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EFFICIENCY RATE AND LOCAL DEFICIENCY OF THE MOST POWERFUL TESTS IN THE MODEL OF CONTAMINACY WITH GENERAL NEIGHBOURHOODS

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Efficiency rate and local deficiency for the most powerful tests in the model of contaminacy with general neighbourhoods are found. These two means compare behaviour of different statistical procedures (in this case — tests) from a family $\mathscr{S}$ with respect to an apriori given probability measure, i.e. they may be applied e.g. to find an optimal member of $\mathscr{S}$ with respect to a given level of contamination (in the case then $\mathscr{S}$ is not a family of optimal procedures with respect to the considered model of contaminacy (see [7])) and characterize losses we incur when selecting the procedure which does not correspond to actual level of contamination.

1. INTRODUCTION

Let us recall that it was in 1964 when P. J. Huber introduced new families of tests and estimators of location. Each member of these families was related (or optimal in some sense) to one fixed contamination level.

Had we learnt the actual contamination level we could have selected the pertaining optimal procedure, if any (with respect to considered model of contaminacy). Generally, we do not know contamination level and hence we select a procedure relying on our idea (or if you want our guess) of contamination level. Then, we could try to introduce a measure which compares the behaviour of the selected statistical procedure with the best one in the case of a fixed contamination level or, more precisely, in the case of known corresponding least favourable distribution. One possible measure of this kind is the local deficiency introduced in Section 3.

Another measure which we shall consider is the efficiency rate; it could be suitably used, e.g. for the study of minimum distance estimators in the framework of the contamination model — see [7].

From what was said follows that efficienty rate and local definciency are characteristics of (robust) procedures dual to the Hampel's influence curve and power and size dependency (see [8]).

While Hampel's characteristics measure the sensitivity of a given statistical procedure to a change in the model, our characteristics compare different statistical procedures in a fixed model (see [7, 8] for a more detail discussion and explanation).

2. NOTATIONS

Let $\boldsymbol{N}$ denote the set of all positive integers and $\boldsymbol{R}$ the real line being endowed with the Borel σ-algebra $\mathscr{B}$. Let $(\mathscr{X}, \mathscr{C})$ be a measurable space and $\mathbb{M}$ the class of all probability measures on it. For any $P \in \mathbb{M}$ let $P^{(n)}$ denote the product measure on the corresponding product space. Finally, A^0 will denote the interior of a subset A of a topological space $\mathscr{Y}$.

3. DEFINITION OF EFFICIENCY RATE AND LOCAL DEFICIENCY

Definition 1. Let Γ and Θ be linearly ordered topological spaces and $\mathscr{S} = \{SP_\gamma\}_{\gamma \in \Gamma}$ and $\mathscr{W} = \{\mathscr{P}_\theta\}_{\theta \in \Theta}$ a family of statistical procedures and a family of subsets of $\mathbb{M}$, respectively, such that for any $\theta_0 \in \Theta^0$ we have

$$\mathscr{P}_{\theta_0} = \bigcup_{\theta \leqq \theta_0} \mathscr{P}_\theta = \bigcap_{\theta \geqq \theta_0} \mathscr{P}_\theta \, .$$

Moreover let $C(\gamma, \theta)$: $\Gamma \times \Theta \to \boldsymbol{R}$ be a statistical characteristic of the statistical procedure SP_γ with respect to a class $\mathscr{P}_\theta$. Then if for some $(\gamma, \theta) \in \Gamma \times \Theta$ there exist a limit

$$\lim_{\nu \to \gamma} \frac{C(\nu, \theta) - C(\gamma, \theta)}{\nu - \gamma}$$

we shall call it the *efficiency rate* of the family $\mathscr{S}$ with respect to $\mathscr{W}$ at the point (γ, θ) and denote it by $ER(\gamma, \theta)$. Moreover let for some $\theta \in \Theta$ there are point(s) $\gamma_i = \gamma_i(\theta) \in \Gamma$ such that $ER(\gamma_i, \theta) = 0$ for $i = 1, \ldots, k$, and neighbourhood(s) $O(\gamma_i)$ such that for all $\nu \in O(\gamma_i)$ $ER(\nu, \theta)$ exists. If then exist(s) limit(s)

$$\lim_{\nu \to \gamma_i} \frac{ER(\nu, \theta)}{\nu - \gamma_i} = l_i(\theta)$$

we shall call $l_j(\theta) = \min_{1 \leqq i \leqq k} l_i(\theta)$ the *local deficiency* of $\mathscr{S}$ with respect to $\mathscr{W}$ at the point θ and denote it by $LF(\theta)$. In this case we shall say that the SP_{γ_j} is the optimal procedure in $\mathscr{S}$ for $\mathscr{P}_\theta$.

Remark 1. As we shall see later the typical case is with $k = 1$, i.e. in the family $\mathscr{S}$ there is just one procedure with efficiency rate equal to zero for given contamination level (see also [7]).

4. MODEL OF CONTAMINACY

In what follows we shall use the general neighbourhoods model of contaminacy (see [3]). Let us recall it in a form suitable for our purposes.

Let for $i = 0, 1$, $P_i \in \mathcal{M}$, $P_0 \neq P_1$ be fixed and let ε_i: $[0, 1) \to [0, 1)$ and $\delta_i[0, 1) \to \to [0, 1)$ be continuously differentiable and strictly increasing mappings such that for any $\tau \in (0, 1)$, $\varepsilon_i(\tau) + \delta_i(\tau) \in (0, 1)$, $\varepsilon_i(0)$ and $\delta_i(0) = 0$. Then put

$$\mathscr{P}_i(\tau) = \{Q \in \mathcal{M}: Q(B) \geqq (1 - \varepsilon_i(\tau)) P_i(B) - \delta_i(\tau) \text{ for all } B \in \mathscr{C}\}.$$

Let us restrict ourselves to $I \subset [0, 1)$ such that, for all $\tau \in I$,

$$\mathscr{P}_0(\tau) \cap \mathscr{P}_1(\tau) = \emptyset .$$

5. CLASS OF TESTS

It has been proved by Rieder in [3] that, in the above framework, there exists a family of the most powerful tests, say $\{\Psi_{n,\alpha,\tau}\}_{n\in\mathbf{N},\,\alpha\in(0,1),\tau\in I}$ ($\Psi_{n,\alpha,\tau}$: $\mathscr{X}^n \to [0, 1]$) for testing $\mathscr{H}_\tau$: $Q \in \mathscr{P}_0(\tau)$ against $\mathscr{A}_\tau$: $Q \in \mathscr{P}_1(\tau)$. Then we have

$$\sup_{Q\in\mathscr{P}_0(\tau)} \mathsf{E}_Q \Psi_{n,\alpha,\tau} \leqq \alpha$$

and

(1)
$$\inf_{Q\in\mathscr{P}_1(\tau)} \mathsf{E}_Q \Psi_{n,\alpha,\tau} = \sup_{\Psi\in\mathscr{F}_{n,\alpha,\tau}} \inf_{Q\in\mathscr{P}_1(\tau)} \mathsf{E}_Q \Psi ,$$

where

$$\mathscr{F}_{n,\alpha,\tau} = \{\Psi: \mathscr{X}^n \to [0, 1] , \sup_{Q\in\mathscr{P}_0(\tau)} \mathsf{E}_Q \Psi \leqq \alpha\}$$

and $\mathsf{E}_Q \Psi_{n,\alpha,\tau}$ denotes the mean value of $\Psi_{n,\alpha,\tau}$ with respect to the product measure $\prod_{i=1}^{n} Q_i$ (defined on the corresponding product space) with $Q_i = Q$ for all i.

Let us recall that $\Psi_{n,\alpha,\tau}$ is in continuous case of the form

$$\Psi_{n,\alpha,\tau}(x) = \begin{cases} 1 & x \in \left\{x \in \mathscr{X}^n: \sum_{i=1}^{n} \log \dfrac{\mathrm{d}Q_{1\tau}(x_i)}{\mathrm{d}Q_{0\tau}(x_i)} \geqq c_{n,\alpha,\tau}\right\}, \\ 0 & \text{otherwise} , \end{cases}$$

where $(Q_{0\tau}, Q_{1\tau})$ is a least favourable pair of distributions with respect to $(\mathscr{P}_0(\tau), \mathscr{P}_1(\tau))$ and $c_{n,\alpha,\tau}$ is chosen so that

$$\mathsf{E}_{Q_{0\tau}} \Psi_{n,\alpha,\tau} = \alpha$$

(For the definition of the least favourable pair see [3]; a way how to find this pair will be recalled later.)

The following illustrative example will give more motivation to what was said in Introduction.

Let $P_0 = \mathcal{N}(0, 1)$, $P_1 = \mathcal{N}(1, 1)$, $\varepsilon_0(\tau) = \varepsilon_1(\tau) = 2\delta_0(\tau) = 2\delta_1(\tau) = \tau$ and $n = 40$.

Now let us assume that $\tau = \cdot 03$. Then we may construct for all $\alpha \in (0, 1)$ the test $\Psi_{40,\alpha,.03}$. The full line in Fig. 1 presents the dependence of the second kind error probabilities on the first ones. (Let us call this curve "the assumed".) But if the actual value of τ is $\cdot 05$ then the test $\Psi_{40,\alpha,.03}$ will have the dependence of the second kind error probabilities on the first one given by dashed line. Finally having learnt that the actual value of τ is .05 we could construct the test $\Psi_{40,\alpha,.05}$ which would have better performance under this actual value of τ then the test $\Psi_{40,\alpha,.03}$. These attainable (i.e. the best possible for $\tau = .05$) values of the second kind error probabilities (in dependence on the value of the first kind ones) are drawn by dotted-and-dashed line.

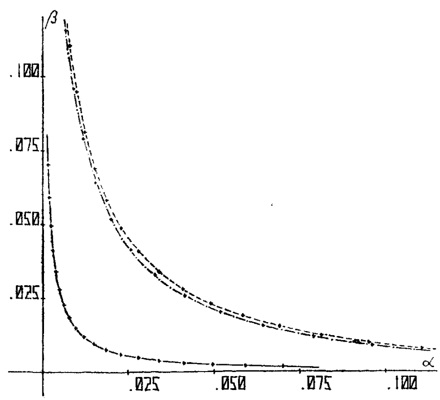

Fig. 1. The assumed (———), actual (— — —) and attainable (— · — ·) dependence of the second kind error probabilities on the first kind errors.

Now we see that while a "difference" between the assumed and actual curve of dependence can be characterized by means of size and power dependency (see [5, 6]), the difference between the actual and attainable curves of dependence can be measured by the local deficiency. It seems that for testing statistical hypotheses the help of the both tools — size and power dependency as well as efficiency rate and local deficiency is of the same importance since we need to have an idea about possible actual and attainable values of the first and second kind error probabilities with respect to the assumed ones. On the other hand, in the case of point estimation, the efficiency rate and the local deficiency may be more useful than the influence curve in the practice (cf. [7, 8]) since it is usually sufficient to know that the performance of selected estimator is not too much worse than the performance of the best possible one without having at hand precise knowledge, e.g., about actual value of the asymptotic variance of the used estimator.

6. EFFICIENCY RATE AND LOCAL DEFICIENCY FOR ROBUST TESTING SCHEME

In this section we shall first describe the concepts introduced in Definition 1 in the robust testing framework. Then we shall derive the formulas for the efficiency rate and the local deficiency and give a graph of the local deficiency in a special case.

We shall see that the supremum of the probability of the second kind error $\beta(v, P)$ over $\mathscr{P}_1(\tau)$ is not convenient as a statistical characteristic $C(v, \tau)$. Actually, using the test $\Psi_{n_0,\alpha_0,v}$ (of sample size n_0 and test size α_0) instead of $\Psi_{n_0,\alpha_0,\tau}$ changes not only the probability of the second kind error but also that of the first kind error. Hence, we have to define a statistical characteristic in a little more complicated way.

Let us define as a statistical procedure $\mathscr{T}_v$ the class of tests $\{\Psi_{n,\alpha,v}\}_{n\in\mathbf{N},\alpha\in(0,1)}$, i.e. $\mathscr{S} = \{\{\Psi_{n,\alpha,v}\}_{n\in\mathbf{N},\alpha\in(0,1)}\}_{v\in I}$. As a family $\mathscr{W}$ we shall consider $\{\mathscr{P}_1(v)\}_{v\in I}$. Now fixing $\alpha_0 \in (0, 1)$ and $n_0 \in \mathbf{N}$ (large enough – it will be specified later) we define $C(v, \tau)$ as follows. Put

$$\beta_{n_0}(v, \tau, P) = 1 - \mathsf{E}_P \Psi_{n_0,\alpha_0(v,\tau),v}$$

where $\alpha_0(v, \tau)$ is given by

$$\alpha_0(v, \tau) = \sup\{\alpha \colon \mathsf{E}_{Q_{0\tau}} \Psi_{n_0,\alpha,v} \leqq \alpha_0\}\,.$$

(Let us recall that $Q_{0\tau}$ is the first component of the least favourable pair $(Q_{0\tau}, Q_{1\tau})$ of the $(\mathscr{P}_0(\tau), \mathscr{P}_1(\tau))$ – see the previous paragraph.) Then we might assume $C(v, \tau)$ to be $\sup_{P\in\mathscr{P}_1(\tau)} \beta_{n_0}(v, \tau, P)$. But this supremum is not easy (if not almost impossible) to evaluate; hence we use the asymptotic approximation of it and denote it by $AP(v, \tau, P)$. So, denoting by Φ the distribution function of the standard normal law and for $\alpha \in (0, 1)$ by u_α its upper α-quantile, we have

$$(2) \qquad AP(v, \tau, P) = \Phi\left(\frac{u_{\alpha_0}\sqrt{[\operatorname{var}_{Q_{0\tau}} w_v(x)]} + n_0^{1/2}\,[\mathsf{E}_{Q_{0\tau}} w_v(x) - \mathsf{E}_P w_v(x)]}{\sqrt{[\operatorname{var}_P w_v(x)]}}\right)$$

where $w_v(x) = \log(\mathrm{d}Q_{1v}(x)/\mathrm{d}Q_{0v}(x))$. Then we put

$$C(v, \tau) = \sup_{P\in\mathscr{P}_1(\tau)} AP(v, \tau, P)\,.$$

Now we need to recall the form of the least favourable pair (see [3], Lemma 4.4 and Theorem 5.2).

Let $\Delta \in \mathrm{d}P_1/\mathrm{d}P_0$ and put

$$v_i(\tau) = \frac{\varepsilon_i(\tau) + \delta_i(\tau)}{1 - \varepsilon_i(\tau)} \quad \text{and} \quad \omega_i(\tau) = \frac{\delta_i(\tau)}{1 - \varepsilon_i(\tau)}, \quad i = 0, 1\,.$$

Then there exist (unique) Δ_0 and Δ_1 satisfying

$$(3) \qquad \Delta_{0\tau} P_0(\Delta < \Delta_{0\tau}) - P_1(\Delta < \Delta_{0\tau}) = v_1 + \omega_0 \Delta_{0\tau}$$

and

$$(4) \qquad P_1(\Delta_{1\tau} < \Delta) - \Delta_{1\tau} P_0(\Delta_{1\tau} < \Delta) = v_0 \Delta_{1\tau} + \omega_1\,.$$

Denoting by p_0, p_1, $q_{0\tau}$ and $q_{1\tau}$ the densities of P_0, P_1, $Q_{0\tau}$ and $Q_{1\tau}$, respectively, with respect to a σ-finite measure μ it holds

$$\frac{q_{1\tau}}{q_{0\tau}} = \min\{\max\{\Delta_{0\tau}, \Delta,\}, \Delta_{1\tau}\} \quad Q_{0\tau} + Q_{1\tau} \text{ a.e.},$$

$$q_{0\tau} = (1 - \varepsilon_0)\, p_0 \quad \mu \quad \text{a.e. on} \quad \{\Delta_{0\tau} \leqq \Delta \leqq \Delta_{1\tau}\}\,,$$

$$(1 - \varepsilon_0)\, p_1\, \Delta_{0\tau}^{-1} \leqq q_{0\tau} \leqq (1 - \varepsilon_0)\, p_0 \quad \mu \text{ a.e. } \{\Delta < \Delta_{0\tau}\}\,,$$

$$(1 - \varepsilon_0)\, p_0 \leqq q_{0\tau} \leqq (1 - \varepsilon_0)\, p_1\, \Delta_{1\tau}^{-1} \quad \mu \text{ a.e. } \{\Delta_{1\tau} < \Delta\}\,,$$

$$Q_{0\tau}(\Delta < \Delta_{0\tau}) = (1 - \varepsilon_0)\, P_0(\Delta < \Delta_{0\tau}) - \delta_0\,,$$

$$Q_{0\tau}(\Delta_{1\tau} < \Delta) = (1 - \varepsilon_0)\, P_0(\Delta_{1\tau} < \Delta) + \varepsilon_0 + \delta_0\,,$$

$$Q_{1\tau}(\Delta < \Delta_{0\tau}) = (1 - \varepsilon_1)\, P_1(\Delta < \Delta_{0\tau}) + \varepsilon_1 + \delta_1\,,$$

and

$$Q_{1\tau}(\Delta_{1\tau} < \Delta) = (1 - \varepsilon_1)\, P_1(\Delta_{1\tau} < \Delta) - \delta_1\,.$$

For evaluation of the efficiency rate (and of the local deficiency) the following lemmas will be helpful.

Lemma 1. $\Delta_{0\tau}$ is increasing in τ ($\Delta_{1\tau}$ decreasing).

Proof. Let $\tau_1 > \tau_0$. Putting

$$\psi_{0\tau}(t) = (v_1(\tau) + \omega_0(\tau)\,.\,t)^{-1}\,[t\,.\,P_0(\Delta < t) - P_1(\Delta < t)]$$

we obtain an increasing function of $t \in \mathbf{R}$ for any $\tau \in I$ (see [3], Lemma 4.4). Moreover we have $\Psi_{0\tau}(\Delta_{0\tau}) = 1$. Since $v_1(\tau)$ and $\omega_0(\tau)$ are strictly increasing in τ we arrive at

$$\psi_{0\tau_1}(\Delta_{0\tau_0}) < \psi_{0\tau_0}(\Delta_{0\tau_0})\,.$$

Now the monotonicity of $\psi_{0\tau_1}$ implies $\Delta_{0\tau_1} > \Delta_{0\tau_0}$. □

Since in what follows we shall integrate repeatedly over the sets $\{t\colon 0 \leqq p_1(t)/p_0(t) \leqq \leqq \Delta_{0\tau_0}\}$ and $\{t\colon \Delta_{0\tau_1} \leqq p_1(t)/p_0(t) \leqq \Delta_{0\tau_0}\}$ for some $\tau_1 < \tau_0$ let us denote them by $S(\tau_0)$ and $S(\tau_1, \tau_0)$, respectively.

Lemma 2. $\Delta_{0\tau}$ (and $\Delta_{1\tau}$) is continuous in τ.

Proof. Let $\tau < \tau_0$. Expressing the defining equation (3) in the form

$$\Delta_{0\tau} \int_{S(\tau)} p_0(t)\, d\mu(t) - \int_{S(\tau)} p_1(t)\, d\mu(t) = v_1(\tau) + \omega_0(\tau)\, \Delta_{0\tau}$$

and subtracting this expression from the analogous one for τ_0 we obtain

$$\Delta_{0\tau_0} \int_{S(\tau_1,\tau_0)} p_0(t)\, d\mu(t) + (\Delta_{0\tau_0} - \Delta_{0\tau}) \int_{S(\tau_0)} p_0(t)\, d\mu(t) - \int_{S(\tau_1,\tau_0)} p_1(t)\, d\mu(t) =$$
$$= v_1(\tau_0) - v_1(\tau) + [\omega_0(\tau_0) - \omega_0(\tau)]\, \Delta_{0\tau_0} + \omega_0(\tau)\, [\Delta_{0\tau_0} - \Delta_{0\tau}]\,.$$

Rewriting the last equation we have

$$(5) \qquad \Delta_{0\tau_0} \int_{S(\tau_1,\tau_0)} p_0(t)\, d\mu(t) + [\Delta_{0\tau_0} - \Delta_{0\tau}]\, \{\int_{S(\tau_0)} p_0(t)\, d\mu(t) - \omega_0(\tau)\} -$$
$$- \int_{S(\tau_1,\tau_0)} p_1(t)\, d\mu(t) = v_1(\tau_0) - v_1(\tau) + [\omega_0(\tau_0) - \omega_0(\tau)]\, \Delta_{0\tau_0}\,.$$

Since for $t \in S(\tau, \tau_0)$ we have $p_1(t) \leqq p_0(t)\, \Delta_{0\tau_0}$ it follows that

$$(6) \qquad \int_{S(\tau_1,\tau_0)} \{p_0(t)\, \Delta_{0\tau_0} - p_1(t)\}\, \mathrm{d}\mu(t) \geqq 0\,.$$

On the other hand from (3) we can derive

$$\Delta_{0\tau} \int_{S(\tau_0)} p_0(t)\, \mathrm{d}\mu(t) \geqq v_1(\tau) + \omega_0(\tau)\, \Delta_{0\tau} \geqq \omega_0(\tau)\, \Delta_{0\tau}\,,$$

i.e.

$$(7) \qquad \int_{S(\tau_0)} p_0(t)\, \mathrm{d}\mu(t) - \omega_0(\tau) \geqq 0\,.$$

Making use of (6), (7) and Lemma 1 we get from (5)

$$(8) \qquad 0 \leqq [\Delta_{0\tau_0} - \Delta_{0\tau}] \{\int_{S(\tau_0)} p_0(t)\, \mathrm{d}\mu(t) - \omega_0(\tau)\} \leqq$$
$$\leqq v_1(\tau_0) - v_1(\tau) + [\omega_0(\tau_0) - \omega_0(\tau)]\, \Delta_{0\tau_0}\,.$$

Since v_1 and ω_0 are continuous we have obtained the continuity of $\Delta_{0\tau}$ from left. Similarly we may find continuity of $\Delta_{0\tau}$ from right as well as the continuity of $\Delta_{1\tau}$. □

Lemma 3. In the above given framework we have

$$(9) \qquad \frac{\mathrm{d}\Delta_{0\tau}}{\mathrm{d}\tau} = \{P_0(\Delta < \Delta_{0\tau}) - \omega_0(\tau)\}^{-1} \left\{\frac{\mathrm{d}v_1}{\mathrm{d}\tau} + \frac{\mathrm{d}\omega_0}{\mathrm{d}\tau}\, \Delta_{0\tau}\right\}$$

and

$$(10) \qquad \frac{\mathrm{d}\Delta_{1\tau}}{\mathrm{d}\tau} = -\{P_0(\Delta > \Delta_{1\tau}) + v_0(\tau)\}^{-1} \left\{\frac{\mathrm{d}v_0}{\mathrm{d}\tau}\, \Delta_{1\tau} + \frac{\mathrm{d}\omega_1}{\mathrm{d}\tau}\right\}.$$

Remark 2. The relations (9) and (10) seem to be at the first glance somewhat asymmetrical (in this form – as we shall see from proof of them they are consequences of (3) and (4)). Using (4) relation (10) can be transformed e.g. on

$$\frac{\mathrm{d}\Delta_{1\tau}}{\mathrm{d}\tau} = -\{P_1(\Delta > \Delta_{1\tau}) - \omega_1(\tau)\}^{-1}\, \Delta_{1\tau} \left\{\frac{\mathrm{d}v_0}{\mathrm{d}\tau}\, \Delta_{1\tau} + \frac{\mathrm{d}\omega_1}{\mathrm{d}\tau}\right\}.$$

Nevertheless this relation is not also completely symmetrical to (9). It is due to fact that $\Delta \in \mathrm{d}P_1/\mathrm{d}P_0$, i.e. $\Delta: \mathbf{R} \to [0, \infty)$.

Proof of Lemma 3. We may rewrite (5) (for $\tau < \tau_0$) into

$$(11) \quad \int_{S(\tau,\tau_0)} (\Delta_{0\tau_0} - \Delta)\, p_0(t)\, \mathrm{d}\mu(t) + [\Delta_{0\tau_0} - \Delta_{0\tau}] \{\int_{S(\tau_0)} p_0(t)\, \mathrm{d}\mu(t) - \omega_0(\tau)\} =$$
$$= v_1(\tau_0) - v_1(\tau) + [\omega_0(\tau_0) - \omega_0(\tau)]\, \Delta_{0\tau_0}\,.$$

Moreover due to continuity and positivity of the function

$$\int_{S(\tau')} p_0(t)\, \mathrm{d}\mu(t) - \omega_0(\tau')$$

in τ' on the interval $[\tau, \tau_0]$ we may find an $M > 0$ such that

$$(12) \qquad \int_{S(\tau')} p_0(t)\, \mathrm{d}\mu(t) - \omega_0(\tau') \geqq M$$

for all $\tau' \in [\tau, \tau_0]$. Now using (8) we obtain

$$0 \leqq [\Delta_{0\tau_0} - \Delta_{0\tau}]\,.\, M \leqq v_1(\tau_0) - v_1(\tau) + [\omega_0(\tau_0) - \omega_0(\tau)]\, \Delta_{0\tau_0}$$

which implies that

$$\limsup_{\tau\to\tau_{0-}} \frac{|\Delta_{0\tau_0} - \Delta_{0\tau}|}{\tau - \tau_0} < \infty .$$

and finally that

$$(13) \qquad \limsup_{\tau\to\tau_{0-}} \int_{S(\tau,\tau_0)} \frac{(\Delta_{0\tau_0} - \Delta)}{\tau_0 - \tau} p_0(t)\, d\mu(t) \leqq$$

$$\leqq \limsup_{\tau\to\tau_{0-}} \frac{\Delta_{0\tau_0} - \Delta_{0\tau}}{\tau_0 - \tau} \int_{S(\tau,\tau_0)} p_0(t)\, d\mu(t) = 0 .$$

Now taking into account (11), (12) and (13) one easy finds that

$$\lim_{\tau\to\tau_{0-}} \frac{\Delta_{0\tau} - \Delta_{0\tau_0}}{\tau - \tau_0} = \{P_0(\Delta < \Delta_{0\tau_0}) - \omega_0(\tau_0)\}^{-1} \left\{\frac{dv_1}{d\tau_0} + \frac{d\omega_0}{d\tau_0} \Delta_{0\tau_0}\right\}.$$

Similarly, derivation of the limit from the right is also straifhtforward and gives the same value. □

Now let us turn to evaluation of the efficiency rate for our class of tests. It follows that for n_0 large enough – due to the factor $\sqrt{n}$ – we reach supremum in (2) for such a $P \in \mathscr{P}_1(\tau)$, if any, for which the difference

$$(14) \qquad \mathsf{E}_{Q_{0\tau}} w_\nu(t) - \mathsf{E}_P w_\nu(t)$$

attains its maximum. To be able to find such probability let us recall that another form of the definition of $\mathscr{P}_i(\tau)$ $(i = 0, 1)$ is the following one:

$$\mathscr{P}_i(\tau) = \{Q \in \mathbb{M}: \|R - Q\| < \delta_i,\ R \in \{S \in \mathbb{M}: S = (1 - \varepsilon_i) P_i + \varepsilon_i H;\ H \in \mathbb{M}\}\}$$

where $\|\ \|$ denotes total variation. For the simplification of notation let us put $z_i(\tau) = \log \Delta_{i\tau}$ for $i = 0, 1$. Rewritting (14) into

$$z_0(\nu)\{Q_{0\tau}(\Delta < \Delta_{0\nu}) - P(\Delta < \Delta_{0\nu})\} + \int_{\{t:\Delta\in[\Delta_{0\nu},\Delta_{1\nu}]\}} \{w_\nu(t)\, dQ_{0\tau}(t) - w_\nu(t)\, dP(t)\} +$$
$$+ z_1(\nu)\{Q_{0\tau}(\Delta_{1\tau} < \Delta) - P(\Delta_{1\tau} < \Delta)\}$$

we may easy find that

$$\sup_{P\in\mathscr{P}_1(\tau)} \{\mathsf{E}_{Q_{0\tau}} w_\nu(t) - \mathsf{E}_P w_\nu(t)\} =$$
$$= z_0(\nu)\{Q_{0\tau}(\Delta < \Delta_{0\nu}) - [1 - \varepsilon_1(\tau)] P_1(\Delta < \Delta_{0\nu}) - \varepsilon_1(\tau) - \delta_1(\tau)\} +$$
$$+ \int_{\{t:\Delta\in[\Delta_{0\nu},\Delta_{1\nu}]\}} \{[1 - \varepsilon(\tau)] . [p_0(t) - p_1(t)]\, w_\nu(t)\, d\mu(t) +$$
$$+ z_1(\nu)\{Q_{0\tau}(\Delta_{1\nu} < \Delta) - [1 - \varepsilon(\tau)] P_1(\Delta_{1\nu} < \Delta) + \delta_1(\tau)\} .$$

Let us denote the corresponding probability measure $R_{\nu,\tau}$, i.e. we have

$$\sup_{P\in\mathscr{P}_1(\tau)} \{\mathsf{E}_{Q_{0\tau}} w_\nu(t) - \mathsf{E}_P w_\nu(t)\} = \mathsf{E}_{Q_{0\tau}} w_\nu(t) - \mathsf{E}_{R_{\nu,\tau}} w_\nu(t) .$$

Now let us denote by $a(\nu, \tau)$

$$(15) \qquad \{u_{\alpha_0} . \operatorname{var}^{1/2}_{Q_{0\tau}} w_\nu(t) + n_0^{1/2}[\mathsf{E}_{Q_{0\tau}} w_\nu(t) - \mathsf{E}_{R_{\nu,\tau}} w_\nu(t)]\} \operatorname{var}^{-1/2}_{R_{\nu,\tau}} w_\nu(t)$$

and by $\phi(t)$ the density of the standard normal distribution Φ. Then we have

$$ER(v, \tau) = \lim_{\theta \to v} \frac{\Phi(a(\theta, \tau)) - \Phi(a(v, \tau))}{\theta - v} = \phi(a(v, \tau)) \frac{da(v, \tau)}{dv} \tag{16}$$

if the last derivative exists. It implies that we shall need to find

$$\lim_{\theta \to v} \frac{\mathsf{E}_{Q_{0\tau}} w_\theta(x) - \mathsf{E}_{Q_{0\tau}} w_v(x)}{\theta - v},$$

$$\lim_{\theta \to v} \frac{\mathsf{E}_{R_{\theta,\tau}} w_\theta(x) - \mathsf{E}_{R_{v,\tau}} w_v(x)}{\theta - v},$$

$$\lim_{\theta \to v} \frac{\mathrm{var}_{Q_{0\tau}} w_\theta(x) - \mathrm{var}_{Q_{0\tau}} w_v(x)}{\theta - v}$$

and

$$\lim_{\theta \to v} \frac{\mathrm{var}_{R_{\theta,\tau}} w_\theta(x) - \mathrm{var}_{R_{v,\tau}} w_v(x)}{\theta - v},$$

(denote these limits $D_0(v, \tau)$, $D_1(v, \tau)$, $W_0(v, \tau)$ and $W_1(v, \tau)$, respectively). Let us evaluate at first $D_0(v, \tau)$. We have for $\theta > v$

$$\mathsf{E}_{Q_{0\tau}} w_\theta(x) - \mathsf{E}_{Q_{0\tau}} w_v(x) =$$
$$= [z_0(\theta) - z_0(v)] \{(1 - \varepsilon_0(\tau)) P_0(\Delta < \Delta_{0v}) - \delta_0(\tau)\}$$
$$(1 - \varepsilon_0(\tau)) \int_{\{t:\Delta \in [\Delta_{0v}, \Delta_{0\theta}]\}} (z_0(\theta) - w_v(x)) p_0(x) \, d\mu(x) +$$
$$+ (1 - \varepsilon_0(\tau)) \int_{\{t:\Delta \in [\Delta_{1\theta}, \Delta_{1v}]\}} (z_1(\theta) - w_v(x)) p_0(x) \, d\mu(x) +$$
$$+ [z_1(\theta) - z_1(v)] \{(1 - \varepsilon_0(\tau)) P_0(\Delta_{1v} < \Delta) + \varepsilon_0(\tau) + \delta_0(\tau)\}.$$

One may verify that

$$\lim_{\theta \to v+} \frac{1}{\theta - v} \int_{\{t:\Delta \in [\Delta_{0v}, \Delta_{0\theta}]\}} (z_0(\theta) - w_v(x)) p_0(x) \, d\mu(x) = 0.$$

Hence

$$\lim_{\theta \to v+} \frac{1}{\theta - v} [\mathsf{E}_{Q_{0\tau}} w_\theta(x) - \mathsf{E}_{Q_{0\tau}} w_v(x)] = \tag{17}$$
$$= \left[\frac{dz_0(\theta)}{d\theta}\right]_{\theta = v} \{(1 - \varepsilon_0(\tau)) P_0(\Delta < \Delta_{0v}) - \delta_0(\tau)\} +$$
$$+ \left[\frac{dz_1(\theta)}{d\theta}\right]_{\theta = v} \{(1 - \varepsilon_0(\tau)) P_0(\Delta_{1v} < \Delta) + \varepsilon_0(\tau) + \delta_0(\tau)\}.$$

Similarly we can find the analogous limit from the left and finally we obtain that $D_0(v, \tau)$ is equal to the right hand side of (17). Almost the same way gives

$$D_1(v, \tau) = \left[\frac{dz_0(\theta)}{d\theta}\right]_{\theta = v} \{(1 - \varepsilon(\tau)) P_1(\Delta < \Delta_{0v}) + \varepsilon_1(\tau) + \delta_1(\tau)\} +$$

$$+ \left[\frac{\mathrm{d}z_1(\theta)}{\mathrm{d}\theta}\right]_{\theta=v} \{(1 - \varepsilon(\tau))\, P_1(\Delta_{1v} < \Delta) - \delta_1(\tau)\} ,$$

$$W_0(v, \tau) = 2\left[\frac{\mathrm{d}z_0(\theta)}{\mathrm{d}\theta}\right]_{\theta=v} z_0(v)\,\{(1 - \varepsilon_0(\tau))\, P_0(\Delta < \Delta_{0v}) - \delta_0(\tau)\} +$$

$$+ 2\left[\frac{\mathrm{d}z_1(\theta)}{\mathrm{d}\theta}\right]_{\theta=v} z_1(v)\,\{(1 - \varepsilon_0(\tau))\, P_0(\Delta_{1v} < \Delta) + \varepsilon_0(\tau) + \delta_0(\tau)\} -$$

$$- 2D_0(v. \tau) \,.\, \mathsf{E}_{Q_{0\tau}}\, w_v(x)$$

and

$$W_1(v, \tau) = 2\left[\frac{\mathrm{d}z_0(\theta)}{\mathrm{d}\theta}\right]_{\theta=v} z_0(v)\,\{(1 - \varepsilon_1(\tau))\, P_1(\Delta < \Delta_{0v}) + \varepsilon_1(\tau) + \delta_1(\tau)\} +$$

$$+ 2\left[\frac{\mathrm{d}z_1(\theta)}{\mathrm{d}\theta}\right]_{\theta=v} z_1(v)\,\{(1 - \varepsilon_1(\tau))\, P_1(\Delta_{1v} < \Delta) - \delta_1(\tau)\} -$$

$$- 2 \,.\, D_1(v, \tau) \,.\, \mathsf{E}_{R_{v,\tau}}\, w_v(x) \,.$$

Theorem 1. The *efficiency rate* for above defined family $\mathscr{S}$ with respect to $\mathscr{W}$ is equal to $\phi(a(v, \tau)) \,.\, (\mathrm{d}a(v, \tau)/\mathrm{d}v)$, where $a(v, \tau)$ is given by (15) and

(18) $$\frac{\mathrm{d}a(v, \tau)}{\mathrm{d}v} = [\mathsf{var}_{R_{v,\tau}}\, w_v(x)]^{-1}\, [\{u_\alpha \,.\, \tfrac{1}{2} \,.\, \mathsf{var}_{Q_{0\tau}}^{-1/2}\, w_v(x) \,.\, W_0(v, \tau) +$$

$$+ n_0^{1/2}[D_0(v, \tau) - D_1(v, \tau)]\}\, \mathsf{var}_{R_{v,\tau}}^{1/2}\, w_v(x) - \{u_\alpha\, \mathsf{var}_{Q_{0\tau}}^{1/2}\, w_v(x) +$$

$$+ n_0^{1/2}[\mathsf{E}_{Q_{0\tau}}\, w_v(x) - \mathsf{E}_{R_{v,\tau}}\, w_v(x)]\} \,.\, \tfrac{1}{2}\, \mathsf{var}_{R_{v,\tau}}^{-1/2}\, w_v(x) \,.\, W_1(v, \tau)] \,.$$

Proof. Using the above arguments the proof follows from (15) and (16). □

Lemma 4. For any $\tau \in I$ there exists τ^* such that $ER(\tau^*, \tau) = 0$.

Proof. Our family $\mathscr{S}$ is optimal for $\mathscr{W}$ in the sense that $\mathscr{S}$ includes the most powerful tests, and hence had we chosen $\tilde{C}(v, \tau)$ to be equal precisely to the supremum over $\mathscr{P}_1(\tau)$ of the second kind error probabilities we would have $E\tilde{R}(\tau, \tau) = 0$. Now, since the convergence of the distribution function of our test statistics to the normal one is uniform (over τ from a compact subset K of I) we have $ER(\tau', \tau) < 0$ and $ER(\tau'', \tau) > 0$ for some $\tau' < \tau < \tau''$ and n_0 sufficiently large. The rest follows from the continuity of all functions in (18). □

Remark 3. From the proof of previous lemma we have $\tau^* \cong \tau$. This fact may be illustrated by the following table in which the values of $ER(\tau, \tau)$ are given for the framework:

$$n_0 = 40 \quad P_0 = \mathscr{N}(0, 1) \quad P_1 = \mathscr{N}(1, 1) \quad \alpha_0 = .10$$

$$\varepsilon(\tau) = \tau \quad \delta(\tau) = \tfrac{1}{2}\tau \quad \text{(both functions defined on } \tilde{I} = [0, .1])$$

Table 1. Efficiency rate $ER(\tau, \tau)$.

τ	·01	·02	·03	·04	·05
$ER(\tau)$	$5{\cdot}0 . 10^{-6}$	$5{\cdot}6 . 10^{-5}$	$3{\cdot}9 . 10^{-4}$	$1{\cdot}8 . 10^{-3}$	$6{\cdot}6 . 10^{-3}$

Theorem 2. The *local deficiency* $LD(\tau)$ exists and is approximately equal to

$$\phi(a(\tau, \tau)) \left[\frac{d^2 a(v, \tau)}{dv^2}\right]_{v=\tau} \tag{19}$$

where $d^2(a(v, \tau))/d\tau^2$ is yielded by a straightforward computation from (15).

The proof is transparent and hence will be omitted.

For the framework given a few lines above the expression (19) was evaluated for a few values of τ from $[0, 1]$ and the corresponding graph is offered in Fig. 2.

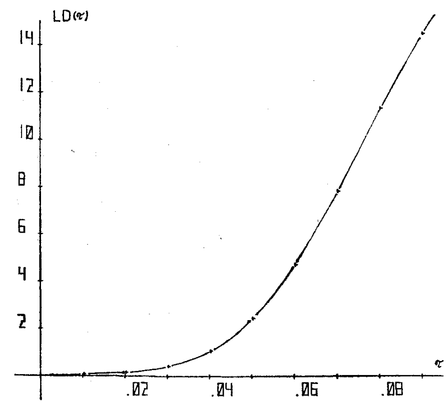

Fig. 2. Dependence of local deficiency $LD(\tau)$ on τ for the most powerful robust test under the framework given before Table 1.

Remark 4. The fact that $LD(\tau)$ is increasing in τ may be surprising noting that $LD(\varepsilon)$ is decreasing in ε in the case of location estimation – see [7]. The source of the mentioned increase lies in the following: Since the "centres" of hypothesis and alternative are fixed (i.e. P_0 and P_1 are fixed) the increase of τ implies the increase of the second kind error probabilities in such a way that it causes increase of $LD(\tau)$.

7. CONCLUDING REMARKS

From the above text it seems that the definition of the efficiency rate is only a technical matter. For an example of practical usefulness the reader is referred to [7].

The local deficiency may be used directly for building up an idea about behaviour of the statistical characteristic $C(\gamma, \theta)$ in a neighbourhood of "the optimal point $\gamma(\theta)$" using an approximate relation

$$2[C(\gamma, \theta) - C(\gamma(\theta), \theta)] \cong LD(\theta)(\nu - \gamma(\theta))^2 .$$

To see a possibility of utilization of the local deficiency for comparing statistical procedures the reader is again referred to [7].

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