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ON THE REPRESENTATION OF TRAJECTORIES OF BILINEAR SYSTEMS AND ITS APPLICATIONS

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Special representation of trajectories of bilinear systems is obtained. On the basis of this representation estimate for continuous dependence of trajectories of bilinear systems on control is developed and properties of the so-called attainable set of these systems are studied. An algorithm for the numerical solution of bilinear systems is suggested. An illustrative example is also included.

1. INTRODUCTION

Let us consider the following control system which we shall call a bilinear one:

$$\dot{x} = Ax + (Bx + c)\, u\,, \quad x(t_0) = x_0\,, \tag{1}$$

where A, B are $(n \times n)$-dimensional constant matrices and c is vector in R^n space. Scalar control u is assumed to be a measurable function on every finite time interval $[t_0, t_1]$ such that almost everywhere on $[t_0, t_1]$ holds

$$u_{\min} \leqq u(t) \leqq u_{\max}\,,$$

where $u_{\min}, u_{\max}$ are given real numbers. Such a control is further denoted as an admissible one. Finally, $x \in R^n$ is the vector of state variables and $x_0 \in R^n$ is the given initial state of the system. More general forms of bilinear systems are described in [3]. Examples of practical application of bilinear systems are presented in [3], [4], [5].

In this paper special representation for trajectories of system (1) is developed. Further, continuous dependence of trajectories resulting from (1) on controls is studied. The following estimate for this continuous dependence was obtained:

$$\max_{t_0 \leqq t \leqq t_1} \|x_1(t) - x_2(t)\|_{R^n} \leqq K \max_{t_0 \leqq t \leqq t_1} \left| \int_{t_0}^{t} (u_1(s) - u_2(s))\, \mathrm{d}s \right|, \tag{2}$$

where $x_1(t)$ and $x_2(t)$ are solutions of (1) for $u_1(t)$ and $u_2(t)$ respectively, K is a constant depending only on $A, B, c, t_0, t_1, u_{\min}$ and $u_{\max}$. Representation of trajectories of system (1) and estimate (2) are used in order to study some important properties of the so-called attainable set of bilinear systems.

In the last section of this paper an algorithm for the numerical solution of bilinear system (1) is suggested and an illustrative example is included.

2. ANALYTICAL REPRESENTATION OF SOLUTION OF BILINEAR SYSTEMS

Let us introduce the following notation

$$w(t) = \int_{t_0}^{t} u(s) \, \mathrm{d}s \,. \tag{3}$$

By $\exp(F)$, where F is some real $(n \times n)$-dimensional matrix, we denote the matrix function defined by:

$$\exp(F) = \sum_{k=0}^{\infty} \frac{F^k}{k!} \,. \tag{4}$$

It can be easily verified that this sum exists for any real matrix F.

Our aim in this section is to find a special representation of $x(t)$, the solution of (1) for a given $u(t)$, which allows us to obtain the estimate (2). First we construct special representation for the fundamental matrix $\Phi(t)$ of system (1) and its inverse $\Phi^{-1}(t)$. Let us recall that the fundamental matrix $\Phi(t)$ of system (1) is a solution to the following matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = (A + B\,u(t))\,X(t)\,, \quad X(t_0) = I\,, \tag{5}$$

and its inverse matrix $\Phi^{-1}(t)$ is a solution to

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = -Y(t)\,(A + B\,u(t))\,, \quad Y(t_0) = I\,, \tag{6}$$

where I denotes the $(n \times n)$-dimensional identity matrix.

Theorem 1. Let us consider system (1). Then we can represent its fundamental matrix $\Phi(t)$ and its inverse matrix $\Phi^{-1}(t)$ as the sums of the following infinite series:

$$\Phi(t) = \exp(B\,w(t)) + \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \dots \int_{t_0}^{\tau_2} \{\prod_{i=0}^{k-1} \exp(B(w(\tau_{k+1-i}) - w(\tau_{k-i})))\,A\} \exp(B\,w(\tau_1))\,\mathrm{d}\tau_1 \dots \mathrm{d}\tau_k \tag{7}$$

and

$$\Phi^{-1}(t) = \exp(-B\,w(t)) + \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \dots \int_{t_0}^{\tau_2} \exp(-B\,w(\tau_1))\,.\,\{\prod_{i=1}^{k} (-A) \exp(B(w(\tau_i) - w(\tau_{i+1})))\}\,\mathrm{d}\tau_1 \dots \mathrm{d}\tau_k\,. \tag{8}$$

Proof. Let us consider the following iterative procedure:

$$\Phi^0(t) = \exp(B\, w(t))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{i+1}(t) = B\, u(t)\, \Phi^{i+1}(t) + A\, \Phi^i(t)\,, \quad \Phi^{i+1}(t_0) = I\,,$$

that is

$$\Phi^{i+1}(t) = \exp(B\, w(t)) + \int_{t_0}^{t} \exp(B(w(t) - w(s)))\, A\, \Phi^i(s)\, \mathrm{d}s$$

or

$$\Phi^{i+1}(t) = \exp(B\, w(t)) + \sum_{k=1}^{i+1} \int_{t_0}^{t=\tau_{k+1}} \dots \int_{t_0}^{\tau_2} \{\prod_{i=0}^{k-1} \exp(B(w(\tau_{k+1-i}) - $$
$$- w(\tau_{k-i})))\, A\}\, \exp(B\, w(\tau_1))\, \mathrm{d}\tau_1 \dots \mathrm{d}\tau_k\,.$$

For the solution of (5) it holds clearly

$$\Phi(t) = \exp(B\, w(t)) + \int_{t_0}^{t} \exp(B(w(t) - w(s)))\, A\, \Phi(s)\, \mathrm{d}s\,.$$

Hence

$$\|\Phi^{i+1}(t) - \Phi(t)\|_s \leqq \int_{t_0}^{t} \mathrm{e}^{(\|B\|_s |w(t) - w(s)|)} \|A\|_s\, \|\Phi(s) - \Phi^i(s)\|_s\, \mathrm{d}s \leqq$$
$$\leqq M \int_{t_0}^{t} \|\Phi(s) - \Phi^i(s)\|_s\, \mathrm{d}s \leqq \frac{(M(t - t_0))^{i+1}}{(i+1)!} \max_{t_0 \leqq t \leqq t_1} \|\Phi(t) - \Phi^0(t)\|_s\,.$$

Here $\|\cdot\|_s$ denotes spectral matrix norm, M is a certain constant.

So, we can see that the sequence $\{\Phi^i\}_{i=0}^{\infty}$ converges to the solution of (5) in the spectral matrix norm, as i tends to infinity. On the other hand, it is obvious that the series on the right hand side of (7) is just the limit of the sequence $\{\Phi^i\}_{i=0}^{\infty}$. Hence, validity of representation (7) was proved.

In the same way we can prove representation (8), the only difference is that the appropriate iterative procedure is of the form:

$$\Phi_0^{-1}(t) = \exp(-B\, w(t))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{i+1}^{-1}(t) = \Phi_{i+1}^{-1}(t)(-B\, u(t)) - \Phi_i^{-1}(t)\, A\,, \quad \Phi_{i+1}^{-1}(t_0) = I\,. \qquad \square$$

The following theorem gives a representation of the solution $x(t)$ of (1) such that $x(t)$ depends only on $w(t)$, $t \in [t_0, t_1]$, with $w(t)$ given by (3).

Theorem 2. Let us consider system (1) with initial state $x(t_0) = x_0$ and control $u(t)$. Then for the appropriate solution of this system $x(t)$ the following formula holds:

$$x(t) = \Phi(t)\left(x_0 + \int_{t_0}^{t} \Phi^{-1}(s)\, A\, \exp(B\, w(s))\, E(w(s))\, c\, \mathrm{d}s\right) + \tag{9}$$
$$+ \exp(B\, w(t))\, E(w(t))\, c\,.$$

Here $E(s)$ denotes the following real matrix function of a real variable:

$$E(s) = \sum_{i=0}^{\infty} (-1)^i B^i \frac{s^{i+1}}{(i+1)!}. \tag{10}$$

Proof. Let us first remark that the infinite series on the right hand side of (10) evidently converges for any real matrix B and any real number s. Furthermore, the following equality holds:

$$\frac{d}{ds} E(s) = \exp(-Bs), \quad E(0) = 0. \tag{11}$$

As it is known from the theory of ordinary differential equations (see e.g. [1], p. 135) solution $x(t)$ of system (1) has the following form:

$$x(t) = \Phi(t)\left(x_0 + \int_{t_0}^{t} u(s)\, \Phi^{-1}(s)\, c\, ds\right). \tag{12}$$

Here $\Phi(t)$ is again the fundamental matrix of (1). From (8) it follows that we can write:

$$\Phi^{-1}(t) = \Phi_A(t) \exp(-B\, w(t)), \tag{13}$$

where

(14)

$$\Phi_A(t) = I + \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \dots \int_{t_0}^{\tau_2} \left\{\prod_{i=1}^{k} \exp(-B\, w(\tau_i))(-A) \exp(B\, w(\tau_i))\right\} d\tau_1 \dots d\tau_k.$$

By direct evaluation we obtain that

$$\frac{d}{dt} \Phi_A(t) = \Phi_A(t) \exp(-B\, w(t))(-A) \exp(B\, w(t)), \tag{15}$$

$$\Phi_A(t_0) = I. \tag{16}$$

Using relations (11), (13)–(16) and integrating by parts we have:

$$\int_{t_0}^{t} u(s)\, \Phi^{-1}(s)\, c\, ds = \int_{t_0}^{t} u(s)\, \Phi_A(s) \exp(-B\, w(s))\, c\, ds =$$

$$= \int_{t_0}^{t} \Phi_A(s) \frac{d}{ds} E(w(s))\, c\, ds =$$

$$= \Phi_A(t)\, E(w(t))\, c - \int_{t_0}^{t} \Phi_A(s) \exp(-B\, w(s))(-A) \exp(B\, w(s))\, E(w(s))\, c\, ds,$$

that is

$$\int_{t_0}^{t} u(s)\, \Phi^{-1}(s)\, c\, ds = \Phi_A(t)\, E(w(t))\, c + \int_{t_0}^{t} \Phi^{-1}(s)\, A \exp(B\, w(s))\, E(w(s))\, c\, ds. \tag{17}$$

When the substitution from (17) in the right-hand side of (12) is performed, taking into the account (13), we obtain representation (9). □

Remark 1. Formula (9) for the solution of system (1) is more complicated than (12), but it gives explicit dependence of $x(t)$ on $w(t)$, defined by (3). From this explicit dependence numerous interesting conclusions follow.

An important class of bilinear systems is the class of the so-called commutative bilinear systems for which condition $AB = BA$ holds. Under this condition we can simplify formula (9).

Corollary 1. Let us consider system (1) for which condition $AB = BA$ holds. Then for the solution $x(t)$ of this system with the initial state $x(t_0) = x_0$ and control $u(t)$ the following formula is true:

$$x(t) = \exp\left(A(t - t_0) + B\, w(t)\right)\left(x_0 + \exp\left(-A(t - t_0)\right) E(w(t))\, c + A\int_{t_0}^{t} \exp\left(-A(s - t_0)\right) E(w(s))\, c\, \mathrm{d}s\right). \tag{18}$$

In order to prove Corollary 1 it is sufficient to realize that in the case $AB = BA$ the fundamental matrix of (1) has the form $\Phi(t) = \exp\left(A(t - t_0) + B\, w(t)\right)$ and its inverse $\Phi^{-1}(t) = \exp\left(-A(t - t_0) - B\, w(t)\right)$. Furthermore, it is easily verified that if $AB = BA$ also the following equality holds:

$$A \exp\left(-B\, w(t)\right) = \exp\left(-B\, w(t)\right) A\,.$$

Remark 2. Let in addition to the assumption of Corollary 1 $Ac = 0$. Then if we take into account that in this case $\exp(-As)\, c = c$ for each real s, we can even more simplify the formula (18)

$$x(t) = \exp\left(A(t - t_0) + B\, w(t)\right)\left(x_0 + E(w(t))\, c\right). \tag{19}$$

3. ESTIMATE FOR CONTINUOUS DEPENDENCE OF TRAJECTORIES OF BILINEAR SYSTEM ON CONTROLS

In this section estimate (2) will be derived.

Theorem 3. Let us consider system (1), defined on the time interval $[t_0, t_1]$, with the initial state $x(t_0) = x_0$ and let $x_1(t)$ and $x_2(t)$ be trajectories of this system for admissible controls $u_1(t)$ and $u_2(t)$, respectively. Then the estimate (2) is valid, where

$$K = K_1 K_2 \|x_0\|_{R^n} + 2K_1 K_2^2 K_3 K_4 \|A\|_s \|c\|_{R^n} (t_1 - t_0) + K_2^2 K_4 \|A\|_s \|c\|_{R^n} (t_1 - t_0) + K_4 \|c\|_{R^n}\,. \tag{20}$$

Here we use the notation:

$$K_1 = \left(2\|A\|_s \|B\|_s (t_1 - t_0) + \|B\|_s\right) \tag{21}$$

$$K_2 = \exp\left(\left(\|B\|_s\, u_p + \|A\|_s\right)(t_1 - t_0)\right) \tag{22}$$

$$K_3 = \left\{\exp\left(\|B\|_s\, u_p (t_1 - t_0)\right) - 1\right\} / \|B\|_s \tag{23}$$

$$K_4 = \exp\left(\|B\|_s\, u_p (t_1 - t_0)\right) \tag{24}$$

$$u_p = \max\left\{|u_{\min}|, |u_{\max}|\right\} \tag{25}$$

Finally $\|\cdot\|_s$ stands for the spectral matrix norm and $\|\cdot\|_{R^n}$ for the Euclidean vector norm.

Proof. Let us denote fundamental matrices of system (1) for controls $u_1(t)$ and $u_2(t)$ by $\Phi_1(t)$ and $\Phi_2(t)$, respectively. First we establish estimates for

$$\max_{t_0 \leqq t \leqq t_1} \|\Phi_1(t) - \Phi_2(t)\|_s \quad \text{and} \quad \max_{t_0 \leqq t \leqq t_1} \|\Phi_1^{-1}(t) - \Phi_2^{-1}(t)\|_s .$$

It is easily verified that for any square matrices $X_1, X_2, \ldots, X_k, Y_1, Y_2, \ldots, Y_k$, the following identity holds

$$\prod_{i=1}^{k} X_i - \prod_{i=1}^{k} Y_i = \sum_{i=1}^{k} \Big(\prod_{j=1}^{i-1} Y_j\Big)(X_i - Y_i)\Big(\prod_{j=i+1}^{k} X_j\Big) . \tag{26}$$

(We define that $\prod_{i=p}^{q} D_i = 1$ for $q < p$.)

Using this identity and formula (7) we obtain:

$$\Phi_1(t) - \Phi_2(t) = \tag{27}$$

$$= F(t) + \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \ldots \int_{t_0}^{\tau_2} \sum_{i=1}^{k} \Big(\prod_{j=0}^{i-2} D_2^{k-j} A\Big)\big(D_1^{k-i+1} - D_2^{k-i+1}\big) A \Big(\prod_{j=i}^{k-1} D_1^{k-j} A\Big) .$$

$$. \exp\big(B\, w(\tau_1)\big)\, d\tau_1 \ldots d\tau_k + \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \ldots \int_{t_0}^{\tau_2} \Big(\prod_{i=0}^{k-1} D_2^{k-i} A\Big) F(\tau_1)\, d\tau_1 \ldots d\tau_k ,$$

where

$$D_p^q = \exp\big(B(w_p(\tau_{q+1}) - w_p(\tau_q))\big) ,$$
$$F(s) = \exp\big(B\, w_1(s)\big) - \exp\big(B\, w_2(s)\big) .$$

The following relation holds

$$\|D_1^q - D_2^q\|_s \leqq \|B\|_s \max_{0 \leqq \Theta \leqq 1} \{\|\exp(Bw^*)\|_s\} \, |w_1(\tau_{q+1}) -$$
$$- w_1(\tau_q) - w_2(\tau_{q+1}) + w_2(\tau_q)| ,$$

where

$$w^* = w_1(\tau_{q+1}) - w_1(\tau_q) + \Theta\big(w_2(\tau_{q+1}) - w_2(\tau_q) - w_1(\tau_{q+1}) + w_2(\tau_q)\big) \quad \text{and}$$
$$0 \leqq \Theta \leqq 1 .$$

Further, from (3) it follows

$$w^* = (1 - \Theta) \int_{\tau_q}^{\tau_{q+1}} u_1(s)\, ds + \Theta \int_{\tau_q}^{\tau_{q+1}} u_2(s)\, ds , \quad 0 \leqq \Theta \leqq 1 .$$

Hence

$$\|D_1^q - D_2^q\|_s \leqq 2\|B\|_s \exp\big(\|B\|_s\, |\tau_{q+1} - \tau_q|\, u_p\big) \max_{t_0 \leqq t \leqq t_1} |w_1(t) - w_2(t)| . \tag{28}$$

Moreover

$$\|D_p^q\|_s \leqq \exp\big(\|B\|_s\, |\tau_{q+1} - \tau_q|\, u_p\big) , \tag{29}$$

$$\|F(s)\|_s \leqq \|B\|_s \exp\big(\|B\|_s (s - t_0)\, u_p\big) \max_{t_0 \leqq t \leqq t_1} |w_1(t) - w_2(t)| . \tag{30}$$

Using relation (27), (28), (29), (30) we obtain:

$$\|\Phi_1(t) - \Phi_2(t)\|_s \leqq \{\|B\|_s \exp(\|B\|_s (t_1 - t_0) u_p) +$$
$$+ 2\|B\|_s \sum_{k=1}^{\infty} k \int_{t_0}^{t=\tau_{k+1}} \dots \int_{t_0}^{\tau_2} \exp(\|B\|_s u_p \sum_{q=1}^{k} |\tau_{q+1} - \tau_q|) \|A\|_s^k .$$
$$. \exp(\|B\|_s u_p(\tau_1 - t_0)) \, d\tau_1 \dots d\tau_k +$$
$$+ \|B\|_s \sum_{k=1}^{\infty} \int_{t_0}^{t=\tau_{k+1}} \dots \int_{t_0}^{\tau_2} \exp(\|B\|_s u_p \sum_{q=1}^{k} |\tau_{q+1} - \tau_q|) \|A\|_s^k .$$
$$. \exp(\|B\|_s u_p(\tau_1 - t_0)) \, d\tau_1 \dots d\tau_k\} \max_{t_0 \leqq t \leqq t_1} |w_1(t) - w_2(t)| .$$

Let us observe that $\tau_{k+1} \geqq \tau_k \geqq \dots \geqq \tau_2 \geqq \tau_1$, hence

$$\sum_{q=1}^{k} |\tau_{q+1} - \tau_q| = \tau_{k+1} - \tau_1 = t - \tau_1 .$$

Thus

$$\|\Phi_1(t) - \Phi_2(t)\|_s \leqq \|B\|_s \exp(\|B\|_s u_p(t_1 - t_0)) \left\{\sum_{k=1}^{\infty} \|A\|_s^k \int_{t_0}^{t=\tau_{k+1}} \dots \int_{t_0}^{\tau_2} d\tau_1 \dots d\tau_k + \right.$$
$$\left. + \sum_{k=1}^{\infty} 2\|A\|_s^k k \int_{t_0}^{t=\tau_{k+1}} \dots \int_{t_0}^{\tau_2} d\tau_1 \dots d\tau_k + 1\right\} \max_{t_0 \leqq t \leqq t_1} |w_1(t) - w_2(t)| \leqq$$
$$\leqq \|B\|_s \exp(\|B\|_s u_p(t_1 - t_0)) \left\{\sum_{k=1}^{\infty} \frac{\|A\|_s^k (t_1 - t_0)^k}{k!} + 1 + \right.$$
$$\left. + 2(t_1 - t_0) \|A\|_s \sum_{k=1}^{\infty} \frac{\|A\|_s^{k-1}}{(k-1)!} (t_1 - t_0)^{k-1}\right\} \max_{t_0 \leqq t \leqq t_1} |w_1(t) - w_2(t)| =$$
$$= \|B\|_s \exp(\|B\|_s u_p(t_1 - t_0)) \exp(\|A\|_s (t_1 - t_0)) (1 + 2(t_1 - t_0) \|A\|_s) .$$
$$. \max_{t_0 \leqq t \leqq t_1} |w_1(t) - w_2(t)| .$$

Finally we can write the following estimate

$$\max_{t_0 \leqq t \leqq t_1} \|\Phi_1(t) - \Phi_2(t)\|_s \leqq K_1 K_2 \max_{t_0 \leqq t \leqq t_1} |w_1(t) - w_2(t)| , \tag{31}$$

where K_1, K_2 are given by (21) and (22).

Analogously as for $\Phi(t)$ we can obtain estimate for $\Phi^{-1}(t)$:

$$\max_{t_0 \leqq t \leqq t_1} \|\Phi_1^{-1}(t) - \Phi_2^{-1}(t)\|_s \leqq K_1 K_2 \max_{t_0 \leqq t \leqq t_1} |w_1(t) - w_2(t)| . \tag{32}$$

Now it follows from (10)

$$\|E(w(s))\|_s \leqq (\exp(\|B\|_s u_p(t - t_0)) - 1)/\|B\|_s . \tag{33}$$

Furthermore

$$\|\exp(B w_1(s)) E(w_1(s)) - \exp(B w_2(s)) E(w_2(s))\|_s \leqq$$
$$\leqq \max_{0 \leqq \Theta(s) \leqq 1} \{\|B \exp(B w^*(s)) E(w^*(s)) + I\|_s\} |w_1(s) - w_2(s)| ,$$

where

$$w^*(s) = w_1(s) + \Theta(s)\,(w_2(s) - w_1(s))\,, \quad 0 \leqq \Theta(s) \leqq 1\,.$$

Taking into account that $B\,E(w^*) = I - \exp(-Bw^*)$, we obtain :

$$\|\exp(B\,w_1(s))\,E(w_1(s)) - \exp(B\,w_2(s))\,E(w_2(s))\| \leqq \tag{34}$$
$$\leqq \exp(\|B\|_s\,u_p(s - t_0))\,|w_1(s) - w_2(s)|\,.$$

Now we can prove estimate (2). From (9) it follows that

$$x_1(t) - x_2(t) = (\Phi_1(t) - \Phi_2(t))\left(x_0 + \int_{t_0}^t \Phi_1^{-1}(s)\,A\exp(B\,w_1(s))\,E(w_1(s))\,c\,\mathrm{d}s\right) +$$
$$+ \Phi_2(t)\int_{t_0}^t (\Phi_1^{-1}(s) - \Phi_2^{-1}(s))\,A\exp(B\,w_1(s))\,E(w_1(s))\,c\,\mathrm{d}s +$$
$$+ \Phi_2(t)\int_{t_0}^t \Phi_2^{-1}(s)\,A(\exp(B\,w_1(s))\,E(w_1(s)) - \exp(B\,w_2(s))\,E(w_2(s)))\,c\,\mathrm{d}s +$$
$$+ (\exp(B\,w_1(t))\,E(w_1(t)) - \exp(B\,w_2(t))\,E(w_2(t)))\,c\,.$$

Using estimates (31), (32), (33), (34), the triangle inequality and the relation between spectral matrix norm of the $(n \times n)$-dimensional matrix F and the Euclidean vector norm of some vector $y \in R^n$ (see e.g. [2]):

$$\|Fy\|_{R^n} \leqq \|F\|_s \cdot \|y\|_{R^n}\,,$$

we obtain estimate (2) with K given by (20)–(24). □

Remark 3. Theorem 3 in fact establishes not only continuous dependence of trajectory of system (1) on control with respect to norms $\max_{t_0 \leqq t \leqq t_1} |\int_{t_0}^t u(s)\,\mathrm{d}s|$ and $\max_{t_0 \leqq t \leqq t_1} \|x(t)\|_{R^n}$, but even Lipschitzean dependence of a trajectory of a bilinear system on control with respect to these norms. This fact can be employed for practical computations as it will be shown.

4. PROPERTIES OF ATTAINABLE SET FOR BILINEAR SYSTEMS

In this section we use the results of preceding sections in order to study some important properties of the so-called attainable set for bilinear systems. First, let us recall the definition of an attainable set.

Definition 1. Let us consider system (1) with initial state $x(t_0) = x_0$. We call an *attainable set* for this system at time $t \in [t_0, t_1]$ the set of all points y from R^n, for which there exists an admissible control $u(s)$ on $[t_0, t]$ such that for the respective trajectory $x(s)$, $s \in [t_0, t]$, holds $x(t) = y$. We denote attainable set at time t by $X(t_0, t, x_0)$.

Definition 2. Let us consider system (1) with initial state $x(t_0) = x_0$. By $X^*(t_0, t, x_0)$

we denote the set of all points y from R^n for which there exists a piecewise constant control $u(s)$ on $[t_0, t_1]$, $u(s) \in \{u_{\min}, u_{\max}\}$ for all $s \in [t_0, t]$, such that for respective trajectory $x(s)$, $s \in [t_0, t]$, holds $x(t) = y$.

It is obvious that $X^*(t_0, t, x_0) \subset X(t_0, t, x_0)$.

The following theorem establishes the fundamental property of an attainable set for bilinear systems (1).

Theorem 4. Let system (1) with initial state $x(t_0) = x_0$ be given. Then for any $t \in [t_0, t_1]$

$$X(t_0, t, x_0) = \overline{X^*(t_0, t, x_0)}$$

that is, $X^*(t_0, t, x_0)$ is dense in $X(t_0, t, x_0)$.

Proof. In order to prove assertion of Theorem 4 we need to establish that for any measurable function $u(s) \in [u_{\min}, u_{\max}]$ a.e. on $[t_0, t]$ and for any $\varepsilon > 0$ there exists $u_\varepsilon(s)$ piecewise constant on $[t_0, t]$ and $u_\varepsilon(s) \in \{u_{\min}, u_{\max}\}$ for all $s \in [t_0, t]$ such that for the appropriate trajectories $x(t)$, $x_\varepsilon(t)$ of system (1) holds

$$\|x(t) - x_\varepsilon(t)\|_{R^n} \leqq \varepsilon .$$

Taking into account Theorem 3 it only sufficies to prove the following assertion which we formulate as a separate lemma.

Lemma 1. Let us consider function $u(s)$ measurable on a closed interval $[t_0, t_1]$ such that $u(s) \in [u_{\min}, u_{\max}]$ a.e. on $[t_0, t_1]$. Let us divide closed interval $[t_0, t_1]$ into k closed equal subintervals $[t_0 + (i-1)h, t_0 + ih]$, $i = 1, 2, \ldots, k$, $h = (t_1 - t_0)/k$. Then there exists a function $u^*(s)$ with the following properties:

1) $u^*(s)$ is constant on each subinterval of the form $[t_0 + (i-1)h, t_0 + ih]$,
2) $u^*(s) \in \{u_{\min}, u_{\max}\}$ for all $s \in [t_0, t_1]$,
3) for all $t \in [t_0, t_1]$

$$\left|\int_{t_0}^{t} u(s)\,\mathrm{d}s - \int_{t_0}^{t} u^*(s)\,\mathrm{d}s\right| \leqq \tfrac{1}{2}(u_{\max} - u_{\min})\,h . \tag{35}$$

Proof. Let us denote

$$U_i = \int_{t_0}^{ih} u(s)\,\mathrm{d}s \quad \text{and} \quad u^*(s) = u_i\,, \quad \text{where} \quad u_i \in \{u_{\min}, u_{\max}\}\,,$$
$$s \in [t_0 + (i-1)h,\ t_0 + ih]\,.$$

We shall construct the function $u^*(s)$, i.e. numbers $u_1, \ldots, u_k$, in the following way:

1)

$$u_1 = u_{\max}\,, \quad \text{if} \quad U_1 \geqq \tfrac{1}{2}(u_{\max} + u_{\min})\,h\,,$$

or

$$u_1 = u_{\min}\,, \quad \text{if} \quad U_1 < \tfrac{1}{2}(u_{\max} + u_{\min})\,h\,.$$

2) Let us suppose that we have already defined $u_1, \ldots, u_i$, where $i < k$, then we define

$$u_{i+1} = u_{\max}\,, \quad \text{if} \quad U_{i+1} - \sum_{j=1}^{i} u_j h \geqq \tfrac{1}{2}(u_{\max} + u_{\min})\, h$$

or

$$u_{i+1} = u_{\min}\,, \quad \text{if} \quad U_{i+1} - \sum_{j=1}^{i} u_j h < \tfrac{1}{2}(u_{\max} + u_{\min})\, h\,.$$

From the definition of U_i it follows that

$$u_{\min} h \leqq U_i - U_{i-1} \leqq u_{\max} h\,.$$

So we can conclude that if we construct numbers u_i, $i = 1, 2, \ldots, k$ according to the preceding procedure the following relations hold:

$$0 \geqq U_1 - u_1 h \geqq -\tfrac{1}{2}(u_{\max} - u_{\min})\, h\,, \quad \text{if} \quad u_1 = u_{\max}$$

$$\tfrac{1}{2}h\,(u_{\max} - u_{\min}) > U_1 - u_1 h \geqq 0\,, \quad \text{if} \quad u_1 = u_{\min}\,.$$

Thus

$$|U_1 - u_1 h| \leqq \tfrac{1}{2}(u_{\max} - u_{\min})\, h\,.$$

Further

$$U_i - \sum_{j=1}^{i} u_j h \geqq U_{i+1} - u_{i+1} h - \sum_{j=1}^{i} u_j h \geqq -\tfrac{1}{2}(u_{\max} - u_{\min})\, h\,, \quad \text{if} \quad u_{i+1} = u_{\max}$$

$$\tfrac{1}{2}h\,(u_{\max} - u_{\min}) > U_{i+1} - u_{i+1} h - \sum_{j=1}^{i} u_j h \geqq U_i - \sum_{j=1}^{i} u_j h\,, \quad \text{if} \quad u_{i+1} = u_{\min}\,.$$

Thus

$$\left|U_{i+1} - \sum_{j=1}^{i+1} u_j h\right| \leqq \max\left\{\tfrac{1}{2}(u_{\max} - u_{\min})\, h,\ \left|U_i - \sum_{j=1}^{i} u_j h\right|\right\}$$

for each $i = 1, 2, \ldots, k - 1$.

These relations imply that inequality (35) holds for every t of the form $t = t_0 + ih$, $i = 1, 2, \ldots, k$, that is for all boundary points of subintervals. It remains to prove inequality (35) for the interior points of subintervals. Let $t \in \operatorname{int}\,[t_0 + (i - 1)\, h, t_0 + ih]$, $i = 1, 2, \ldots, k$. Then

$$(36) \quad \left|\int_{t_0}^{t} u(s)\,\mathrm{d}s - \int_{t_0}^{t} u^*(s)\,\mathrm{d}s\right| = \left|U_{i-1} - \sum_{j=1}^{i-1} u_j h + \int_{t_0+(i-1)h}^{t} (u(s) - u_i)\,\mathrm{d}s\right|.$$

Let us analyze the term $\int_{t_0+(i-1)h}^{t} (u(s) - u_i)\,\mathrm{d}s$.

There are two possibilities.

1) $u_i = u_{\max}$. Then we have

$$(37) \quad 0 \geqq \int_{t_0+(i-1)h}^{t} (u(s) - u_i)\,\mathrm{d}s \geqq \int_{t_0+(i-1)h}^{t_0+ih} (u(s) - u_i)\,\mathrm{d}s = U_i - U_{i-1} - hu_i\,.$$

2) $u_i = u_{\min}$. Then we have

$$(38) \quad 0 \leqq \int_{t_0+(i-1)h}^{t} (u(s) - u_i)\,\mathrm{d}s \leqq \int_{t_0+(i-1)h}^{t_0+ih} (u(s) - u_i)\,\mathrm{d}s = U_i - U_{i-1} - hu_i$$

From (37) it follows

$$U_i - \sum_{j=1}^{i} hu_j \leqq U_{i-1} - \sum_{j=1}^{i-1} hu_j + \int_{t_0+(i-1)h}^{t} (u(s) - u_i) \, ds \leqq U_{i-1} - \sum_{j=1}^{i-1} hu_j .$$

On the other hand from (38)

$$U_{i-1} - \sum_{j=1}^{i-1} hu_j \leqq U_{i-1} - \sum_{j=1}^{i-1} hu_j + \int_{t_0+(i-1)h}^{t} (u'(s) - u_i) \, ds \leqq U_i - \sum_{j=1}^{i} hu_j .$$

From these relations and from (36) we obtain

$$\left| \int_{t_0}^{t} u'(s) \, ds - \int_{t_0}^{t} u^*(s) \, ds \right| \leqq \max \left\{ \left|U_i - \sum_{j=1}^{i} hu_j\right|, \left|U_{i-1} - \sum_{j=1}^{i-1} hu_j\right| \right\},$$

when $t \in [t_0 + (i-1) h, t_0 + ih]$. This relation means that if inequality (35) holds for all t of the form $t = t_0 + ih$, $i = 1, 2, \ldots, k$, then it holds for all $t \in [t_0, t_1]$, too. □

Lemma 1 will be further used in the next section where an algorithm for numerical solution of bilinear systems will be constructed.

Now we establish an interesting property of attainable sets for some commutative bilinear systems.

Theorem 5. Let us consider system (1) with initial state $x(t_0) = x_0$, where $AB = BA$ and $Ac = 0$. Then the point $y \in X(t_0, t, x_0)$ if and only if there exists constant control $u_c \in [u_{\min}, u_{\max}]$, such that for the appropriate trajectory $x(s)$, $s \in [t_0, t]$, holds $x(t) = y$.

In other words, Theorem 5 asserts that under the given condition state y is reachable by some trajectory of system (1) if and only if it is reachable by a trajectory corresponding to a constant control. The proof of this theorem clearly follows from equality (19) (Remark 2 to Theorem 2) and from the fact that for any admissible control $u(s)$ there exists real number $u_c \in [_{\min}, u_{\max}]$ such that

$$w(t) = \int_{t_0}^{t} u(s) \, ds = u_c(t - t_0) .$$

5. ALGORITHM FOR THE NUMERICAL SOLUTION OF BILINEAR SYSTEMS

In this section we shall use the previous results to construct an algorithm for the approximate solution of system (1). First we formulate the following theorem which is a corollary to Lemma 1 and Theorem 3.

Theorem 6. Let us consider an arbitrary admissible control $u(s)$ for system (1) on the time interval $[t_0, t_1]$ and let us denote the corresponding trajectory of system (1) with initial condition $x(t_0) = x_0$ by $x(t)$. Further, let $[t_0, t_1]$ be divided into k subintervals as in Lemma 1, let $u^*(s)$ be a control constructed to the control $u(s)$ by Lemma 1 and let $x^*(t)$ be the corresponding trajectory of system (1) with initial condition $x(t_0) = x_0$. Then

$$\max_{t_0 \leqq t \leqq t_1} \|x(t) - x^*(t)\|_{R^n} \leqq K \frac{u_{\max} - u_{\min}}{2} \cdot \frac{t_1 - t_0}{k} \tag{39}$$

Here K is given by (20)–(24).

The proof of Lemma 1 suggests directly a simple algorithm, how to construct for any given admissible control $u(s)$ the appropriate control $u^*(s)$. Furthermore, the corresponding trajectory $x^*(t)$, which we can naturally consider as an approximate solution of system (1), is easily being computed in the following way.

Let us introduce two operators L_h^+, L_h^- which act from R^n to R^n:

$$L_h^+ \, x = \exp\left((A + Bu_{\max})\, h\right) x + u_{\max} \int_0^h \exp\left((A + Bu_{\max})(h - s)\right) c \, \mathrm{d}s \,,$$

$$L_h^- \, x = \exp\left((A + Bu_{\min})\, h\right) x + u_{\min} \int_0^h \exp\left((A + Bu_{\min})(h - s)\right) c \, \mathrm{d}s \,.$$

The matrices on the right hand sides can be easily analytically computed for arbitrary A, B, c and h.

Then

$$x^*(t_0 + ih) = L_1 L_2 \ldots L_i \, x(t_0) \,, \quad i = 1, 2, \ldots, k \,,$$

where

$$L_j = L_h^+ \,, \quad \text{if} \quad u^*(s) = u_{\max} \quad \text{for} \quad s \in [t_0 + (j-1)\, h,\ t_0 + jh]$$

and

$$L_j = L_h^- \,, \quad \text{if} \quad u^*(s) = u_{\min} \quad \text{for} \quad s \in [t_0 + (j-1)\, h,\ t_0 + jh] \,,\ j = 1, 2, \ldots, i \,.$$

It is not difficult to see that necessary computation grows linearly with k.

The corresponding computer code was prepared, which for the given system, initial state and control computes and approximate trajectory. The method was tested on systems with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad u_{\min} = -1 \,, \quad u_{\max} = 1 \,,$$

$$t_0 = 0 \,, \quad t_1 = 2\pi \,,$$

for controls $u_1(t) = 0$ and $u_2(t) = \cos t$ and initial states $x_0^1 = (5\ 5)^{\mathrm{T}}$ and $x_0^2 = (3\ 3)^{\mathrm{T}}$, respectively. Computations were performed for $k = 100$ and $k = 1\,000$. Results are shown in Figures 1–3; for $u_1(t) = 0$ the exact solution is a circle with centre at the point $(0, 0)$ and radius $5\sqrt{2}$; for $u_2(t) = \cos t$ the exact solution is a certain closed curve passing through the point (3,3). The achieved accuracy in the case $u_1(t) = 0$ was $5 \, . \, 10^{-1}$ for $k = 100$ and $5 \, . \, 10^{-2}$ for $k = 1\,000$.

In order to improve accuracy and have the trajectory $x^*(t)$ more smooth, special

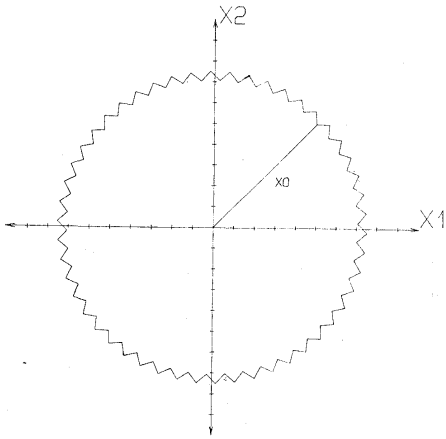

Fig. 1. Trajectory for $u_1(t) = 0$, $k = 100$, $x(t_0) = (5,5)$.

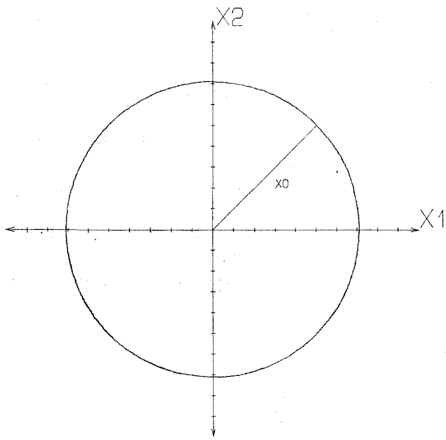

Fig. 2. Trajectory for $u_1(t) = 0$, $k = 1000$, $x(t_0) = (5,5)$.

interpolation procedure was implemented. While earlier computed points $x^*(t_0 + ih)$ and $x^*(t_0 + (i + 1)\, h)$ were connected by a line, this procedure searches line l_i passing through the point $x^*(t_0 + ih)$ such that

$$d(x^*(t_0 + (i + 1)\, h), l_i)^2 + d(x^*(t_0 + (i + 2)\, h), l_i)^2 \quad \text{is minimal}.$$

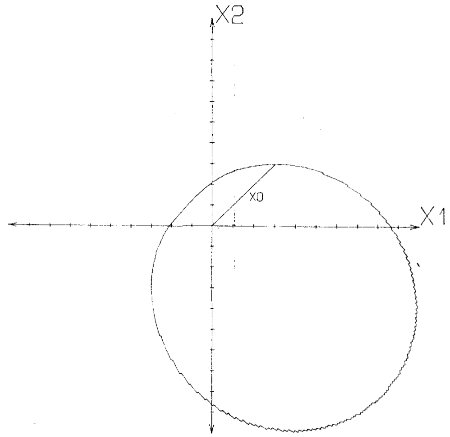

Fig. 3. Trajectory for $u_2(t) = \cos t$, $k = 1000$, $x(t_0) = (3,3)$.

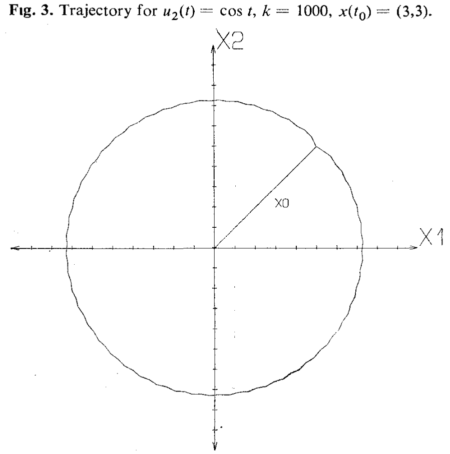

Fig. 4. Trajectory for $u_1(t) = 0$, $k = 100$, $x(t_0) = (5,5)$ — special interpolation.

Here $d(\cdot, l_i)$ is the distance to line l_i. Then the point $x^*(t_0 + (i + 1) h)$ is replaced by its nearest point on line l_i and the next step searches line l_{i+1} passing through this point, such that

$$d(x^*(t_0 + (i + 2) h), l_{i+1})^2 + d(x^*(t_0 + (i + 3), l_{i+1})^2 \quad \text{is minimal}.$$

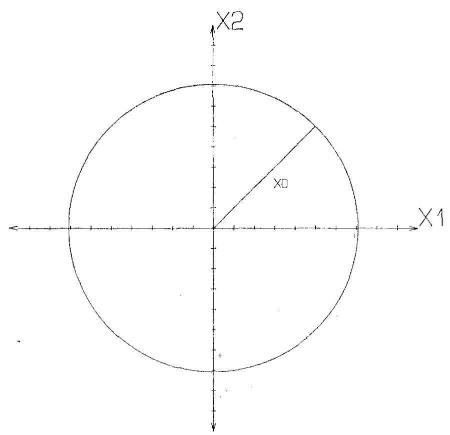

Fig. 5. Trajectory for $u_1(t) = 0$, $k = 1000$, $x(t_0) = (5,5)$ — special interpolation.

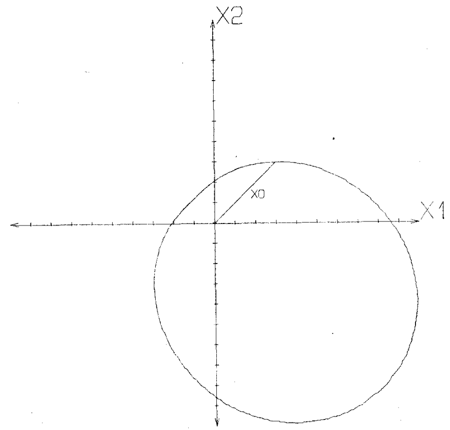

Fig. 6. Trajectory for $u_2(t) = \cos t$, $k = 1000$, $x(t_0) = (3,3)$ — special interpolation.

This is done gradually for $i = 1, 2, \ldots, k - 1$. The points $x^*(t_0)$ and $x^*(t_1)$ remains without changes. Results of application of this procedure are shown in Fig. 4–6.

Comparison with Fig. 1–3 shows that this procedure improves approximation

$x^*(t)$, although there is no strict mathematical proof of this fact. Accuracy in the case $u_1(t) = 0$ was $3 \,.\, 10^{-1}$ for $k = 100$ and $3 \,.\, 10^{-2}$ for $k = 1\,000$.

In Fig. 7 is depicted a part of trajectory $x^*(t)$ in the case $u_2(t) = \cos t$, $x(t_0) = (3,3)$, for $k = 12\,000$ with t in the range from $t' = 0.42\pi$ to $t'' = (0{,}42 + \frac{1}{6})\,\pi$. Also here the special interpolation procedure was applied. Observe the gain in accuracy with respect to the case in Fig. 6.

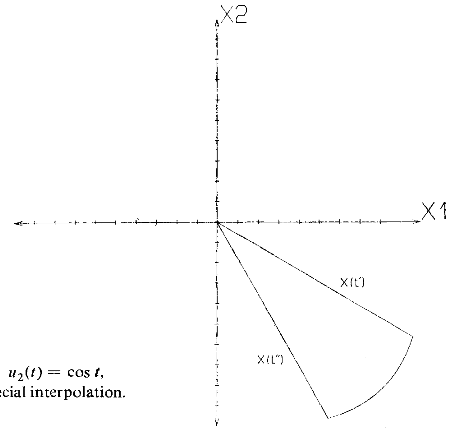

Fig. 7. Part of trajectory for $u_2(t) = \cos t$, $k = 12\,000$, $x(t_0) = (3,3)$ — special interpolation.

The suggested algorithm of numerical solution of bilinear systems is attractive by its simplicity. Moreover it gives trajectories corresponding only to two values of control; this fact may be sometimes useful for applications. This algorithm is fast, cases with $k = 1\,000$ were computed on IBM 370/135 per 10 second CPU time including using rather slow graphics Calcomp. Last, but not least, there are no requirements on smoothness of function $u(s)$, which may be even only measurable.

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