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ISOMORPHISM OF MEASURE PRESERVING TRANSFORMATIONS

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The δ-entropy of endomorphism has been defined. This entropy reduces to the Shannon entropy of endomorphism for $\delta = 1$. The two isomorphic measure preserving transformations have the same δ-entropy of endomorphism.

0. MEASURE-PRESERVING TRANSFORMATIONS

In what follows, we assume that $(\Omega, \mathscr{R}, P)$ is a probability space, that is, Ω is an arbitrary non-empty set, called the set of elementary events, $\mathscr{R}$ a σ-field of subsets of Ω and P a probability measure defined on $\mathscr{R}$.

Let $(\Omega, \mathscr{R}, P)$ and $(\Omega', \mathscr{R}', P')$ be two probability spaces. The transformation T from the set Ω into the set Ω' is called *measurable* if the inverse image $T^{-1} A \in \mathscr{R}$ for each measurable set $A \in \mathscr{R}'$. A measurable transformation T is *measure-preserving* if, for every $A \in \mathscr{R}'$, the sets A and $T^{-1}A$ have the same measure, i.e. $P(T^{-1}A) = P(A)$ for each $A \in \mathscr{R}'$. A measurable (but not necessarily measure-preserving) transformation $T : \Omega \to \Omega'$ is invertible if there exists a (necessarily unique) measurable transformation $T^{-1}: \Omega' \to \Omega$ such that each of the composites $T^{-1}T$ and TT^{-1} is equal to the identity on its domain. If T is measure-preserving and invertible, then T^{-1} is measure-preserving also. Most of the transformations that we shall consider in this work are transformations from a probability space $(\Omega, \mathscr{R}, P)$ into itself or in other words transformations on $(\Omega, \mathscr{R}, P)$. A measure-preserving transformation on $(\Omega, \mathscr{R}, P)$ is also called *endomorphism* and invertible measure-preserving transformation on $(\Omega, \mathscr{R}, P)$ is called *automorphism*. The quadruple $(\Omega, \mathscr{R}, P, T)$ where T is an endomorphism on $(\Omega, \mathscr{R}, P)$ is called the *dynamical system* (a detailed discussion on measure-preserving transformations can be found in Halmos [4, 5, 6, 7] and Billingsley [1]).

For advances in entropy theory of measure theoretic dynamical systems, with particular emphasis on ideas and results relevant from the point of view of information theory refer to the work of Štefan Šujan [8, 9, 10, 11, 12].

1. δ-ENTROPY OF ENDOMORPHISM

Definition 1.1. If $\mathscr{R}_0$ is a finite sub-σ-field of $\mathscr{R}$ whose atoms are $A_1, A_2, \ldots, A_l$; then the δ-entropy of $\mathscr{R}_0$ is given by

$$H_\delta(\mathscr{R}_0) = \sum_{i=1}^{l} P(A_i) \, |\log P(A_i)|^\delta , \quad 0 < \delta \leqq 1 .$$

The concept has been used earlier by the author in [2, 3].

Definition 1.2. If $\mathscr{R}_0$ is an arbitrary finite sub-σ-field of $\mathscr{R}$ and T an endomorphism on Ω, then the δ-entropy of endomorphism is defined as:

$$h_\delta(T) = \sup_{\mathscr{R}_0 \in Z} h_\delta(\mathscr{R}_0, T) , \quad 0 < \delta \leqq 1 , \tag{1.1}$$

where Z is the set of all finite sub-σ-fields of $\mathscr{R}$, and

$$h_\delta(\mathscr{R}_0, T) = \limsup_{n \to \infty} \{ n^{-\delta} H_\delta(\mathscr{R}_0 \vee T^{-1}\mathscr{R}_0 \vee \ldots \vee T^{-(n-1)}\mathscr{R}_0) \} . \tag{1.2}$$

Theorem 1.1. If $\mathscr{R}_0$ and $\mathscr{R}_1$ are finite sub-σ-fields of $\mathscr{R}$ such that $\mathscr{R}_0 \subset \mathscr{R}_1$, then

$$h_\delta(\mathscr{R}_0, T) \leqq h_\delta(\mathscr{R}_1, T) , \tag{1.3}$$

where T is an endomorphism on Ω.

Proof. If $\mathscr{R}_0 \subset \mathscr{R}_1$, then clearly,

$$\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0 \subset \bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_1 .$$

If $\mathscr{A} \subset \mathscr{B}$, then it can be shown that $H_\delta(\mathscr{A}) \leqq H_\delta(\mathscr{B})$. Hence,

$$H_\delta(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0) \leqq H_\delta(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_1) \Rightarrow$$

$$\Rightarrow \frac{1}{n^\delta} H_\delta(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0) \leqq \frac{1}{n^\delta} H_\delta(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0) \Rightarrow$$

$$\Rightarrow \limsup_{n \to \infty} \frac{1}{n^\delta} H_\delta(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0) \leqq \limsup_{n \to \infty} \frac{1}{n^\delta} H_\delta(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_1) \Rightarrow$$

$$\Rightarrow h_\delta(\mathscr{R}_0, T) \leqq h_\delta(\mathscr{R}_1, T) . \qquad \square$$

Theorem 1.2. If $\mathscr{R}_0$ and $\mathscr{R}_1$ are two finite sub-σ-fields of $\mathscr{R}$ such that for some N and T $\mathscr{R}_0 \subset \bigvee_{i=-N}^{N} T^{-i}\mathscr{R}_1$ is an automorphism on Ω, then

$$h_\delta(\mathscr{R}_0, T) \leqq h_\delta(\mathscr{R}_1, T) . \tag{1.4}$$

Proof. Clearly,

$$\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0 \subset \bigvee_{i=-N}^{N+n-1} T^{-i}\mathscr{R}_1 .$$

This gives

$$(1.5)\qquad H_\delta\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0\Big) \leqq H_\delta\Big(\bigvee_{i=-N}^{N+n-1} T^{-i}\mathscr{R}_1\Big).$$

Thus,

$$h_\delta(\mathscr{R}_0, T) = \limsup_{n\to\infty} \frac{1}{n^\delta} H_\delta\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0\Big) = \limsup_{n\to\infty} \frac{1}{(n+2N)^\delta} H_\delta\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0\Big).$$

Using (1.5), we have

$$h_\delta(\mathscr{R}_0, T) \leqq \limsup_{n\to\infty} \frac{1}{(n+2N)^\delta} H_\delta\Big(\bigvee_{i=-N}^{N+n-1} T^{-i}\mathscr{R}_1\Big) =$$

$$= \limsup_{n\to\infty} \frac{1}{(n+2N)^\delta} H_\delta\Big(\bigvee_{i=0}^{2N+n-1} T^{-i}T^N\mathscr{R}_1\Big) = h_\delta(T^N\mathscr{R}_1, T) = h_\delta(\mathscr{R}_1, T). \qquad \square$$

Theorem 1.3. If $\mathscr{R}_0$ is a finite sub-σ-field of $\mathscr{R}$ and k is a positive integer, then

$$(1.6)\qquad h_\delta(\mathscr{R}_0, T^k) \leqq k^\delta h_\delta(\mathscr{R}_0, T)$$

where T is an endomorphism on Ω.

Proof. Let $\mathscr{R}_1 = \bigvee_{i=0}^{k-1} T^{-i}\mathscr{R}_0$. Clearly $\mathscr{R}_0 \subset \mathscr{R}_1$ and thus

$$h_\delta(\mathscr{R}_0, T^k) \leqq h_\delta(\mathscr{R}_1, T^k) = \limsup_{n\to\infty} \frac{1}{n^\delta} H_\delta\Big(\bigvee_{i=0}^{n-1} T^{-ki}\Big(\bigvee_{j=0}^{k-1} T^{-j}\mathscr{R}_0\Big)\Big).$$

Since, $\bigvee_{i=0}^{n-1} T^{-ki}\Big(\bigvee_{j=0}^{k-1} T^{-j}\mathscr{R}_0\Big) = \bigvee_{i=0}^{nk-1} T^{-i}\mathscr{R}_0$, we have

$$h_\delta(\mathscr{R}_0, T^k) \leqq \limsup_{n\to\infty} \frac{1}{n^\delta} H_\delta\Big(\bigvee_{i=0}^{nk-1} T^{-i}\mathscr{R}_0\Big) = k^\delta \limsup_{n\to\infty} \frac{1}{(nk)^\delta} H_\delta\Big(\bigvee_{i=0}^{nk-1} T^{-i}\mathscr{R}_0\Big) \leqq$$

$$\leqq k^\delta h_\delta(\mathscr{R}_0, T). \qquad \square$$

Theorem 1.4. If T is an automorphism on Ω and $\mathscr{R}_0$ is a finite sub-σ-field, then

$$(1.7)\qquad h_\delta(\mathscr{R}_0, T^k) \leqq |k|^\delta\, h_\delta(\mathscr{R}_0, T)$$

for all integers k.

Proof. We have already proved the result for positive k. If $k = 0$, then $T^0 = I$, the identity transformation. Thus, in this case

$$(1.8)\qquad H_\delta\Big(\bigvee_{i=0}^{n-1} T^{-i}\mathscr{R}_0\Big) = H_\delta(\mathscr{R}_0),$$

and hence

$$(1.9)\qquad h_\delta(\mathscr{R}_0, I) = \limsup_{n\to\infty} \frac{1}{h^\delta} H_\delta(\mathscr{R}_0) = 0.$$

If $k = -1$, note that

$$H_\delta\Big(\bigvee_{i=0}^{n-1} T^i \mathscr{R}_0\Big) = H_\delta\Big(T^{n-1} \bigvee_{i=0}^{n-1} T^{-i} \mathscr{R}_0\Big) = H_\delta\Big(\bigvee_{i=0}^{n-1} T^{-i} \mathscr{R}_0\Big).$$

Thus,

$$h_\delta(\mathscr{R}_0, T^{-1}) = h_\delta(\mathscr{R}_0, T). \tag{1.10}$$

If k is a negative integer, then from (1.6) it follows that

$$h_\delta(\mathscr{R}_0, T^k) = h_\delta(\mathscr{R}_0, (T^{-1})^{-k}) \leqq (-k)^\delta\, h_\delta(\mathscr{R}_0, T^{-1}).$$

Thus,

$$h_\delta(\mathscr{R}_0, T^k) \leqq |k|^\delta\, h_\delta(\mathscr{R}_0, T^{-1}) = |k|^\delta\, h_\delta(\mathscr{R}_0, T).$$

□

Theorem 1.5. If T is an endomorphism on Ω, then

$$h_\delta(T^k) = k^\delta\, h_\delta(T), \tag{1.11}$$

for k a positive integer and if T is an automorphism on Ω, then

$$h_\delta(T^k) = |k|^\delta\, h_\delta(T), \tag{1.12}$$

for k to be an integer.

We first prove the following lemma:

Lemma 1.1. If $\mathscr{R}_0$ is a finite sub-σ-field of $\mathscr{R}$ and $\mathscr{R}_1 = \bigvee_{i=0}^{k-1} T^{-i} \mathscr{R}_0$, then

$$h_\delta(\mathscr{R}_0, T) = \frac{1}{k^\delta}\, h_\delta(\mathscr{R}_1, T^k),$$

where T is an endomorphism on Ω and k, a positive integer.

Proof.

$$h_\delta(\mathscr{R}_1, T^k) = \limsup_{n\to\infty} \frac{1}{n^\delta} H_\delta\Big(\bigvee_{i=0}^{n-1} (T^k)^{-i} \mathscr{R}_1\Big) =$$

$$= \limsup_{n\to\infty} \frac{1}{n^\delta} H_\delta\Big(\bigvee_{i=0}^{n-1} T^{-ki}\Big(\bigvee_{j=0}^{k-1} T^{-j} \mathscr{R}_0\Big)\Big).$$

Since, $\bigvee_{i=0}^{n-1} T^{-ki}\Big(\bigvee_{j=0}^{k-1} T^{-j}\mathscr{R}_0\Big) = \bigvee_{i=0}^{nk-1} T^{-i}\mathscr{R}_0$, we have

$$h_\delta(\mathscr{R}_1, T^k) = \limsup_{n\to\infty} \frac{1}{n^\delta} H_\delta\Big(\bigvee_{i=0}^{nk-1} T^{-i} \mathscr{R}_0\Big) =$$

$$= k^\delta \limsup_{n\to\infty} \frac{1}{(nk)^\delta} H_\delta\Big(\bigvee_{i=0}^{nk-1} T^{-i} \mathscr{R}_0\Big) \leqq k^\delta\, h_\delta(\mathscr{R}_0, T).$$

Next, since $H_\delta(\bigvee_{i=0}^{nk-1} T^{-i}\mathscr{R}_0)$ is a monotone increasing sequence, we have

$$h_\delta(\mathscr{R}_1, T^k) = k^\delta \limsup_{n\to\infty} \frac{1}{(nk)^\delta} H_\delta(\bigvee_{i=0}^{nk-1} T^{-i}\mathscr{R}_0) =$$

$$= k^\delta \limsup_{n\to\infty} \frac{(nk-j)^\delta}{(nk)^\delta} \cdot \frac{1}{(nk-j)^\delta} H_\delta(\bigvee_{i=0}^{nk-1} T^{-i}\mathscr{R}_0)\,; \quad 0 \leqq j \leqq k-1$$

$$= k^\delta \limsup_{n\to\infty} \frac{1}{(nk-j)^\delta} H_\delta(\bigvee_{i=0}^{nk-1} T^{-i}\mathscr{R}_0)\,; \quad 0 \leqq j \leqq k-1$$

$$\geqq k^\delta \limsup_{n\to\infty} \frac{1}{(nk-j)^\delta} H_\delta(\bigvee_{i=0}^{nk-j} T^{-i}\mathscr{R}_0)\,; \quad 0 \leqq j \leqq k-1\,.$$

Thus,

$$h_\delta(\mathscr{R}_1, T^k) \geqq k^\delta\, h_\delta(\mathscr{R}_0, T)\,.$$

This completes the proof of the lemma. □

Proof of the Theorem. From Theorem 1.4, it follows that the result holds for $k = 0$ and $h_\delta(T) = h_\delta(T^{-1})$ if T is an automorphism. Hence we need prove the theorem only for $k > 0$. Let $\mathscr{R}_0$ be any finite sub-σ-field of $\mathscr{R}$ and $\mathscr{R}_1 = \bigvee_{i=0}^{k-1} T^{-i}\mathscr{R}_0$. From Theorem 1.3, it follows that

$$h_\delta(\mathscr{R}_0, T^k) \leqq k^\delta\, h_\delta(\mathscr{R}_0, T)\,,$$

and thus

$$h_\delta(T^k) \leqq k^\delta\, h_\delta(T)\,. \tag{1.13}$$

Now, from Lemma 1.1, we have

$$h_\delta(\mathscr{R}_0, T) = \frac{1}{k^\delta}\, h_\delta(\mathscr{R}_1, T^k)\,,$$

where $\mathscr{R}_1 = \bigvee_{i=0}^{k-1} T^{-i}\mathscr{R}_0$. Therefore,

$$h_\delta(T) = \sup_{\mathscr{R}_0 \in Z} h_\delta(\mathscr{R}_0, T) = \frac{1}{k^\delta} \sup_{\mathscr{R}_0 \in Z} h_\delta(\bigvee_{i=0}^{k-1} T^{-i}\mathscr{R}_0, T^k) \leqq \frac{1}{k^\delta}\, h_\delta(T^k)\,.$$

Thus,

$$h_\delta(T^k) \geqq k^\delta\, h_\delta(T)\,. \tag{1.14}$$

Now, from (1.13) and (1.14) the required result follows. □

2. THE PROBLEM OF ISOMORPHISM

It is known that if two measure-preserving transformations are isomorphic, then they have the same Shannon entropy of endomorphism. In the following theorem it has been shown that this result also holds for the δ-entropy of endomorphism.

Theorem 2.1. The δ-entropy of endomorphism is invariant under isomorphism.

Proof. Consider the two dynamical systems $(\Omega_i, \mathscr{R}_i, P_i, T_i)$ where T_i is an endomorphism on Ω_i, $i = 1, 2$, Let Φ be the isomorphism between T_1 and T_2. Thus to every finite sub-σ-field $\mathscr{B}_1$, of $\mathscr{R}_1$ there corresponds a finite sub-σ-field

$$\mathscr{B}_2 = \Phi\mathscr{B}_1 = \{\Phi B_1 : B_1 \in \mathscr{B}_1\} \quad \text{of} \quad \mathscr{R}_2 \quad \text{and conversely}.$$

$$\frac{1}{n^\delta} H_\delta\left(\bigvee_{i=0}^{n-1} T_2^{-i}\mathscr{B}_2\right) = \frac{1}{n^\delta} H_\delta\left(\bigvee_{i=0}^{n-1} \Phi^{-1} T_2^{-i}\mathscr{B}_2\right) = \frac{1}{n^\delta} H_\delta\left(\bigvee_{i=0}^{n-1} T_1^{-i}\Phi^{-1}\mathscr{B}_2\right) =$$

$$= \frac{1}{n^\delta} H_\delta\left(\bigvee_{i=0}^{n-1} T_1^{-i}\mathscr{B}_1\right).$$

Taking lim sup on both sides with respect to n, we obtain

$$h_\delta(\mathscr{B}_2, T_2) = h_\delta(\mathscr{B}_1, T_1).$$

Thus for each $\mathscr{B}_1$ there is a $\mathscr{B}_2$ with $h_\delta(\mathscr{B}_1, T_1) = h_\delta(\mathscr{B}_2, T_2)$ and vice-versa.

Hence,

$$h_\delta(T_1) = h_\delta(T_2).$$

This completes the proof of the theorem. □

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