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A UNIFIED APPROACH TO MULTIVARIABLE DISCRETE-TIME FILTERING BASED ON THE WIENER THEORY

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Several theories of discrete-time optimal filtering are in current use but the relationships between these theories is not always appreciated. By using a slightly modified form of the Wiener theory, a unification of the various methods is obtainable.

In the present paper this approach is used to discuss in the multivariable case, instantaneous filtering, one-step delayed filtering, fixed-lag filtering, and prediction. It is shown that the one-step delay filter has a natural feedback structure related to the Kalman filter. The relation of the theory with the Hagander-Wittenmark filter is also discussed.

1. INTRODUCTION

This article arose from the observation of an apparent discrepancy between, on the one hand, the expression for the optimal discrete-time Kalman filter for separating signal from white noise (Shaked [1]) and, on the other hand, the expression for the self-tuning Hagander-Wittenmark filter [2] in the multidimensional version obtained by Moir [3] (see also Moir & Grimble [4]). On investigation it became clear that the apparent discrepancy occurs because in the first case, filtering is performed with one-step delay while in the second case, filtering is performed instantaneously, that is, with zero-step delay.

The present paper describes a unified theory covering both cases as well as smoothing and prediction. The theory is developed from the multidimensional form of the Wiener filter and extends previous work where the corresponding continuous-time case was considered (Barrett [5]). Of course, in the continuous-time case the distinction between one-step delay and instantaneous filtering does not occur.

Further, the theory is related to the polynomial approach of Kučera [6] resulting in a unification within which the relations between the different ideas may be discussed. The work described here is related to the recent independent contributions of Deng [7], [8].

2. NOTATION

z	message variable
y	signal variable
$\hat{y}_{k_1 \mid k}$	estimator of signal at instant k_1 using message process up to instant k i.e. $\{z'_k, k' \leqq k\}$
v, w	white noise variables
$\mathsf{E}[\cdot]$	expectation operator
q	lead operator
z	Z-transform variable
$S(z)$	bilateral Z-transform of autocorrelation function
$\Lambda(z)$	spectral factor

3. FORMULATION

We first give a statement of the general form of the Wiener filtering theory used in the paper.

Problem. Consider optimal linear least-square sequential estimation of a stationary discrete-time vector stochastic process $\{y_k\}$ called the *signal process* from another such process $\{z_k\}$ of the same dimension called the *message process* which is sequentially related to it. The estimator $\hat{y}_k$ at instant k is determined as

$$\hat{y}_k = \sum_{r=-\infty}^{\infty} h_r z_{k-r} \tag{3.1}$$

and the estimation error at instant k by

$$e_k = y_k - \hat{y}_k \,. \tag{3.2}$$

For physical realizability certain of the h_r will have to vanish. The interval of values of r for which h_r is nonzero will be denoted by $\mathfrak{I}$. The nonzero values of h_r, $r \in \mathfrak{I}$ are to be chosen to minimize.

$$\operatorname{tr} \mathsf{E}\{e_k e_k^{\mathrm{T}}\} \tag{3.3}$$

where E is the statistical expectation operator.

This estimator $\hat{y}_k$ may be written as

$$\hat{y}_k = H(q^{-1})\, z_k \tag{3.4}$$

where q^{-1} denotes lag operator and

$$H(q^{-1}) = \sum_{r=-\infty}^{\infty} h_r q^{-r} \,. \tag{3.5}$$

Thus $H(z^{-1})$ is the transfer function matrix (t.f.m.) of the filter defined as

$$H(z^{-1}) = \sum_{r=-\infty}^{\infty} h_r z^{-r} \,. \tag{3.6}$$

The relationship between the variables is depicted in Fig. 1*.

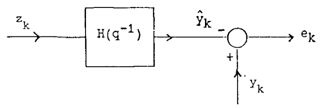

Fig. 1. Inter-relation of problem variables.

It should be emphasised that there is no loss of generality in adopting this notation since specific case relating to time shifts between message and estimate can be accommodated by delay terms within $H(z^{-1})$.

Standard minimization procedure leads immediately to the following general form of the *Wiener-Hopf equation*

$$\sum_r h_r R_{z,z}[s - r] = R_{y,z}[s] \qquad s \in \mathfrak{I}, \quad r \in \mathfrak{I} \tag{3.7}$$

where $R_{y,z}$, $R_{z,z}$ are the *cross* and *auto-correlation* functions viz

$$R_{y,z}[s] = \mathsf{E}\{(y_{k+s} - \mathsf{E}[y])(y_k - \mathsf{E}[y])^{\mathrm{T}}\} \tag{3.8}$$

$$R_{z,z}[s] = \mathsf{E}\{(z_{k+s} - \mathsf{E}[z])(z_k - \mathsf{E}[z])^{\mathrm{T}}\} \tag{3.9}$$

The Wiener-Hopf equation is solved using Z-transforms. For this purpose there are defined the *z-spectral function matrices*

$$S_{y,z}(z^{-1}) = \sum_{s=-\infty}^{\infty} R_{y,z}[s]\, z^{-s} \tag{3.10}$$

$$S_{z,z}(z^{-1}) = \sum_{s=-\infty}^{\infty} R_{z,z}[s]\, z^{-s} \tag{3.11}$$

For the class of stochastic processes normally considered in filtering theory, these are rational functions of z. This is because the stochastic processes are considered to be generated by acting on vector white noise sources with filters having rational transfer functions. For simplicity attention will be restricted to stochastic processes for which the dimensionality of the processes is the same as that of the generating white noise source. For such processes the spectral function matrix $S(z^{-1})$ has full rank almost everywhere and the corresponding stochastic process is said to be of *full rank*. The following theorem then applies (Rozanov [9]).

Theorem. If $S(z^{-1})$ is rational and of full rank then it possesses a spectral factorization

$$S(z^{-1}) = \Lambda(z^{-1})\, \Lambda(z)^{\mathrm{T}} \tag{3.12}$$

in terms of the spectral factor $\Lambda(z^{-1})$ where $\Lambda(z^{-1})$ is a rational function of z^{-1}

* Note that a distinction is being made between the lag operator q^{-1} and the Z-transform variable z^{-1}. The script z is used to avoid notational confusion with the z used for the message.

which is *minimum phase* i.e. $\Lambda(\varkappa^{-1})$ is regular in the region $|\varkappa^{-1}| < 1$ and it possesses an inverse $\Lambda(\varkappa^{-1})^{-1}$ which is also regular in the region $|\varkappa^{-1}| < 1$.

Here the regularity condition implies that $\Lambda(\varkappa^{-1})$ and its inverse are both stable.

In this theorem the spectral factorization is not unique although the spectral factor Λ is determined up to a right multiplication by a constant (non-singular) orthogonal matrix.

On using the spectral factorization

(3.13) $$S_{z,z}(\varkappa^{-1}) = \Lambda_z(\varkappa^{-1})\,\Lambda_z(\varkappa)^{\mathrm{T}}$$

of the message process, the solution of the Wiener-Hopf equation provides the following optimal filter t.f.m.

(3.14) $$H_{\mathrm{opt}}(\varkappa^{-1}) = [S_{y,z}(\varkappa^{-1})\,\{\Lambda_z^{\mathrm{T}}(\varkappa)\}^{-1}]_+\,\Lambda_z(\varkappa^{-1})^{-1}$$

where the bracket $[\;]_+$ denotes the realizable part corresponding to the interval $\mathfrak{I}$ of the message.

4. THE INNOVATIONS REPRESENTATION

Following normal usage we introduce the term *shaping-filter t.f.m.* to denote a t.f.m. which produces the spectral density of a given process when acting on white noise. Such a shaping filter is not unique.

The *innovations shaping-filter t.f.m.* is defined as the t.f.m. which produces the spectral density of a given process by acting on the innovations process associated with this process. We now show that the innovations shaping-filter t.f.m. is given uniquely by

(4.1) $$Z(\varkappa^{-1}) = \Lambda_z(\varkappa^{-1})\,\Lambda_z(0)^{-1}$$

in the case when $\Lambda_z(0)$ is nonsingular – a condition equivalent to assuming the process to be of full-rank.

We first note that $Z(\cdot)$ is not affected by right multiplication of $\Lambda(\cdot)$ by a constant non-singular matrix so that the non-uniqueness of $\Lambda(\cdot)$ does not appear as a corresponding non-uniqueness of $Z(\cdot)$.

Clearly $Z(\varkappa^{-1})$ satisfies the two *innovations shaping-filter conditions*:

(4.2) (i) $Z(\varkappa^{-1})$ is minimum-phase.

(ii) $Z(0)$ is the identity matrix.

From the minimum-phase condition it follows that $Z(\varkappa^{-1})^{-1}$ is a stable t.f.m. and so the process

(4.4) $$\varepsilon_k = Z(q^{-1})^{-1}\,z_k$$

is well defined. Its spectral density is constant:

(4.5) $$\begin{aligned} S_{\varepsilon,\varepsilon}(\varkappa^{-1}) &= Z(\varkappa^{-1})^{-1}\,S_{z,z}(\varkappa^{-1})\,(Z(\varkappa)^{\mathrm{T}})^{-1} \\ &= Z(\varkappa^{-1})^{-1}\,\Lambda(\varkappa^{-1})\,\Lambda_z(\varkappa)^{\mathrm{T}}\,(Z(\varkappa)^{\mathrm{T}})^{-1} \\ &= \Lambda_z(0)\,\Lambda_z(0)^{\mathrm{T}} \end{aligned}$$

Hence $\{\varepsilon_k\}$ is a white noise process having variance matrix

(4.6) $$R_{\varepsilon,\varepsilon} = \Lambda_z(0)\,\Lambda_z(0)^{\mathrm{T}}\,.$$

The t.f.m. $Z(\mathscr{z}^{-1})^{-1}$ is called the *(innovations) whitening filter t.f.m.* of the message process. Application of it to the message results in the process $\{\varepsilon_k\}$ which may be identified as the *innovations process* of $\{z_k\}$. The identification follows from the expansion

(4.7) $$Z(\mathscr{z}^{-1})^{-1} = I + \text{terms in powers of } \mathscr{z}^{-1}$$

which from the minimum-phase condition (ii) is valid when $|\mathscr{z}^{-1}| < 1$. Then it follows from (4.4) that,

(4.8) $$\varepsilon_k = z_k + \text{linear terms in } z_{k-1}, z_{k-2}, \ldots$$

which, by transposition of terms, becomes the autoregressive representation

(4.9) $$z_k = B_1 z_{k-1} + B_2 z_{k-2} + \ldots + \varepsilon_k$$

using the regression coefficients $B_1, B_2, \ldots$. So $\{\varepsilon_k\}$ is seen to coincide with the innovations process as normally defined in the theory originally due to Wold. From (4.7), (4.8), (4.9) follows

(4.10) $$Z(\mathscr{z}^{-1})^{-1} = I - B_1\mathscr{z}^{-1} - B_2\mathscr{z}^{-2} - \ldots.$$

Finally on inverting (4.4) we have arrived at the unique representation

(4.11) $$z_k = Z(q^{-1})\,\varepsilon_k$$

which may be called the *t.f.m. innovations representation* of the message. It is seen that the condition that $\Lambda_z(0)$ should be non-singular is equivalent to the condition that $R_{\varepsilon,\varepsilon}$ is non-singular i.e. that the processes $\{\varepsilon_k\}$, $\{z_k\}$ are of full rank. This will be assumed to be so in this paper.

Since $Z(0)$ reduces to the identity, $Z(\mathscr{z}^{-1})$ has the form

(4.12) $$Z(\mathscr{z}^{-1}) = I + G_1\mathscr{z}^{-1} + G_2\mathscr{z}^{-2} + \ldots$$

(4.13) $$= I + G(\mathscr{z}^{-1})$$

The function $G(\mathscr{z}^{-1})$ will be important in what follows. The coefficients $G_1, G_2, \ldots$ may easily be calculated from the regression coefficients $B_1, B_2, \ldots$ by inversion:

(4.14) $$I + G_1\mathscr{z}^{-1} + G_2\mathscr{z}^{-2} + \ldots = (I - B_1\mathscr{z}^{-1} - B_2\mathscr{z}^{-2} - \ldots)^{-1}$$

so that

(4.15) $$I = (I - B_1\mathscr{z}^{-1} - B_2\mathscr{z}^{-2} - \ldots)(I + G_1\mathscr{z}^{-1} + G_2\mathscr{z}^{-2} + \ldots)$$

from which it is easy to deduce the recursive relation

(4.16) $$G_i = \sum_{j=1}^{i} B_j G_{i-j} \quad i = 1, 2, \ldots$$

where here G_0 is taken as the identity matrix.

5. OPTIMAL ESTIMATION USING INNOVATIONS

On substituting the expression for $\Lambda_z(\cdot)$ in terms of $Z(\cdot)$ in the Wiener-Hopf solution formula (3.14) we find for the optimal filter t.f.m.

(5.1) $$H(\varkappa^{-1}) = [S_{y,z}(\varkappa^{-1})(R_{\varepsilon,\varepsilon}\, Z(\varkappa)^{\mathrm{T}})^{-1}]_+\, Z(\varkappa^{-1})^{-1}\,.$$

It is convenient to write this as

(5.2) $$H(\varkappa^{-1}) = [F(\varkappa^{-1})]_+\, Z(\varkappa^{-1})^{-1}$$

where

(5.3) $$F(\varkappa^{-1}) = S_{y,z}(\varkappa^{-1})(R_{\varepsilon,\varepsilon}\, Z(\varkappa)^{\mathrm{T}})^{-1}\,.$$

The structure of the optimal filter is that shown in Fig. 2 below.

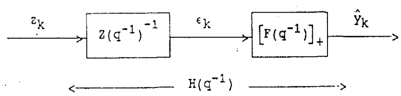

Fig. 2. Structure of the optimal filter.

The incoming message process $\{z_k\}$ is first whitened by use of the *innovations-whitening filter* t.f.m. $Z(\varkappa^{-1})^{-1}$ to give $\{\varepsilon_k\}$, the innovations process. This is then passed through the filter with t.f.m. $[F(\varkappa^{-1})]_+$ to give an estimator $\hat{y}_k$ of the signal.

The structure of the optimal filter shown in Fig. 2 is basic to filter theory and originates with the work of Wold [10] and Kolmogorov [11]. The idea is clearly explained by Whittle [12] who derives the Wiener solution from it. For a more recent review of the ideas, see Wouters and Gevers [13].

$F(q^{-1})$ itself may be thought of as the *optimal nonrealizable estimator* of the signal from the innovations process, the restriction to past values resulting in the restriction implied by the bracket $[\ ]_+$. If $F(\varkappa^{-1})$ has Laurent expansion

(5.4) $$F(\varkappa^{-1}) = \sum_{n=-\infty}^{\infty} F_n \varkappa^{-n}$$

this restriction is conveniently made by truncation of the series. Here $\{F_n\}$ is the time-domain representation of the estimator filter $F(\cdot)$. So truncation of the series corresponds to estimation over different time periods. If truncation is made to correspond to time lag l then $\mathfrak{I} = \{n;\, n \geqq l\}$ and we shall have

(5.5) $$[F(\varkappa^{-1})]_+ = \sum_{n=l}^{\infty} F_n \varkappa^{-n}$$

(5.6) $$[F(\varkappa^{-1})]_- = \sum_{n=-\infty}^{l-1} F_n \varkappa^{-n}$$

Cases of particular interest are:

(a) $l = 0$: *instantaneous estimation.*

Here $\mathfrak{I} = \{0, 1, 2, \ldots\}$ and

$$[F(z^{-1})]_+ = F_0 + F_1 z^{-1} + F_2 z^{-2} + \ldots . \tag{5.7}$$

This corresponds to the case when the message is used to provide an instantaneous estimate based on previous and current values of the message.

(b) $l > 0$: *estimate in advance of message.*

Here $\mathfrak{I} = \{l, l + 1, \ldots\}$ and

$$[F(z^{-1})]_+ = F_l z^{-l} + F_{l+1} z^{-(l+1)} + \ldots . \tag{5.8}$$

The estimate is l steps in advance of the most recent value of the message. This may correspond either for example, to an l-step predictor based on past and present values of the message or alternatively, to estimation of present values of the signal based on l step delayed message.

(c) $l < 0$: *message in advance of signal estimate.*

Letting $l_1 = -l$ we have $\mathfrak{I} = \{-l_1, -l_1 + 1, \ldots, 0, \ldots\}$ and

$$[F(z^{-1})]_+ = F_{-l_1} z^{l_1} + F_{-l_1+1} z^{(l_1-1)} + \ldots + F_{-1} z + \\ + F_0 + F_1 z^{-1} + F_2 z^{-2} + \ldots . \tag{5.9}$$

The estimate being delayed with respect the the message, this corresponds to the fixed-lag smoother with delay l_1.

The process of truncation of the nonrealizable filter impulse response to give the actual impulse response is depicted in the figures below.

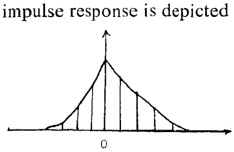

Fig. 3(a). Impulse response of ideal nonrealizable filter.

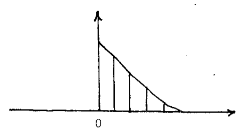

Fig. 3(b). Impulse response of filter when $l = 0$.

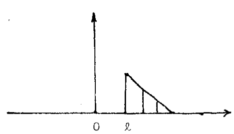

Fig. 3(c). Impulse response of filter when $l > 0$.

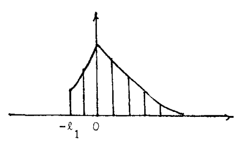

Fig. 3(d). Impulse response of filter when $l < 0$.

In the published literature a distinction between instantaneous and delayed forms of the Wiener filter was not always made clear even fairly recently – see e.g. Chi Tsong Chen [14]. The ideas are however quite old, e.g. the distinction was clearly made in unpublished notes by Grensted [15].

6. SIGNAL PLUS WHITE NOISE: SPECTRAL FACTORIZATION

Now consider the process of spectral factorization for the additive white noise case represented by the block diagram of Fig. 4.

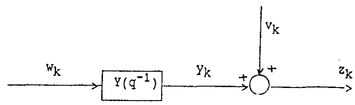

Fig. 4. Block diagram of message generating process of signal plus white noise.

The message is

$$z_k = y_k + v_k = Y(q^{-1})\, w_k + v_k \tag{6.1}$$

where $\{w_k\}, \{v_k\}$ are mutually uncorrelated, zero-mean white noise processes and $Y(\varkappa^{-1})$ is the shaping filter t.f.m. of the signal process. The spectral function for z is, from this equation,

$$S_{z,z}(\varkappa) = Y(\varkappa^{-1})\, Q Y(\varkappa)^{\mathrm{T}} + R \tag{6.2}$$

where Q and R is the covariance matrix of the white noises w and v:

$$Q = \mathsf{E}\{w_k w_k^{\mathrm{T}}\}\,, \quad R = \mathsf{E}\{v_k v_k^{\mathrm{T}}\}\,. \tag{6.3}$$

On the other hand, from the innovations representation for z:

$$S_{z,z}(\varkappa) = Z(\varkappa^{-1})\, R_{\varepsilon,\varepsilon}\, Z(\varkappa)^{\mathrm{T}}\,. \tag{6.4}$$

On equating the two expressions for $S_{zz}(\varkappa)$ follows

$$Z(\varkappa^{-1})\, R_{\varepsilon,\varepsilon}\, Z(\varkappa)^{\mathrm{T}} = Y(\varkappa^{-1})\, Q Y(\varkappa)^{\mathrm{T}} + R \tag{6.5}$$

the *spectral factorization* equation for determining $Z(\cdot)$.

7. OPTIMAL ESTIMATION FOR SIGNAL PLUS WHITE NOISE

Substitution of

$$S_{y,z}(\varkappa^{-1}) = S_{y,y}(\varkappa^{-1}) = S_{z,z}(\varkappa^{-1}) - R \tag{7.1}$$

in the expression (5.3) for the optimal unrealizable filter t.f.m. $F(\varkappa^{-1})$ gives

$$F(\varkappa^{-1}) = S_{y,z}(\varkappa^{-1})\,(R_{\varepsilon,\varepsilon}\, Z(\varkappa)^{\mathrm{T}})^{-1} \tag{7.2}$$

$$= S_{y,y}(\varkappa^{-1})\,(R_{\varepsilon,\varepsilon}\, Z(\varkappa)^{\mathrm{T}})^{-1} \tag{7.3}$$

$$= Z(\varkappa^{-1}) - R(R_{\varepsilon,\varepsilon}\, Z(\varkappa)^{\mathrm{T}})^{-1} \tag{7.4}$$

On the right hand side, the first term is regular when $|\varkappa| > 1$ and the second term regular when $|\varkappa| < 1$ as can be seen from the expansions

$$Z(\varkappa^{-1}) = I + G_1 \varkappa^{-1} + G_2 \varkappa^{-2} + \dots \tag{7.5}$$

$$-R(R_{\varepsilon,\varepsilon}^{-1}\, Z(z)^{\mathrm{T}})^{-1} = -R(Z(z)^{-1})^{\mathrm{T}}\, R_{\varepsilon,\varepsilon}^{-1}$$

$$= -RR_{\varepsilon,\varepsilon}^{-1} + RB_1^{\mathrm{T}} R_{\varepsilon,\varepsilon}^{-1} z + RB_2^{\mathrm{T}} R_{\varepsilon,\varepsilon}^{-1} z^2 + \dots \qquad (7.6)$$

These two series thus give the terms of the Laurent expansions corresponding to past and future values respectively. Both series include a constant term referring to present value.

(a) *Instantaneous operation*: here the constant term is included in $[F]_+$ so that F has decomposition with

$$[F(z^{-1})]_+ = Z(z^{-1}) - RR_{\varepsilon,\varepsilon}^{-1} \qquad (7.7)$$

$$[F(z^{-1})]_- = RR_{\varepsilon,\varepsilon}^{-1} - R(R_{\varepsilon,\varepsilon}\, Z(z)^{\mathrm{T}})^{-1} \qquad (7.8)$$

The optimal filter t.f.m. is

$$H_{\mathrm{opt}}(z^{-1}) = (Z(z^{-1}) - RR_{\varepsilon,\varepsilon}^{-1})\, Z(z^{-1})^{-1} \qquad (7.9)$$

$$= I - RR_{\varepsilon,\varepsilon}^{-1}\, Z(z^{-1})^{-1}\,. \qquad (7.10)$$

(b) *One-step delayed filter*: here the constant term is not included in $[F]_+$ so that

$$[F(z^{-1})]_+ = Z(z^{-1}) - I \qquad (7.11)$$

$$[F(z^{-1})]_- = I - R(R_{\varepsilon,\varepsilon}\, Z(z)^{\mathrm{T}})^{-1}\,. \qquad (7.12)$$

The optimal filter t.f.m. is

$$H_{\mathrm{opt}}(z^{-1}) = (Z(z^{-1}) - I)\, Z(z^{-1})^{-1} \qquad (7.13)$$

$$= I - Z(z^{-1})^{-1} \qquad (7.14)$$

8. FEEDBACK REPRESENTATION OF OPTIMAL FILTER

Recall that

$$Z(z^{-1}) = I + G(z^{-1}) = I + G_1 z^{-1} + G_2 z^{-2} + \dots \qquad (8.1)$$

so that $G(z^{-1})$ represents the terms in powers of z^{-1} in the expansion of $Z(z^{-1})$. $G(\cdot)$ has the following significance:

$$G(q^{-1})\, \varepsilon_k = (Z(q^{-1}) - I)\, \varepsilon_k \qquad (8.2)$$

$$= z_k - \varepsilon_k \qquad (8.3)$$

$$= \hat{z}_{k|k-1} \qquad (8.4)$$

the one step estimate of z_k based on the previous message values. In the case when the message is signal plus white noise, as is here assumed, this estimate will coincide with the estimate of signal from previous message value. Thus

$$\hat{y}_{k|k-1} = G(q^{-1})\, \varepsilon_k\,. \qquad (8.5)$$

Now consider:

(a) *One-step delayed operation*: the optimal filter t.f.m. is from (7.13)

(8.6) $$H(z^{-1}) = (Z(z^{-1}) - I)\, Z(z^{-1})^{-1}$$

(8.7) $$= G(z^{-1})\,(I + G(z^{-1}))^{-1}$$

which is the equation relating feedback and open-loop transfer function matrices. The optimal filter has the feedback realization shown below with open-loop t.f.m. $G(z^{-1})$:

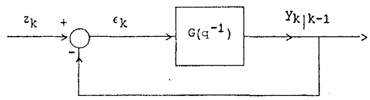

Fig. 5. Feedback realization of the optimal one-step-delay filter of signal from additive white noise.

The feedback scheme is due basically to the formation at the innovations process by the equation.

(8.8) $$\varepsilon_k = z_k - \hat{y}_{k|k-1}\,.$$

The return-difference t.f.m. of the filter is

(8.9) $$I + G(z^{-1}) = Z(z^{-1})\,.$$

(b) *Instantaneous filter*: using the expression (7.10) follows

(8.10) $$\hat{y}_{k|k} = H(q^{-1})\, z_k$$

(8.11) $$= z_k - RR_{\varepsilon,\varepsilon}^{-1}\, Z(q^{-1})^{-1}\, z_k$$

(8.12) $$= z_k - RR_{\varepsilon,\varepsilon}^{-1}\varepsilon_k$$

from which

(8.13) $$\varepsilon_k = R_{\varepsilon,\varepsilon}R^{-1}(z_k - \hat{y}_{k|k})$$

assuming R nonsingular. The feedback realization of the filter involves an extra

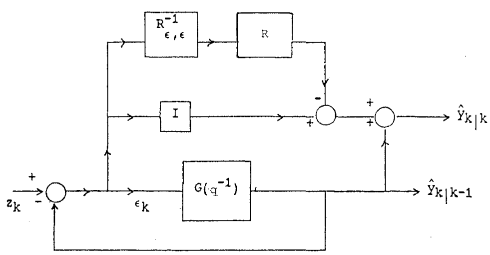

Fig. 6. Modified filter for instantaneous estimation.

multiplicative factor in the forward path and is not so convenient as in Fig. 5. However from (8.8), (8.12) is found

(8.14) $$\hat{y}_{k|k} = \hat{y}_{k|k-1} + (I - RR_{\varepsilon,\varepsilon}^{-1})\,\varepsilon_k$$

and consequently the estimate $\hat{y}_{k|k}$ may be found by adding a correction term to $\hat{y}_{k|k-1}$ by the method shown in Fig. 6 below.

9. THE KALMAN THEORY

Here the signal process $\{y_k\}$ has state space representation

(9.1) $$\begin{aligned} x_{k+1} &= Ax_k + Bw_k \\ y_k &= Hx_k \end{aligned}$$

where A, B, H are matrices of appropriate dimension with A stable. The message process $\{z_k\}$ is

(9.2) $$z_k = y_k + v_k$$

$\{y_k\}$ will be the projection of a stationary Gauss-Markov process. The system has t.f.m. representation

(9.3) $$y_k = Y(q^{-1})\,w_k$$

where

(9.4) $$Y(z^{-1}) = H(Iz - A)^{-1}\,B = H(I - z^{-1}A)^{-1}\,z^{-1}B\,.$$

The spectral factorization equation is well known in the Kalman theory and it may be shown by purely algebraic manipulation (Mee [16], Arcasoy [17]) that

(9.5) $$Y(z^{-1})\,Q\,Y(z)^{\mathrm{T}} + R = $$
$$= (I + H(I - z^{-1}A)^{-1}\,z^{-1}K_p)\,(R + HPH^{\mathrm{T}})\,(I + H(I - zA)^{-1}\,zK_p)^{\mathrm{T}}$$

where K_p is the Kalman predictor gain:

(9.6) $$K_p = APH^{\mathrm{T}}(R + HPH^{\mathrm{T}})^{-1}$$

and P is the unique symmetrical solution of the discrete matrix Riccati equation:

(9.7) $$P = APA^{\mathrm{T}} - APH^{\mathrm{T}}(R + HPH^{\mathrm{T}})^{-1}\,HPA^{\mathrm{T}} + BQB^{\mathrm{T}}$$

A being assumed stable, it has eigenvalues of modulus less than unity and so the expansion

(9.8) $$(I - z^{-1}A)^{-1} = I + z^{-1}A + z^{-2}A^2 + \ldots$$

is valid for $|z^{-1}| < 1$. Consequently

(9.9) $$Z(z^{-1}) = I + H(I - z^{-1}A)^{-1}\,z^{-1}K_p$$

is analytic within $|z^{-1}| < 1$. Since it also satisfies

(9.10) $$Z(0) = I$$

it is the message innovations-shaping filter t.f.m. and so the message innovations covariance matrix is

$$R_{\varepsilon,\varepsilon} = R + HPH^{T} . \tag{9.11}$$

The open loop t.f.m. is now seen to be

$$G(z^{-1}) = H(I - z^{-1}A)^{-1} K_p z^{-1} . \tag{9.12}$$

The one-step delay feedback filter of Fig. 3 is seen to coincide with the usual form for the stationary Kalman filter shown in Fig. 7 below.

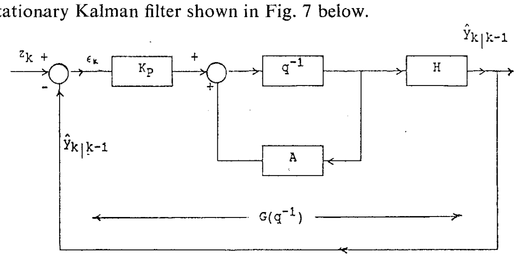

Fig. 7. Stationary discrete-time Kalman filter.

10. THE MULTIVARIABLE HAGANDER-WITTENMARK FILTER

Here the signal is defined by the vector ARMA model

$$\begin{aligned} y_k = {} & -A_1 y_{k-1} - A_2 y_{k-2} - \ldots - A_n y_{k-n} + \\ & + C_1 w_{k-1} + C_2 w_{k-2} + \ldots + C_n w_{k-n} \end{aligned} \tag{10.1}$$

without loss of generality since some of the C coefficient matrices can be assumed zero. $\{w_k\}$ is a white noise process having zero mean and given covariance matrix Q. On the right-hand side a term in w_k will not occur since y_k is assumed to have no white noise component – see [2].

Equation (10.1) may be written as

$$A(q^{-1})\, y_k = C(q^{-1})\, w_k \tag{10.2}$$

where $A(\cdot)$ and $C(\cdot)$ are the matrix polynomials

$$A(z^{-1}) = I + A_1 z^{-1} + \ldots + A_n z^{-n} \tag{10.3}$$

$$C(z^{-1}) = \quad C_1 z^{-1} + \ldots + C_n z^{-n} \tag{10.4}$$

$A(\cdot)$ and $C(\cdot)$ are assumed left coprime and $A(z^{-1})$ is assumed stable i.e. det $A(z^{-1})$ is assumed to have no zeros when $|z^{-1}| < 1$. Then $A(z^{-1})^{-1}$ is regular in this region and so $A(q^{-1})^{-1}$ defines a stable operator. Hence we may write

$$y_k = A(q^{-1})^{-1}\, C(q^{-1})\, w_k \tag{10.5}$$

so that the signal shaping t.f.m.* is

$$Y(z^{-1}) = A(z^{-1})^{-1}\, C(z^{-1})\,. \tag{10.6}$$

The message process is

$$z_k = Y(q^{-1})\, w_k + v_k \tag{10.7}$$

$$= A(q^{-1})^{-1} \{C(q^{-1})\, w_k + A(q^{-1})\, v_k\} \tag{10.8}$$

where as before $\{v_k\}$ is white noise having zero mean and covariance matrix R. The expression

$$C(q^{-1})\, w_k + A(q^{-1})\, v_k \tag{10.9}$$

occurring here may be put into innovations form

$$D(q^{-1})\, \varepsilon_k \tag{10.10}$$

on solving the spectral factorization equation

$$C(z^{-1})\, QC(z)^{\mathrm{T}} + A(z^{-1})\, RA(z)^{\mathrm{T}} = D(z^{-1})\, R_{\varepsilon,\varepsilon}\, D(z)^{\mathrm{T}} \tag{10.11}$$

for $D(\cdot)$ as a stable nth degree polynomial having the form

$$D(z^{-1}) = I + D_1 z^{-1} + \ldots + D_n z^{-n}\,. \tag{10.12}$$

In full the spectral factorization equation is

$$\begin{aligned}(C_1 z^{-1} + \ldots + C_n z^{-n})\, Q(C_1 z + \ldots + C_n z^n)^{\mathrm{T}} &+ \\ + (I + A_1 z^{-1} + \ldots + A_n z^{-n})\, R(I + A_1 z + \ldots + A_n z^n)^{\mathrm{T}} &= \\ = (I + D_1 z^{-1} + \ldots + D_n z^{-n})\, R_{\varepsilon,\varepsilon}(I + D_1 z + \ldots + D_n z^n)^{\mathrm{T}}&\,.\end{aligned} \tag{10.13}$$

On comparing the coefficients of z^{-1} we find

$$A_n R = D_n R_{\varepsilon,\varepsilon}\,. \tag{10.14}$$

On substitution of the value found for $D(\cdot)$, the message process may be written

$$z_k = A(q^{-1})^{-1}\, D(q^{-1})\, \varepsilon_k \tag{10.15}$$

which in fact is the innovations representation for z_k since

$$Z(z^{-1}) = A(z^{-1})^{-1}\, D(z^{-1}) \tag{10.16}$$

is minimum phase and satisfies the condition

$$Z(0) = I\,. \tag{10.17}$$

The one-step-delay filter: This will have feedback realization with open-loop t.f.m.

$$G(z^{-1}) = Z(z^{-1}) - I \tag{10.18}$$

$$= A(z^{-1})^{-1} \{D(z^{-1}) - A(z^{-1})\} \tag{10.19}$$

The feedback realization will consequently be as shown below.

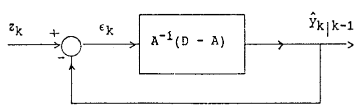

Fig. 8. One-step-delay filter of signal in ARMA representation from white noise.

* Not unique since w_k is not an innovations process.

Instantaneous filter: The transfer function is

(10.20) $$H(z^{-1}) = I - RR_{\varepsilon,\varepsilon}^{-1} D(z^{-1})^{-1} A(z^{-1})$$

and multiplying by z_k we get

(10.21) $$\hat{y}_{k|k} = z_k - RR_{\varepsilon,\varepsilon}^{-1} \varepsilon_k .$$

When D_n is nonsingular, this equation can be written using (10.14) as

(10.22) $$\hat{y}_{k|k} = z_k - A_n^{-1} D_n \varepsilon_k$$

which corresponds to the multidimensional form of the Hagander-Wittenmark self-tuning filter.

11. ONE-STEP-DELAY ARMA FILTER AS A KALMAN FILTER

By using an auxiliary state vector

(11.1) $$x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(n)} \end{bmatrix}$$

$x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ being vectors of the same type as y, the ARMA representation of the previous section may be written in state-space form

(11.2) $$x_{k+1} = Ax_k + Cw_k$$

(11.3) $$y_k = Hx_k$$

(11.4) $$z_k = y_k + v_k$$

where

(11.5) $$A = \begin{bmatrix} -A_1 & I & 0 & 0 \\ -A_2 & 0 & I & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -A_{n-1} & 0 & 0 & I \\ -A_n & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{n-1} \\ C_n \end{bmatrix}$$

(11.6) $$H = [I \ 0 \ \ldots \ 0]$$

the matrix A being in canonical observable form (Anderson & Moore [18]). The condition that $A(z^{-1})$ is stable clearly implies that A is also stable.

Similarly the one-step-delay filter of Fig. 8 described by

(11.7) $$\hat{y}_{k|k-1} = A(q^{-1})^{-1} \{D(q^{-1}) - A(q^{-1})\} \varepsilon_k$$

(11.8) $$\varepsilon_k = z_k - \hat{y}_{k|k-1}$$

may be written in the normal state-space form for a Kalman filter for the estimates $\hat{x}, \hat{y}$ as

(11.9) $$\hat{x}_{k+1} = A\hat{x}_k + K_p\varepsilon_k$$

(11.10) $$\hat{y}_{k+1|k} = H\hat{x}_k$$

where A, H have the above values and the Kalman predictor gain K_p is

(11.11) $$K_P = \begin{bmatrix} D_1 - A_1 \\ D_2 - A_2 \\ \vdots \\ D_n - A_n \end{bmatrix}$$

(see Deng [7])

Instantaneous Kalman filter as Hagander-Wittenmark filter: The instantaneous Kalman filter follows from the equations

(11.12) $$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_f\varepsilon_k$$

(11.13) $$\hat{y}_{k|k} = H\hat{x}_{k|k}$$

where

(11.14) $$AK_f = K_P\,.$$

From here the Hagander-Wittenmark filter may be derived by two methods. The first is that of Deng [7].

Method 1: if A is nonsingular then

(11.15) $$K_f = A^{-1}K_P$$

By computing A^{-1}, Deng shows that

(11.16) $$HK_f = HA^{-1}K_P = I - A_n^{-1}D_n\,.$$

Hence the instantaneous filter is

(11.17) $$y_{k|k} = \hat{y}_{k|k-1} + (I - A_n^{-1}D_n)\,\varepsilon_k = z_k - A_n^{-1}D_n\varepsilon_k\,.$$

Method 2:

(11.18) $$\begin{aligned} \hat{y}_{k|k} &= \hat{y}_{k|k-1} + HK_f\varepsilon_k \\ &= \hat{y}_{k|k-1} + HP\,H^{\mathrm{T}}\,R_{\varepsilon,\varepsilon}^{-1}\varepsilon_k \\ &= \hat{y}_{k|k} + (R_{\varepsilon,\varepsilon} - R)\,R_{\varepsilon,\varepsilon}^{-1}\varepsilon_k \\ &= \hat{y}_{k|k-1} + \varepsilon_k - RR_{\varepsilon,\varepsilon}^{-1}\varepsilon_k \\ &= z_k - RR_{\varepsilon,\varepsilon}^{-1}\varepsilon_k \\ &= z_k - A_n^{-1}\,D_n\,\varepsilon_k \end{aligned}$$

12. THE FIXED-LAG SMOOTHING PROBLEM

Now we shall briefly consider the multivariable fixed lag smoother. This corresponds to case (c) of Section 5 where estimation of the signal is delayed l_1 steps so that the message is l_1 steps in advance of the signal estimate ($l_1 > 0$).

In this case the Laurent expansion (5.9) contains positive powers of z up to z^{l_1}, which terms come from the relation (7.6). Now from (4.10),

$$-R(R_{\varepsilon,\varepsilon} Z(z)^{\mathrm{T}})^{-1} = -R(Z(z)^{-1})^{\mathrm{T}} R_{\varepsilon,\varepsilon}^{-1} \tag{12.1}$$
$$= -R(I - B_1^{\mathrm{T}} z - B_2^{\mathrm{T}} z^2 \dots) R_{\varepsilon,\varepsilon}^{-1}$$

Taking terms up to degree l_1 we get

$$[F(z^{-1})]_+ = Z(z^{-1}) + \left(-RR_{\varepsilon,\varepsilon} + \sum_{i=1}^{l_1} RB_i^{\mathrm{T}} R_{\varepsilon,\varepsilon}^{-1} z^i\right). \tag{12.2}$$

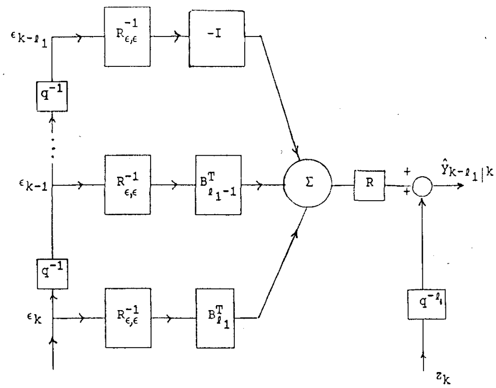

Fig. 9. Synthesis of optimal fixed lag filter for lag l_1.

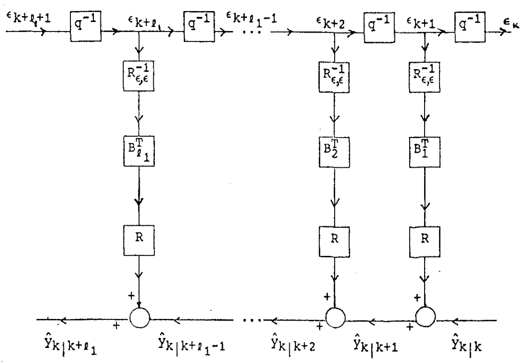

Fig. 10. Structure of estimation procedure.

If this filter acts on l-step delayed innovations to give required estimate (as in Fig. 2) it is seen that

$$(12.3) \qquad \hat{y}_{k-l_1}|_k = z_{k-l_1} - RR_{\varepsilon,\varepsilon}^{-1}\varepsilon_{k-l_1} + \sum_{i=1}^{l_1} RB_i^{\mathrm{T}}R_{\varepsilon,\varepsilon}^{-1}\varepsilon_{k-l_1+i}\,.$$

The estimation may be performed as shown in Fig. 9 below. In this figure the innovations may be generated from the incoming message by the feedback filter of Fig. 5. The estimate may also be thought of as being built up by successive modifications to the one-step delay estimate as shown in Fig. 10. Similar schemes have been proposed in the literature using Kalman filtering (see e.g. Moore [19]).

13. THE PREDICTION PROBLEM

Finally we consider the prediction problem where the message is being delayed by l-steps behind the signal estimate. In this case the Laurent expansion (5.4) is cut off as in (5.8). In the case of signal plus white noise the coefficients are

$$(13.1) \qquad F_i = G_i\,, \quad i = 1, 2, 3, \ldots$$

(5.8) becomes

$$(13.2) \qquad [F(z^{-1})]_+ = G_l z^{-l} + G_{l+1} z^{-(l+1)} + \ldots$$

$$(13.3) \qquad = Z(z^{-1}) - (I + G_1 z^{-1} + G_2 z^{-2} + \ldots + G_{l-1} z^{-(l-1)})\,.$$

In the prediction problem the direct attempt to realize (13.3) would result in a non-causal filter. However further progress may be made if $Z(\cdot)$ is expressed as the ratio of polynomial matrices:

$$(13.4) \qquad Z(z^{-1}) = A(z^{-1})^{-1}\, D(z^{-1})$$

as in Section 10 on the Hagander-Wittenmark theory. Expansion in powers of z^{-1} gives

$$(13.5) \qquad A(z^{-1})^{-1}\, D(z^{-1}) = I + G_1 z^{-1} + G_2 z^{-2} + \ldots$$

where the G's are easily determined recursively from

$$(13.6) \qquad (I + D_1 z^{-1} + D_2 z^{-2} + \ldots + D_n z^{-n}) =$$
$$= (I + A_1 z^{-1} + A_2 z^{-2} + \ldots + A_n z^{-n})(I + G_1 z^{-1} + G_2 z^{-2} + \ldots)$$

as

$$(13.7) \qquad G_i = \begin{cases} D_i - \sum\limits_{j=1}^{\min(i,n)} A_j G_{i-j} & \text{for} \quad 1 \leqq i \leqq n \\ \quad - \sum\limits_{j=1}^{n} A_j G_{i-j} & \text{for} \quad i > n \end{cases}$$

(with G_0 as I).

Then the right-hand side of (13.3) is

$$(13.8) \qquad A(z^{-1})^{-1}\, D(z^{-1}) - (I + G_1 z^{-1} + G_2 z^{-2} + \ldots + G_{l-1} z^{-(l-1)})$$

i.e. the expansion (13.5) with the initial l terms removed. This gives

(13.9) $$[F(z^{-1})]_+ = A(z^{-1})^{-1} D_1(z^{-1})$$

where

(13.10) $$D_1(z^{-1}) = D(z^{-1}) - A(z^{-1})(I + G_1 z^{-1} + G_2 z^{-2} + \ldots + G_{l-1} z^{-(l-1)}).$$

The filter is as shown in Fig. 11 below. Note that $[F(z^{-1})]_+$ contains a delay of l

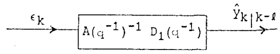

Fig. 11. The l-step predictor.

steps so that $\hat{y}_{k|k-l}$ is estimated from values of message up to z_{k-l}. As before the innovations process may be generated either by direct whitening of the message of by the feedback arrangement of the one-step predictor.

The procedure here described leads to the expression for a predictor using only simple algebraic computations and avoids the customary Diophantine formulation.

The successive estimates may be calculated using the equation

(13.11) $$A(q^{-1})\,\hat{y}_{k|k-l} = D_1(q^{-1})\,\varepsilon_k$$

which leads to a recursive ARMA model for the estimates, in which each estimate is calculated in terms of lagged innovations and l-step predictions.

14. CONCLUSION

A unified approach to the discrete-time stationary filtering problem has been presented based on the multivariable Wiener theory.

This approach covers various aspects of filtering using instantaneous, delayed or future information; it includes, with others, the Kalman filter and the Hagander-Wittenmark filtering used in self-tuning. The advantage of the approach is that interrelationships between the various filters can be easily appreciated. The approach also leads in a natural way to the feedback form of the basic one-step delay filter i.e. Kalman filter.

The apporach also extends to stochastic control but this will be the subject of future work.

15. APPENDIX: DERIVATION OF THE OPTIMAL WIENER FILTER BY "COMPLETING THE SQUARE"

A convenient way of solving the Wiener filtering problem is by "completing the square". In the multivariable continuous-time case this method was deseribed by Kučera [20] and Barrett [21]. The scalar discrete-time case was described by Grensted

[15] being adapted from the scalar continuous-time case described by Barrett [22].

The derivation goes as follows. Using the notation of the text, the estimation error is

$$e_k = y_k - H(q^{-1})\, z_k$$

and has spectral function

$$S_{e,e}(z^{-1}) = S_{y,y}(z^{-1}) - H(z^{-1})\, S_{z,y}(z^{-1}) - \\ - S_{y,z}(z^{-1})\, H(z)^{\mathrm{T}} + H(z^{-1})\, S_{z,z}(z^{-1})\, H(z)^{\mathrm{T}}.$$

The analogue of the familiar "completing the square" procedure used for solving quadratic equations gives

$$S_{e,e}(z^{-1}) = \{S_{y,y}(z^{-1}) - S_{y,z}(z^{-1})\, S_{z,z}(z^{-1})\, S_{z,y}(z^{-1})\} + \\ + \{F(z^{-1}) - H(z^{-1})\, Z(z^{-1})\}\, R_{\varepsilon,\varepsilon}\{F(z) - H(z)\, Z(z)\}^{\mathrm{T}}$$

where

$$F(z) = S_{y,z}(z^{-1})\,(R_{\varepsilon,\varepsilon}\, Z(z)^{\mathrm{T}})^{-1}$$

$F(z)$ is here the ideal t.f.m. used in the text (equation (5.3)).

Now follows using the variance formula,

$$\operatorname{tr} \mathsf{E}\,\{e_k e_k^{\mathrm{T}}\} = \frac{1}{2\pi \mathrm{j}} \oint_{|z|=1} \operatorname{tr} S_{e,e}(z^{-1}) \frac{\mathrm{d}z}{z} = I_1 + I_2$$

where

$$I_1 = \frac{1}{2\pi \mathrm{j}} \oint_{|z|=1} \operatorname{tr} \{S_{y,y}(z^{-1}) - S_{y,z}(z^{-1})\, S_{z,z}(z^{-1})\, S_{z,y}(z)\} \frac{\mathrm{d}z}{z}$$

$$I_2 = \frac{1}{2\pi \mathrm{j}} \oint_{|z|=1} \operatorname{tr} [\{F(z^{-1}) - H(z^{-1})\, Z(z^{-1})\}\, R_{\varepsilon,\varepsilon}\{F(z) - H(z)\, Z(z)\}^{\mathrm{T}}] \frac{\mathrm{d}z}{z}.$$

The expression I_2 here is non-negative as may be seen on putting $z = \mathrm{e}^{\mathrm{j}\theta}$ and writing it as

$$I_2 = \frac{1}{2\pi} \oint_0^{2\pi} \operatorname{tr} [A(\mathrm{e}^{\mathrm{j}\theta})\, R_{\varepsilon,\varepsilon} A(\mathrm{e}^{-\mathrm{j}\theta})^{\mathrm{T}}]\, \mathrm{d}\theta$$

where

$$A(z^{-1}) = F(z^{-1}) - H(z^{-1})\, Z(z^{-1}).$$

Since the trace operator acts on a Hermitian matrix, I_2 vanishes only when

$$H(z^{-1}) = F(z^{-1})\, Z(z^{-1})^{-1}$$

i.e. when $H(z^{-1})$ is the ideal optimal t.f.m. acting on all values of the message, past, present, and future.

Otherwise if $A(z^{-1})$ is expanded as a Laurent series and truncation is made (for any delay) giving a realizable and nonrealizable part:

$$A(z^{-1}) = [A(z^{-1})]_+ + [A(z^{-1})]_-$$

as in Section 5 of the text then it is easily verified that

$$I_2 = \frac{1}{2\pi j} \oint_{|z|=1} \operatorname{tr} [A_+(z^{-1}) R_{\varepsilon,\varepsilon} A_+(z)^T] \frac{dz}{z} +$$

$$+ \frac{1}{2\pi j} \oint_{|z|=1} \operatorname{tr} [A_-(z^{-1}) R_{\varepsilon,\varepsilon} A_-(z)^T] \frac{dz}{z}.$$

Since the two terms involving the cross-products:

$$A_+(z^{-1}) R_{\varepsilon,\varepsilon} A_-(z)^T, \quad A_-(z^{-1}) R_{\varepsilon,\varepsilon} A_+(z)^T$$

vanish because the series expansions of these cross-products do not contain constant terms and so the products give zero residue for the pole at $z = 0$.

Now it is found that

$$A_+(z^{-1}) = [F(z^{-1})]_+ - H(z^{-1}) Z(z^{-1})$$
$$A_-(z^{-1}) = [F(z^{-1})]_- .$$

Then the first term of I_2 vanishes when and only when

$$H(z^{-1}) = [F(z^{-1})]_+ Z(z^{-1})^{-1}$$

and there is an additional contribution to $\operatorname{tr} \mathsf{E}\{ee^T\}$ of

$$\frac{1}{2\pi j} \oint_{|z|=1} \operatorname{tr} [\{F(z^{-1})\}_+ R_{\varepsilon,\varepsilon} \{F(z)^T\}^+] \frac{dz}{z}.$$

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