [6]).

Proof. The proof of the first identity is trivial:

$$\forall f \in C^G, \quad \forall a \in G: \quad (\hat{f})^\vee (a) = \sum_{I \in \mathscr{A}} \sum_{b \in G} f(b)\, \delta_I(a^I - b^I) =$$

$$= \sum_{b \in G} f(b) \,.\, \sum_{I \in \mathscr{A}} \delta^I(a - b) = \sum_{b \in G} f(b)\, \varphi(a - b) = (f * \varphi)(a)\,.$$

For $g \in \mathscr{R}_\wedge$ we may find $f \in C^G$ such that $\hat{f} = g$ and then:

$$(\check{g})^\wedge = ((\hat{f})^\vee)^\wedge = (f * \varphi)^\wedge = \hat{f} * \hat{\varphi} = g * \hat{\varphi} \qquad \square$$

We have seen the function φ plays an interesting role in the last identities, we shall need its Fourier transform. As the Fourier transform of δ^I is $m_{N-I}\delta^{N-I}$ we have:

$$F\varphi = \sum_{I \in \mathscr{A}} m_{N-I} \delta^{N-I} \tag{7}$$

Let $U = \{a \in G; \exists I \in \mathscr{A}: a^{N-I} = 0\}$ then

$$F\varphi(a) = 0 \quad a \in U$$

$$F\varphi(a) = 0 \quad a \notin U$$

i.e., U is the "support" of $F\varphi$.

4. KERNELS AND RANGES

Let $\mathscr{N}_\wedge$, $\mathscr{R}_\wedge$ ($\mathscr{N}_\vee$, $\mathscr{R}_\vee$) be the kernel and the range of the discrete Radon transform (its dual); we present their bases using functions of the following type: for $b_I \in G_I$ let $\xi^{b_I} \in C^{G_I}$:

$$\forall a_I \in G_I: \quad \xi^{b_I}(a_I) = \frac{1}{\sqrt{m_I}} \exp - [a_I, b_I]$$

The Fourier transform of ξ^{b_I}:

$$F_I \xi^{b_I}(a_I) = \frac{1}{\sqrt{m_I}} \sum_{c_I \in G_I} \exp\,[c_I, a_I - b_I] = \sqrt{(m_I)}\, \delta_I(a_I - b_I)$$

helps us to establish orthogonality:

$$(\xi^{b_I}, \xi^{a_I}) = \frac{1}{m_I} (F_I \xi^{b_I}, F_I \xi^{a_I}) = \delta_I(b_I - a_I)\,,$$

i.e. $\{\xi^b, b \in G\}$ is a base of C^G.

The Radon transform of ξ^b is:

$$\forall I \in \mathscr{A} \quad \forall a_I \in G_I: \quad \hat{\xi}^b_I(a_I) = \sum_{c \in G} \frac{1}{\sqrt{m_N}} \exp - [c, b] \,.\, \delta_I(a_I - c^I) =$$

$$= \frac{1}{\sqrt{m_N}} \sum_{c_I \in G_I} \sum_{d_{N-I} \in G_{N-I}} \exp - [c_I, b^I] \,.\, \exp - [d_{N-I}, b^{N-I}] \,.\, \delta_I(a_I - c_I) =$$

$$= \frac{1}{\sqrt{m_N}} \exp - [a_I, b^I] \sum_{d_{N-I} \in G_{N-I}} \exp - [d_{N-I}, b^{N-I}] = \sqrt{(m_{N-I})}\, \xi^{b^I}(a_I)\, \delta^{N-I}(b),$$

what gives immediately $\hat{\xi}^b = 0$ for $b \notin U$, moreover for $a, b \in U$:

$$(\hat{\xi}^b, \hat{\xi}^a) = \sum_{I \in \mathscr{A}} (\hat{\xi}^b_I, \hat{\xi}^a_I) = \sum_{I \in \mathscr{A}} m_{N-I}\, \delta^{N-I}(a)\, \delta^{N-I}(b)\, (\xi^{b^I}, \xi^{a^I}) =$$
$$= \sum_{I \in \mathscr{A}} m_{N-I}\, \delta^{N-I}(a)\, \delta^{N-I}(b)\, \delta^I(a-b) = \delta(a-b)\, F\varphi(a)$$

Another way of phrasing this result is the following

Lemma 3. $C^G = \mathscr{N}_\wedge \oplus \mathscr{R}_\vee$ where $\{\xi^b, b \in U\}$ is an orthonormal base of $\mathscr{R}_\vee$ and $\{\xi^b, b \in G - U\}$ is an orthonormal base of $\mathscr{N}_\wedge$, (the orthogonal decomposition is trivial).

The situation on the space $C^{G\mathscr{A}}$ is less clear. We propose to use the functions $\eta^{b,J} = (\eta^{b,J}_I)_{I \in \mathscr{A}} \in C^{G\mathscr{A}}$ supposing first $b \in G\ J \in \mathscr{A}$:

$$\eta^{b,J}_I = \xi^{b^J} . \delta^{N-J}(b) \quad I = J$$
$$= 0 \qquad\qquad I \neq J$$

Evidently $\eta^{b,J} \neq 0$ if and only if $b \in U$ and $J \in \mathscr{A}_b = \{K \in \mathscr{A}, b^{N-K} = 0\}$; for $a, b \in U$, $I \in \mathscr{A}_a$, $J \in \mathscr{A}_b$:

$$(\eta^{a,I}, \eta^{b,J}) = \sum_{K \in \mathscr{A}} (\eta^{a,I}_K, \eta^{b,J}_K) = 0 \qquad I \neq J$$
$$= (\xi^{a^I}, \xi^{b^I}) \quad I = J \in \mathscr{A}_a \cap \mathscr{A}_b\,,$$

in the second case we became 1 only if $a^I = b^I$ but $a^{N-I} = 0 = b^{N-I}$ what shows orthonormality of the set $\{\eta^{b,J};\ b \in U,\ J \in \mathscr{A}_b\}$. This set has

$$\sum_{b \in U} |\mathscr{A}_b| = \sum_{b \in U} \sum_{I \in \mathscr{A}} \delta^{N-I}(b) = \sum_{I \in \mathscr{A}} m_I = m_{\mathscr{A}} \quad \text{elements}$$

so that it is a base of $C^{G\mathscr{A}}$.

Thus, if $\mathscr{H}^b$ $(b \in U)$ denotes the linear span of $\{\eta^{b,J}, J \in \mathscr{A}_b\}$ then $\hat{\xi}^b = \sum_{I \in \mathscr{A}_b} \sqrt{(m_{N-I})}$. . $\eta^{b,I} \in \mathscr{H}^b$. Let finally:

$$\mathscr{H}^b = \mathscr{R}^b_\wedge \oplus \mathscr{N}^b_\vee$$

be the orthogonal decomposition of $\mathscr{H}^b$, where $\mathscr{R}^b_\wedge$ is the linear span of $\hat{\xi}^b$. We can summarize:

Lemma 4. $C^{G\mathscr{A}} = \mathscr{N}_\vee \oplus \mathscr{R}_\wedge$, where $\{\hat{\xi}^b; b \in U\}$ is an orthogonal base of $\mathscr{R}_\wedge$ and $\mathscr{N}_\vee = \bigoplus_{b \in U} \mathscr{N}^b_\vee$, where $\mathscr{N}^b_\vee$ is the set of all functions from $\mathscr{H}^b$ orthogonal to $\hat{\xi}^b$.

Remark. Considering the compositions of the Radon transform and the back-projection we see from $(b \in U)$

$$\forall a \in G: \quad (\hat{\xi}^b)^\vee (a) = \sum_{I \in \mathscr{A}_b} \sqrt{(m_{N-I})}\, \xi^{b^I}(a^I) = \xi^b(a) \sum_{I \in \mathscr{A}_b} m_{N-I}$$

and from

$$((\xi^b)^\vee)^\wedge = \hat{\xi}^b \sum_{I \in \mathscr{A}_b} m_{N-I}$$

that $\sum_{I \in \mathscr{A}_b} m_{N-I}$ $(b \in U)$ are nonzero eigenvalues and ξ^b, $\hat{\xi}^b$ are the corresponding eigenvectors of these compositions.

5. SOLUTION OF THE PROBLEM

Another characterization of $\mathcal{N}_\wedge$ and $\mathcal{R}_\vee$ follows from Lemma 3 using the Fourier transform

$$\mathcal{N}_\wedge = \{f \in C^G; Ff|_U = 0\}$$

$$\mathcal{R}_\vee = \{f \in C^G; Ff|_{G-U} = 0\}$$

Let $\omega \in \mathcal{R}_\vee$ be the function (unambiguously) given by $F\omega . F\varphi = \chi_U$ where χ_U is the characteristic function of the set U, i.e. ω may be expressed as

$$\forall a \in G: \; \omega(a) = \frac{1}{m_N} \sum_{b \in U} \frac{1}{F\varphi(b)} \exp - [a, b] \tag{8}$$

and its Radon transform $\hat{\omega}$ is

$$\forall I \in \mathcal{A}, \quad \forall a_I \in G_I: \quad \hat{\omega}_I(a_I) = \sum_{c \in G} \frac{1}{m_N} \sum_{b \in U} \frac{1}{F\varphi(b)} \exp - [c, b] . \delta_I(c^I - a_I) =$$

$$= \frac{1}{m_I} \sum_{b \in U} \frac{1}{F\varphi(b)} \delta^{N-I}(b) \exp - [a_I, b^I] = \frac{1}{m_I} \sum_{b_I \in G_I} \frac{\exp - [a_I, b_I]}{\sum_{J \in \mathcal{A}} m_{N-J} \, \delta_I^{I-J}(b_I)} \tag{9}$$

From the facts $\forall b \in G: F\varphi(b) = F\varphi(-b)$ and $b \in U \Leftrightarrow -b \in U$ we get simply $\omega(a) = \overline{\omega(a)}$ $(a \in G)$, i.e. ω is a real function; ω, $\hat{\omega}$ will be called the convolving functions.

Lemma 5.

$$\forall f \in \mathcal{R}_\vee: \; f = \omega * (\hat{f})^\vee \tag{10}$$

$$\forall g \in \mathcal{R}_\wedge: \; g = \hat{\omega} * (\check{g})^\wedge \tag{11}$$

Proof. Let $f \in \mathcal{R}_\vee$, we apply the Fourier transform to $\omega * (\hat{f})^\vee$ and then we use Lemma 2. We get

$$F(\omega * (\hat{f})^\vee) = F(\omega * \varphi * f) = F\omega . F\varphi . Ff = Ff \tag{12}$$

because it is

$$0 = Ff|_{G-U} = F\omega|_{G-U} = F\varphi|_{G-U} \quad \text{and} \quad (F\omega . F\varphi)|_U = 1 .$$

Let $g \in \mathcal{R}_\wedge$, then there is $f \in \mathcal{R}_\vee$, $\hat{f} = g$ and using (3) and (10)

$$\hat{\omega} * (\check{g})^\wedge = (\omega * \check{g})^\wedge = (\omega * (\hat{f})^\vee)^\wedge = \hat{f} = g . \qquad \square$$

Theorem 1. The equation $\hat{f} = g$, where $g \in C^{G_\mathcal{A}}$ is known, is solvable if and only if for g (11) holds, and in this case the set of all solutions is given by

$$\omega * \check{g} + \mathcal{N}_\wedge . \tag{13}$$

Proof. If the equation is solvable then $g \in \mathcal{R}_\wedge$ and (11) holds due to Lemma 5. Conversely, let (11) hold and $g \notin \mathcal{R}_\wedge$ then g can be written as the orthogonal sum:

$$g = g' + g'' , \quad g' \in \mathcal{R}_\wedge , \quad 0 \neq g'' \in \mathcal{N}_\vee , \quad (g', g'') = 0 .$$

Using Lemma 5 (for g') we get

$$g = \hat{\omega} * ((g' + g'')^{\vee})^{\wedge} = \hat{\omega} * (\check{g}')^{\wedge} = g'$$

what is a contradiction.

Let now $g \in \mathscr{R}_{\wedge}$ then any element of $\omega * \check{g} + \mathscr{N}_{\wedge}$ is evidently a desirable solution according to (3) and (11), and if f solves this equation then

$$(f - \omega * \check{g})^{\wedge} = g - g = 0\,, \quad \text{i.e.} \quad f - \omega * \check{g} \in \mathscr{N}_{\wedge}\,. \qquad \square$$

Theorem 2. The equation $\check{g} = f$, where $f \in C^G$ is known, is solvable if and only if for f (10) holds, and in this case the set of all solutions is given by

$$\hat{\omega} * \hat{f} + \mathscr{N}_{\vee}\,. \tag{14}$$

The proof can be omitted as it is "quite dual" to the preceding one.

Corollary. Due to Lemma 1 $\omega * \check{g} = (\hat{\omega} * g)^{\vee} \in \mathscr{R}_{\vee}$, i.e. this solution of the discrete marginal problem has the smallest norm among all its solutions. "Dual" corollary is true too.

Remark 1. Kellerer in [1] formulates the following necessary and sufficient condition for solvability:

$$\forall I, J \in \mathscr{A}\,, \quad \forall a_{I \cap J} \in G_{I \cap J}\colon \sum_{b_I \in G_I} g_I(b_I)\, \delta_{I \cap J}(a_{I \cap J} - b_I^{I \cap J}) = \sum_{b_J \in G_J} g_J(b_J)\, \delta_{I \cap J}(a_{I \cap J} - b_J^{I \cap J}) \tag{15}$$

In our notation it can be proved directly without any difficulty (for $g = \sum_{c \in U} \sum_{K \in \mathscr{A}_c} g^{c,K} \eta^{c,K}$ ($g^{c,K}$ – complex numbers) we have $g_I = \sum_{c \in U} g^{c,I} \xi^{c^I} \delta^{N-I}(c)$ and (15) is equivalent to

$$\forall c \in U \quad \forall I, J \in \mathscr{A}_c\colon \quad g^{c,I} \sqrt{(m_{N-J})} = g^{c,J} \sqrt{(m_{N-I})}$$

i.e. $\sum_{K \in \mathscr{A}_c} g^{c,K} \eta^{c,K} \in \mathscr{R}^c_{\wedge}$, $g \in \mathscr{R}_{\wedge}$; necessity is trivial).

Remark 2. Evidently $\dim \mathscr{R}_{\wedge} = \dim \mathscr{R}_{\vee} = |U| = \sum_{0 \neq \mathscr{B} \subset \mathscr{A}} (-1)^{|\mathscr{B}|-1} m^{\cap \mathscr{B}}$ (in [1] this is proved by induction).

Remark 3. Kellerer found in [1] this solution f of the problem:

$$\forall a \in G\colon \; f(a) = \sum_{0 \neq \mathscr{B} \subset \mathscr{A}} (-1)^{|\mathscr{B}|-1} \frac{1}{m_{N - \cap \mathscr{B}}}\, g_{\cap \mathscr{B}}(a^{\cap \mathscr{B}}) \tag{16}$$

where $g_{\cap \mathscr{B}} \in C^{G_{\cap \mathscr{B}}}$ and if $I \in \mathscr{B} \subset \mathscr{A}$ then $g_{\cap \mathscr{B}}(a_{\cap \mathscr{B}}) = \sum_{b_I \in G_I} g_I(b_I)\, \delta_{\cap \mathscr{B}}(a_{\cap \mathscr{B}} - b_I^{\cap \mathscr{B}})$ are for $g \in \mathscr{R}_{\wedge}$ unambiguously defined functions.

Then

$$f(a) = \sum_{I \in \mathscr{A}} \sum_{I \in \mathscr{B} \subset \mathscr{A}} \frac{(-1)^{|\mathscr{B}|-1}}{|\mathscr{B}|} \frac{1}{m_{N - \cap \mathscr{B}}}\, g_{\cap \mathscr{B}}(a^{\cap \mathscr{B}}) =$$

$$= \sum_{I\in\mathscr{A}} \sum_{b_I\in G_I} g_I(b_I) \sum_{I\in\mathscr{B}\subset\mathscr{A}} \frac{(-1)^{|\mathscr{B}|-1}}{|\mathscr{B}|} \frac{1}{m_{N-\cap\mathscr{B}}} \delta_{\cap\mathscr{B}}(a^{\cap\mathscr{B}} - b_I^{\cap\mathscr{B}}) =$$

$$= \sum_{I\in\mathscr{A}} \sum_{b_I\in G_I} g_I(b_I) \,.\, \varrho_I(a^I - b_I) = (g * \varrho)^{\vee}(a)$$

where

$$\varrho \in C^{G_{\mathscr{A}}}, \quad \varrho_I = \sum_{I\in\mathscr{B}\subset\mathscr{A}} \frac{(-1)^{|\mathscr{B}|-1}}{|\mathscr{B}|} \frac{\delta_I^{\cap\mathscr{B}}}{m_{N-\cap\mathscr{B}}} \quad \forall I \in \mathscr{A},$$

i.e., Kellerer's solution $(g * \varrho)^{\vee} \in \mathscr{R}_{\vee}$ is the same as that one $(g * \hat{\omega})^{\vee}$ in Theorem 1.

Remark 4. As ω and $\hat{\omega}$ are real functions, all solutions of the discrete marginal problem for signed measures may be obtained as the real parts of the found solutions.

Remark 5. When another Radon transform $(\omega * \check{g})^{\cap}$ (for given $\mathscr{B} \supset \mathscr{A}$) of the solution $\omega * \check{g}$ is needed only, it may be computed by:

$$\forall I \in \mathscr{B}, \quad \forall a_I \in G_I\colon \; (\omega * \check{g})_I^{\cap}(a_I) = \sum_{b\in G} (\hat{\omega} * g)^{\vee}(b) \,.\, \delta_I(a_I - b^I) =$$

$$= \sum_{b\in\mathscr{A}} \sum_{J\in\mathscr{A}} (\hat{\omega} * g)_J (b^J) \,.\, \delta_I(a_I - b^I) =$$

$$= \sum_{J\in\mathscr{A}} m_{N-(I\cup J)} \sum_{b_{I\cup J}\in G_{I\cup J}} (\hat{\omega} * g)_J (b_{I\cup J}^J) \,.\, \delta_I(a_I - b_{I\cup J}^I) =$$

$$= \sum_{J\in\mathscr{A}} m_{N-(I\cup J)} \sum_{b_J\in G_J} (\hat{\omega} * g)_J (b_J) \,.\, \delta_{I\cap J}(a_I^{I\cap J} - b_J^{I\cap J}) \,.$$

6. EXAMPLES

1. Let $n \geqq 2$, $m_1 = m_2 = \ldots = m_n = 2$ and $\mathscr{A} = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$. For $I = \{i_1, i_2\} \in \mathscr{A}$, $a_I = (a_{i_1}, a_{i_2}) \in G_I$ due to (9) we have:

$$\hat{\omega}_I(a_I) = \frac{1}{2^2} \sum_{b_I\in G_I} \frac{(-1)^{a_{i_1}b_{i_1} + a_{i_2}b_{i_2}}}{\sum_{J\in\mathscr{A}} 2^{n-2}\, \delta_I^{I-J}(b_I)} =$$

$$= \frac{1}{2^n} \left(\frac{(-1)^{a_{i_2}}}{\sum_{J\in\mathscr{A}} \delta_I^{I-J}(0, 1)} + \frac{(-1)^{a_{i_1}}}{\sum_{J\in\mathscr{A}} \delta_I^{I-J}(1, 0)} + \frac{(-1)^{a_{i_1}+a_{i_2}}}{\sum_{J\in\mathscr{A}} \delta_I^{I-J}(1, 1)} + \frac{1}{\sum_{J\in\mathscr{A}} \delta_I^{I-J}(0, 0)} \right) =$$

$$= \frac{1}{2^n} \left(\frac{1}{n} + \frac{1}{2} ((-1)^{a_{i_1}} + (-1)^{a_{i_2}}) + (-1)^{a_{i_1}+a_{i_2}} \right)$$

i.e.

$$\hat{\omega}_I\colon \quad \hat{\omega}_I(1, 0) = \frac{-1 + 1/n}{2^n} \qquad \hat{\omega}_I(1, 1) = \frac{1/n}{2^n}$$

$$\hat{\omega}_I(0, 0) = \frac{2 + 1/n}{2^n} \qquad \hat{\omega}_I(0, 1) = \frac{-1 + 1/n}{2^n}$$

Let $g \in C^{G,\mathscr{A}}$,

$$g_I\colon \begin{matrix} \beta & \gamma \\ \alpha & \beta \end{matrix}$$

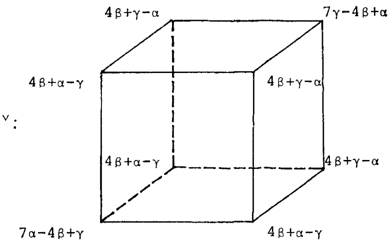
(α, β, γ real nonnegative numbers, $\alpha + 2\beta + \gamma = 1$), be briefly given similarly as in the foregoing table, then (15) holds, i.e. $g \in \mathscr{R}_\wedge$, and

$(g * \hat{\omega})_I$:

$$\begin{matrix} \dfrac{\alpha+\gamma}{2^n}(-1 + 1/n) + \dfrac{\beta}{2^{n-1}}(1 + 1/n) & \dfrac{\alpha}{n\,2^n} + \dfrac{\beta}{2^{n-1}}(-1 + 1/n) + \dfrac{\gamma}{2^n}(2 + 1/n) \\[2ex] \dfrac{\alpha}{2^n}(2 + 1/n) + \dfrac{\beta}{2^{n-1}}(-1 + 1/n) + \dfrac{\gamma}{n\,2^n} & \dfrac{\alpha+\gamma}{2^n}(-1 + 1/n) + \dfrac{\beta}{2^{n-1}}(1 + 1/n) \end{matrix}$$

Due to Theorem 1 $(g * \hat{\omega})^\vee = \omega * \check{g}$ solves the equation $\hat{f} = g$. For $n = 3$ this solution may be represented by the cube:

8. $(g * \hat{\omega})^\vee$:



We see although g represents probability measures, $(g * \hat{\omega})^\vee$ in general does not.

2. Let k be an integer, $n \geqq k + 1$, $m_1 = m_2 = \ldots = m_n = 2$ and $\mathscr{A} = \{I \subset N, |I| = k\}$. For $I \in \mathscr{A}$ $a_I \in G_I$ (we assume that exactly $0 \leqq l \leqq k$ coordinates of a_I are equal to 1) we have:

$$\hat{\omega}_I(a_I) = \frac{1}{2^{|I|}} \sum_{b_I \in G_I} \frac{\exp - [a_I, b_I]}{\sum\limits_{J \in \mathscr{A}} 2^{n-|J|}\, \delta_I^{I-J}(b_I)} = \frac{1}{2^n} \sum_{b_I \in G_I} \frac{\exp - [a_I, b_I]}{\sum\limits_{J \in \mathscr{A}} \delta_I^{I-J}(b_I)}\,.$$

If b_I has exactly j coordinates equal to 1, the denominator is $\sum_m \binom{k-j}{m}\binom{n-k}{m}$ and then

$$\hat{\omega}_I(a_I) = \frac{1}{2^n} \sum_{j=0}^{k} \frac{\sum\limits_m (-1)^m \binom{l}{m}\binom{k-l}{j-m}}{\sum\limits_m \binom{k-j}{m}\binom{n-k}{m}}$$

For $k = 1$:

$$\hat{\omega}_I(0) = \frac{1}{2^n}\frac{1+n}{n} \qquad \hat{\omega}_I(1) = \frac{1}{2^n}\frac{1-n}{n}$$

and for $k = 2$:

$$n(n-1)\,2^n\hat{\omega}_I: \quad \begin{matrix} -n^2+n+2 & n^2-3n+2 \\ n^2+n+2 & -n^2+n+2 \end{matrix}$$

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