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HELLINGER INTEGRALS, CONTIGUITY AND ENTIRE SEPARATION

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Hellinger integrals of distribution laws are estimated in terms of Hellinger integrals of the corresponding conditional distributions belonging to an increasing sequence of sub-σ-algebras. The estimates are employed for a new approach to the problem of contiguity and entire separation of sequences of distribution laws. New sufficient conditions in predictable terms both for contiguity and entire separation are derived.

0. INTRODUCTION

In the present paper we investigate Hellinger integrals of distribution laws P, Q defined on a probability space equipped with a filtration $\mathfrak{F}_0 \subseteq \mathfrak{F}_1 \subseteq \ldots$. A key role plays Theorem 1 where estimates for Hellinger integrals and Hellinger measures in terms of conditional distributions are stated. These estimates generalize results obtained earlier in [4]. There the problem of convergence in variational distance and error probabilities in the problem of testing statistical hypotheses were treated. In the present paper Theorem 1 is employed for deriving necessary and sufficient conditions for contiguity and entire separation of two sequences of probability measures. On the one side the well-known conditions [6] for contiguity and entire separation will be deduced from Theorem 1. On the other side, new conditions will be given. These conditions are formulated in terms of the Hellinger integrals of the conditional distributions. Thereby the parameter of the Hellinger integrals tends to 1. Applying these results in statistics one sequence (denoted by Q_n) corresponds to null hypothesis whereas the other sequence P_n belongs to a sequence of alternatives which has shown to be contiguous with respect to Q_n. Both, the conditions in [6] and the new conditions in Theorem 2 are formulated in such a way that certain conditions concerning the conditional distributions are assumed to be fulfilled P_n — a.s. for n large. But in many cases the structure of the alternatives P_n is more complicated than that of the null hypotheses Q_n. Consequently, it is desirable to state sufficient conditions for contiguity formulated in terms of Q_n. Assertions of this kind are established in Theorem 4 and 5.

We apply the general estatimates to the special case where P_n corresponds to a sequence of Markov processes and Q_n are the distribution laws of a sequence of independent random variables. As an example we study Gaussian first order autoregressive processes.

1. NOTATIONS AND RESULTS

Let $[\Omega, \mathfrak{F}]$ be a measurable space and P, Q probability measures on $[\Omega, \mathfrak{F}]$. Suppose R is a probability measure dominating P and Q. Denote by X, Y the Radon-Nikodym derivatives with respect to R. As in [9] we introduce the Hellinger measure $\mathscr{H}_{s,P,Q}$ by

$$\mathscr{H}_{s,P,Q}(A) = \int_A X^s Y^{1-s} \, dR \,, \quad A \in \mathfrak{F} \,, \quad 0 < s < 1 \,.$$

The functional

$$H_s(P, Q) = \int X^s Y^{1-s} \, dR = \mathscr{H}_{s,P,Q}(\Omega) \,, \quad 0 < s < 1 \,,$$

is called Hellinger integral of order s. Suppose $\mathfrak{F}_0 \subseteq \mathfrak{F}_1 \subseteq \ldots$ is non-decreasing sequence of sub-σ-algebras generating $\mathfrak{F}$ where $\mathfrak{F}_0 = \{\emptyset, \Omega\}$. Denote by P_k, Q_k, R_k the restrictions of P, Q, R to $\mathfrak{F}_k$. Put $X_k = dP_k/dR_k$, $Y_k = dQ_k/dR_k$.
Then

$$X_k = E_R(X \mid \mathfrak{F}_k) \,, \quad Y_k = E_R(Y \mid \mathfrak{F}_k) \,.$$

We write $A \subseteq B$ R-a.s. if $R(A \cap B) = 0$. For real numbers a, b the symbol $ab^{\oplus}$ denotes ab^{-1} if $b \neq 0$ and 0 if $b = 0$. For every non-negative supermartingale ξ_k, $k = 0, 1, \ldots$, the following inclusions hold R-a.s.

$$(1) \qquad \{\xi_{k-1} = 0\} \subseteq \{E_R(\xi_k \mid \mathfrak{F}_{k-1}) = 0\} \subseteq \{\xi_k = 0\} \,.$$

Since $x^s y^{1-s}$ is a concave function $X_k^s Y_k^{1-s}$ is a non-negative supermartingale.
Put

$$U_k = X_k X_{k-1}^{\oplus} \,, \quad V_k = Y_k Y_{k-1}^{\oplus}$$

$$H_{s,k} = E_R(U_k^s V_k^{1-s} \mid \mathfrak{F}_{k-1})$$

for $k \geqq 1$ and $H_{s,0} = 1$. Then

$$(2) \qquad H_{s,k} = (E_R(X_k^s Y_k^{1-s} \mid \mathfrak{F}_{k-1}))\,(X_{k-1}^s Y_{k-1}^{1-s})^{\oplus} \quad \text{R-a.s.}$$

Jensen's inequality for conditional expectation implies

$$0 \leqq H_{s,k} \leqq 1 \quad \text{R-a.s.} \,.$$

Introduce $G_{s,k}$, $0 \leqq k \leqq \infty$, by

$$G_{s,k} = \prod_{l=0}^{k} H_{s,l}$$

and note that because of $0 \leqq H_{s,k} \leqq 1$ the possibly infinite product is well-defined. $G_{s,k}$ is closely related to $G_{t,k}$, $t \neq s$, and to $J_{s,k}$ defined by

$$J_{s,k} = \sum_{l=0}^{k} (1 - H_{s,l}) .$$

In the case $s = \frac{1}{2}$ we omit the index $\frac{1}{2}$. Given real numbers a, b we set $a \wedge b = \operatorname{Min}(a, b)$, $a \vee b = \operatorname{Max}(a, b)$.

Proposition 1. Suppose $0 < s < 1$, $0 < s_1 < s_2 < 1$ and $0 \leqq k \leqq \infty$. Then R-a.s.

$$1 - J_{s,k} \leqq G_{s,k} \leqq \exp\{-J_{s,k}\}$$

$$G_{s_2,k}^{(1-s_1)/(1-s_2)} \leqq G_{s_1,k} \leqq G_{s_2,k}^{s_1/s_2} .$$

Notice that both $G_{s,k}$ and $J_{s,k}$ are $\mathfrak{F}_{k-1}$ measurable. Consequently as $\{k: 0 \leqq k < \infty, G_{s,k} > 0\} \neq \emptyset$ in view of $G_{s,0} = 1$

$$T_s = \sup\{k: G_{s,k} > 0\}$$

is well-defined and it is a stopping time. The second inequality in Proposition 1 shows that for $0 < s_1 < s_2 < 1$ it holds $T_{s_1} = T_{s_2}$ R-a.s. Consequently the index s can be omitted in the sequel. Put $\xi_k = X_k^s Y_k^{1-s}$. The inclusions (1) show that both X_{k-1} and Y_{k-1} are strictly positive on $\{T \geqq k\}$. Hence

$$\mathsf{E}_\mathsf{R}(U_k \mid \mathfrak{F}_{k-1}) = \mathsf{E}_\mathsf{R}(V_k \mid \mathfrak{F}_{k-1}) = 1$$

and

$$1 - H_{s,k} = \mathsf{E}_\mathsf{R}(sU_k + (1-s)\, V_k - U_k^s V_k^{1-s} \mid \mathfrak{F}_{k-1})$$

R-a.s. on $\{T \geqq k\}$. This yields

$$J_{s,k \wedge T} = \sum_{l=1}^{k \wedge T} \mathsf{E}_\mathsf{R}(sU_l + (1-s)\, V_l - U_l^s V_l^{1-s} \mid \mathfrak{F}_{l-1})$$

and in the special case $s = \frac{1}{2}$

$$J_{k \wedge T} = \tfrac{1}{2} \sum_{l=1}^{k \wedge T} \mathsf{E}_\mathsf{R}((\sqrt{(U_l)} - \sqrt{(V_l)})^2 \mid \mathfrak{F}_{l-1}) .$$

The process $J_{k \wedge T}$ has been introduced in [6] and [2], where conditions for contiguity and entire separation have been established in terms of $J_{k \wedge T}$.

If Ω is a product space and regular conditional distributions do exist then, roughly speaking, $H_{s,k}$ is the Hellinger integral of the conditional distributions belonging to P and Q, respectively. To be more precise, let $[\Omega_1, \mathfrak{A}_1]$, $[\Omega_2, \mathfrak{A}_2]$, ... be a sequence of measurable spaces which are assumed to be of type (B) in the sense of [3]. Suppose the σ-algebras $\mathfrak{A}_i$ are countably generated, $i = 1, 2, \ldots$. Put $\Omega = \bigtimes_{i=1}^{m} \Omega_i$,

$\mathfrak{F} = \bigotimes_{i=1}^{m} \mathfrak{A}_i$, $m \leqq \infty$ and denote by $\mathfrak{F}_k$ the sub-σ-algebra of $\mathfrak{F}$ generated by the projections up to k. Set $\mathfrak{F}_0 = \{\emptyset, \Omega\}$. Assume P, Q are probability measures on $[\Omega, \mathfrak{F}]$ and R is a dominating probability measure. Denote by K_1, L_1, M_1 the distributions of the first coordinate and by $K_i(\omega_1, \omega_2, \ldots, \omega_{i-1}, A_i)$, $L_i(\omega_1, \omega_2, \ldots, \omega_{i-1}, A_i)$, $M_i(\omega_1, \omega_2, \ldots, \omega_{i-1}, V_i)$, $\omega_k \in \Omega_k$, $A_i \in \mathfrak{A}_i$ regular conditional distributions belonging to P, Q, R.

As the σ-algebras $\mathfrak{A}_i$ are countably generated we find (see [10]) $\mathfrak{A}_1 \otimes \ldots \otimes \mathfrak{A}_i$ measurable functions $p_i(\omega_1, \ldots, \omega_i)$, $q_i(\omega_1, \ldots, \omega_i)$ which fulfil the conditions

$$\int_{A_i} p_i(\omega_1, \ldots, \omega_i)\, M_i(\omega_1, \ldots, \omega_{i-1}, d\omega_i) = K_i(\omega_1, \ldots, \omega_{i-1}, A_i)$$

$$\int_{A_i} q_i(\omega_1, \ldots, \omega_i)\, M_i(\omega_1, \ldots, \omega_{i-1}, d\omega_i) = L_i(\omega_1, \ldots, \omega_{i-1}, A_i)$$

for every $\omega_1 \in \Omega_1, \ldots, \omega_{i-1} \in \Omega_{i-1}$, $A_i \in \mathfrak{A}_i$, $i = 2, 3, \ldots$. Consider p_i, q_i to be defined on $[\Omega, \mathfrak{F}]$. An easy calculation shows

$$X_k = \prod_{i=1}^{k} p_i\,, \quad Y_k = \prod_{i=1}^{k} q_i\,, \quad k < \infty\,, \quad \text{R-a.s.}$$

Hence $U_l = p_l$, $V_l = q_l$ R-a.s. on the set $\{T \geqq l\} \subseteq \{X_{l-1} > 0, Y_{l-1} > 0\}$. This leads to

$$H_{s,l} = \mathsf{E}_{\mathsf{R}}(U_l^s V_l^{1-s} \mid \mathfrak{F}_{l-1}) = \int p_l^s q_l^{1-s} M_l(\cdot, d\omega_l) =$$

$$= H_s(K_l, L_l) \quad \text{R-a.s. on } \{T \geqq l\}\,.$$

Note that $T \geqq 1$ if K_1 and L_1 are not singular and $H_{s,1} = 0$ if K_1 and L_1 are singular. Hence

$$G_{s,k \wedge T} = \prod_{l=1}^{k \wedge T} H_s(K_l, L_l) \quad \text{R-a.s.} \tag{3}$$

where the convention $\prod_{l=1}^{0} H_s(K_l, L_l) = 1$ is used.

Relation (3) can be applied to product measures $\mathsf{P} = \bigtimes_{i=1}^{\infty} \mu_i$, $\mathsf{Q} = \bigtimes_{i=1}^{\infty} \nu_i$. But in this case, without any additional assumptions concerning the measurable spaces, an easy consideration shows

$$G_{s,k \wedge T} = \prod_{i=1}^{k \wedge T} H_s(\mu_i, \nu_i) \quad \text{R-a.s.}$$

We now return to the general situation. In the following theorem estimates of Hellinger measures and Hellinger integrals, respectively, will be established. The bounds will be formulated in terms of $G_{s,k}$. Denote by $I(A)$ the indicator function of the set A.

Given real numbers with

$$0 < s_1 < s < s_2 < 1\,, \quad \text{or} \quad 0 < s_2 < s < s_1 < 1$$

we put

$$\alpha_1 = \frac{s_1(1-s)}{s(s-s_1)}, \quad \beta_1 = \frac{s-s_1}{1-s_1}$$

$$\alpha_2 = \frac{(1-s_2)\,s}{(1-s)(s_2-s)}, \quad \beta_2 = \frac{s_2-s}{s_2}\,.$$

We make use of the conventions $0^\alpha = \infty$, $\infty^\alpha = 0$ for $\alpha < 0$ in Theorem 1 and in the sequel.

Theorem 1. Suppose P, Q are probability measures on $[\Omega, \mathfrak{F}]$ which is equipped with a non-decreasing sequence $\mathfrak{F}_0 \subseteq \mathfrak{F}_1 \subseteq \ldots$ of sub-σ-algebras generating $\mathfrak{F}$ and $\mathfrak{F}_0 = \{\emptyset, \Omega\}$. Assume S is a stopping time with respect to $\mathfrak{F}_0 \subseteq \mathfrak{F}_1 \subseteq \ldots$ and $0 < s_2 < s < s_1 < 1$. Then

(4) $$H_s(\mathsf{P}, \mathsf{Q}) \geqq [\mathsf{E}_\mathsf{P} G_{s,S}^{\alpha_1}]^{\beta_1} \vee [\mathsf{E}_\mathsf{Q} G_{s,S}^{\alpha_2}]^{\beta_2} - \mathsf{P}^s(S < \infty)\, \mathsf{Q}^{1-s}(S < \infty)\,.$$

Alternatively, for every $A \in \mathfrak{F}$, $0 < s_1 < s < s_2 < 1$

(5) $$\mathscr{H}_{s,\mathsf{P},\mathsf{Q}}(A) \leqq [\mathsf{E}_\mathsf{P}\, I(A)\; G_{s,\infty}^{\alpha_1}]^{\beta_1} \wedge [\mathsf{E}_\mathsf{Q}\, I(A)\; G_{s,\infty}^{\alpha_2}]^{\beta_2}\,.$$

Remark. In case of product measures $\mathsf{P} = \bigtimes_{i=1}^{\infty} \mu_i$, $\mathsf{Q} = \bigtimes_{i=1}^{\infty} \nu_i$ with equivalent components $\mu_i \sim \nu_i$ we see that $T = \infty$ and

$$G_{s,\infty} = \prod_{i=1}^{\infty} H_s(\mu_i, \nu_i)\,.$$

Putting $S = \infty$ and letting $s_1 \downarrow s$, $s_2 \uparrow s$ we achieve equality in (4) since [8]

$$H_s(\mathsf{P}, \mathsf{Q}) = \prod_{i=1}^{\infty} H_s(\mu_i, \nu_i)\,.$$

Analogously, if $A = \Omega$, $s_1 \uparrow s$, $s_2 \downarrow s$ equality in (5) is attained.

Corollary. For every $0 < \gamma < 1$

(6) $$H_s(\mathsf{P}, \mathsf{Q}) \leqq \gamma + \mathscr{H}_{s,\mathsf{P},\mathsf{Q}}(G_{s,\infty} > \gamma)$$

(7) $$H_s(\mathsf{P}, \mathsf{Q}) \geqq \gamma - \mathsf{P}^s(G_{s,\infty} < \gamma)\, \mathsf{Q}^{1-s}(G_{s,\infty} < \gamma)\,.$$

Inequality (7) and inequality (5) for $A = \Omega$ have been already obtained in [4] where the distributions P, Q are defined on product spaces and constructed by regular conditional distributions.

Now we will turn to the problem of contiguity and entire separation of two sequences of probability measures.

Definition. Suppose $[\Omega_n, \mathfrak{F}_n]$ is a sequence of measurable spaces and P_n, Q_n probability measures on $[\Omega_n, \mathfrak{F}_n]$. P_n is said to be contiguous with respect to $\mathsf{Q}_n (\mathsf{P}_n \lhd \mathsf{Q}_n)$ if for every sequence $A_k \in \mathfrak{F}_k$, $\mathsf{Q}_k(A_k) \to 0$ implies $\mathsf{P}_k(A_k) \to 0$ as $k \to \infty$. The sequences P_k, Q_k are said to be entirely separated ($P_k \Delta Q_k$) if there exists a sequence $B_k \in \mathfrak{F}_k$ with

$$\lim_{k \to \infty} [P_k(B_k) + Q_k(B_k^c)] = 0 .$$

Suppose now each measurable space $[\Omega_n, \mathfrak{F}_n]$ is equipped with a non-decreasing sequence $\mathfrak{F}_{0,n} \subseteq \mathfrak{F}_{1,n} \subseteq \ldots$ of sub-σ-algebras with $\mathfrak{F}_{0,n} = \{\emptyset, \Omega_n\}$, $\mathfrak{F}_n = \sigma(\bigcup_{k=0}^{\infty} \mathfrak{F}_{k,n})$.

Let for every n the measures P_n, Q_n be dominated by the probability measure R_n. Denote by $P_{k,n}$, $Q_{k,n}$, $R_{k,n}$ the restrictions of P_n, Q_n, R_n to $\mathfrak{F}_{k,n}$. Introduce $X_{k,n}$, $Y_{k,n}$, $U_{k,n}$, $V_{k,n}$, T_n, $G_{s,k,n}$, $J_{k,n}$ analogous to X_k, Y_k, U_k, V_k, T, $G_{s,k}$, J_k. Put

$$L_{k,n} = \begin{cases} \infty & U_{k,n} > 0 , \quad V_{k,n} = 0 \\ \dfrac{U_{k,n}}{V_{k,n}} & V_{k,n} > 0 \\ 0 & U_{k,n} = 0 , \quad V_{k,n} = 0 \end{cases}$$

$L_{k,n}$ is the likelihood-ratio of the conditional distributions with respect to $\mathfrak{F}_{k,n}$.

We now formulate criteria for contiguity.

Theorem 2. The following assertions are equivalent

(8) $$P_n \lhd Q_n$$

(9) $$\lim_{s \uparrow 1} \lim_{n \to 0} H_s(P_n, Q_n) = 1$$

(10) $$\lim_{s \uparrow 1} \lim_{n \to \infty} E_{P_n} G_{s,\infty,n} = 1$$

(11) $$\overline{\lim_{c \to \infty}}\, \overline{\lim_{n \to \infty}} P_n(\sup_{1 \leq k < \infty} L_{k,n} > c) = 0$$

and

$$\overline{\lim_{c \to \infty}}\, \overline{\lim_{n \to \infty}} P_n(J_{\infty,n} > c) = 0 .$$

Remark. The implication (8) ↔ (9) is a general criterion for contiguity established in [5]. The implication (8) ↔ (11) has been proved in [2], [6] by entirely different methods.

Now we will formulate necessary and sufficient conditions for entire separation Put

$$(P_n \wedge Q_n)(A) = \int I(A)(X_n \wedge Y_n) \, dR_n .$$

Theorem 3. The following assertions are equivalent:

(12) $$P_n \Delta Q_n$$

(13) there exists $0 < s < 1$ with $\lim_{n \to \infty} H_s(P_n, Q_n) = 0$

(14) there exists $0 < s < 1$ with $\underline{\lim}_{n\to\infty} \int G_{s,\infty,n} \, d\mathcal{H}_{s,P_n,Q_n} = 0$

(15) there exists $0 < s < 1$ with $\underline{\lim}_{n\to\infty} \int G_{s,\infty,n} \, d(P_n \wedge Q_n) = 0$.

The implication $(12) \leftrightarrow (13)$ is a general criterion for entire separation established in [5].

From Hölder's inequality one easily concludes

$$\mathcal{H}_{s_2,P,Q}^{(1-s_1)/(1-s_2)}(A) \leqq \mathcal{H}_{s_1,P,Q}(A) \leqq \mathcal{H}_{s_2,P,Q}^{s_1/s_2}(A)$$

$0 < s_1 < s_2 < 1$. This inequality and the second inequality in Proposition 1 show that (13), (14) and (15) hold for every $0 < s < 1$ provided that they hold for some $0 < < s < 1$.

Corollary 1. If there exists $0 < s < 1$ so that

$$\underline{\lim}_{n\to\infty} (E_{P_n} G_{s,\infty,n}) \wedge (E_{Q_n} G_{s,\infty,n}) = 0$$

then $P_n \, \Delta Q_n$.

Corollary 2. Suppose there exists $0 < s < 1$ such that

(16) $$\underline{\lim}_{n\to\infty} E_{P_n} G_{s,\infty,n} = 0\,.$$

Then $P_n \, \Delta Q_n$.

If additionally $$\overline{\lim}_{c\to\infty} \overline{\lim}_{n\to\infty} P_n(\sup_{1\leqq k<\infty} L_{k,n} > c) = 0$$

then $P_n \, \Delta Q_n$ implies (16) for every $0 < s < 1$.

Corollary 2 has been stated in a slightly different form in [6] where entirely different methods have been used. In [6] the condition

$$\underline{\lim}_{c\to\infty} \underline{\lim}_{n\to\infty} P_n(J_{\infty,n} < c) = 0$$

is used instead of (16). By Proposition 1 this is stronger than (16). But under the additional condition above they are equivalent (see the proof of Corollary 2).

In Theorem 2 the conditions for contiguity are formulated with respect to P_n. In statistics P_n plays the role of a contiguous alternative whereas Q_n is the null hypothesis. In many cases the structure of the measure Q_n is essentially more transparent than that of P_n. Consequently, criteria for contiguity formulated in terms of Q_n would be useful for applications in statistics. A result in this direction is Theorem 4 which follows from (4) and (9).

Theorem 4. Suppose there exists a function $\psi(s)$ on $(\alpha, 1)$, $0 < \alpha < 1$, such that

$0 < \psi(s) < s$ for every $\alpha < s < 1$ and

$$\varliminf_{s \uparrow 1} \varliminf_{n \to \infty} \left[\mathsf{E}_{\mathsf{Q}_n} G_{s,\infty,n}^{-(\psi_1(s))} \right]^{-(\psi_2(s))} = 1$$

where

$$\psi_1(s) = \frac{(1 - \psi(s))\, s}{(1 - s)(s - \psi(s))}, \quad \psi_2(s) = \frac{s - \psi(s)}{\psi(s)}.$$

Then $\mathsf{P}_n \lhd \mathsf{Q}_n$.

In order to derive further conditions which are sufficient for contiguity we need some notations.

Put

$$\varphi(u, v) = \begin{cases} u \ln u/v \quad -u + v & u \geqq 0, \quad v > 0 \\ \infty & u > 0, \quad v = 0 \\ 0 & u = 0, \quad v = 0 \end{cases}$$

then $\varphi(u, v) \geqq 0$. Hence $\mathsf{E}_{\mathsf{R}_n}(\varphi(U_{k,n}, V_{k,n}) \mid \mathfrak{F}_{k-1})$ is well-defined. Set

$$I_{k,n} = \sum_{l=1}^{k} \mathsf{E}_{\mathsf{R}_n}(\varphi(U_{l,n}, V_{l,n}) \mid \mathfrak{F}_{l-1}), \quad k \leqq \infty .$$

Theorem 5. Suppose the restrictions of P_n and Q_n to $\mathfrak{F}_{k,n}$, $k = 1, 2, \ldots, n = 1, 2, \ldots,$ are equivalent. If there exists an $\varepsilon > 0$ with

$$\varlimsup_{n \to \infty} \mathsf{E}_{\mathsf{Q}_n} \exp \{(1 + \varepsilon)\, I_{\infty,n}\} < \infty$$

then

$$\mathsf{P}_n \lhd \mathsf{Q}_n .$$

2. APPLICATIONS AND EXAMPLES

The results concerning contiguity and entire separation established in Theorem 2 and 4, respectively, lead to transparent conditions if P_n is the distribution of Markov process and Q_n is the distribution of independent random variables. Suppose the measurable spaces $[\Omega_{l,n}, \mathfrak{A}_{l,n}]$, $l = 1, 2, \ldots, m_n$, $n = 1, 2, \ldots$ are of type (B) ([3]). Assume the σ-algebras $\mathfrak{A}_{l,n}$ are countably generated. Let us be given distribution laws $K_{1,n}$ on $[\Omega_{1,n}, \mathfrak{A}_{1,n}]$ and stochastic kernels

$$K_{l,n}(\omega_{l-1,n}, A_{l,n}), \quad \omega_{l-1,n} \in \Omega_{l-1,n}, A_{l,n} \in \mathfrak{A}_{l,n}, \quad l = 2, \ldots, m_n .$$

Put $[\Omega_n, \mathfrak{F}_n] = \bigotimes_{l=1}^{m_n} [\Omega_{l,n}, \mathfrak{A}_{l,n}]$, $\mathfrak{F}_{0,n} = \{\emptyset, \Omega_n\}$ and denote by $\mathfrak{F}_{k,n}$ the sub-σ-algebra of $\mathfrak{F}_n$ generated by the projections up to k, $k \leqq m_n$ and $\mathfrak{F}_{k,n} = \mathfrak{F}_n$ for $k \geqq m_k$. Denote by P_n the probability measure defined by the initial distribution $K_{1,n}$ and the stochastic kernels $K_{l,n}$. Furthermore, assume that $\nu_{l,n}$ are probability measures on $[\Omega_{l,n}, \mathfrak{A}_{l,n}]$. Put $\mathsf{Q}_n = \nu_{1,n} \times \ldots \times \nu_{m_n,n}$.

Relation (3) yields

$$G_{s,T_n,n} = \prod_{l=1}^{m_n \wedge T_n} H_s(K_{l,n}, \nu_{l,n}) .$$

Because of $G_{s,\infty,n} = 0$ on $\{T\} < \infty\}$ we get

$$G_{s,\infty,n} \leqq \prod_{l=1}^{m_n} H_s(K_{l,n}, \nu_{l,n}) .$$

$H_s(K_{1,n}, \nu_{1,n})$ is a real number and $H_s(K_{l,n}, \nu_{l,n})$, $l = 2, \ldots, m_n$, are random variables which are independent under Q_n. Hence

$$E_{Q_n} G_{s,\infty,n} \leqq \prod_{l=1}^{m_n} \Delta_{l,n}(s)$$

where

$$\Delta_{l,n}(s) = \int H_s(K_{l,n}(\omega_{l-1,n}, \cdot), \nu_{l,n}) \, \nu_{l-1,n}(d\omega_{l-1,n}) ,$$

$l = 2, \ldots, m_n$, and $\Delta_{1,n}(s) = H_s(K_{1,n}, \nu_{1,n})$.

The above inequality and Corollary 1 to Theorem 3 provide

Proposition 2. If

(17) $$\varliminf_{n\to\infty} \prod_{l=1}^{m_n} \Delta_{l,n}(\tfrac{1}{2}) = 0$$

then

$$P_n \, \Delta Q_n .$$

Now we deal with contiguity. Suppose $K_{1,n} \sim \nu_{1,n}$ and $K_{l,n}(\omega_{l-1,n}, \cdot) \sim \nu_{l-1,n}$ for every $\omega_{l-1,n} \in \Omega_{l-1,n}$ $l = 2, \ldots, m_n$. Put

$$p_{l,n} = \frac{dK_{l,n}}{d\nu_{l,n}}$$

where it is assumed that $p_{l,n}$, $l \geqq 2$, is chosen such that it is a measurable function of $(\omega_{l-1,n}, \omega_{l,n})$.

Set

$$\gamma_{1,n} = \int p_{1,n}(\omega_{1,n}) \ln p_{1n}(\omega_{1,n}) \, \nu_{1,n}(d\omega_{1,n})$$

$$\gamma_{l,n}(\omega_{l-1,n}) = \int p_{l,n}(\omega_{l-1,n}, \omega_{l,n}) \ln p_{l,n}(\omega_{l-1,n}, \omega_{l,n}) \, \nu_{l,n}(d\omega_{l,n})$$

for $l = 2, \ldots, m_n$.

As the random variables $\exp\{\gamma_{l,n}\}$, are independent under Q_n we get for $a > 0$

$$E_{Q_n} \exp\{a l_{\infty,n}\} = \prod_{l=1}^{m_n} \chi_{l,n}(a)$$

where

$$\chi_{l,n}(a) = \int \exp\{a\,\gamma_{l,n}(\omega_{l-1,n})\}\,\nu_{l-1,n}(\mathrm{d}\omega_{l-1,n}),$$

$l = 2, \ldots, m_n$, and $\chi_{1,n}(a) = \exp\{a\,\gamma_{1,n}\}$.

In this way we obtain from Theorem 5

Proposition 3. If there exists an $\varepsilon > 0$ so that

$$\overline{\lim_{n\to\infty}} \prod_{l=1}^{m_n} \chi_{l,n}(1+\varepsilon) < \infty$$

then

$$\mathsf{P}_n \lhd \mathsf{Q}_n .$$

As an application of the Propositions we deal with Gaussian first order autoregressive processes. Denote by $N(a, \sigma^2)$ the normal distribution with expectation a and variance σ^2. Suppose $Z_1, Z_2, \ldots$, are i.i.d. random variables with common distribution $N(0, 1)$. Put

$$X_{1,n} = Z_1\,, \quad X_{k+1,n} = \varrho_{k,n} X_{k,n} + \sqrt{(1-\varrho_{k,n}^2)}\, Z_{k+1}, \quad k = 1, 2, \ldots, m_n - 1$$

where $0 \leqq |\varrho_{k,n}| < 1$.

Denote by P_n the distribution of the vector $X_{1,n}, \ldots, X_{m_n,n}$ and by Q_n the distribution of the vector $Z_1, \ldots, Z_{m_n}$. That means in the framework formulated above

$$K_{l,n} = \nu_{l,n} = N(0, 1)\,, \quad K_{l+1,n}(x_l, \cdot) = N(\varrho_{l,n} x_l, 1 - \varrho_{l,n}^2)\,.$$

Since $\Delta_n(s)$ and $\chi_n(a)$ can be calculated explicitly we are able to show

Proposition 4. It holds $\mathsf{P}_n \lhd \mathsf{Q}_n$ iff

$$\overline{\lim_{n\to\infty}} \sup_{1\leqq l\leqq m_n} |\varrho_{l,n}| < 1 \quad \text{and} \quad \overline{\lim_{n\to\infty}} \sum_{l=1}^{m_n} \varrho_{l,n}^2 < \infty\,.$$

$\mathsf{P}_n \,\Delta\, \mathsf{Q}_n$ iff at least one of the following conditions is fulfilled:

$$\overline{\lim_{n\to\infty}} \sup_{1\leqq l\leqq m_n} |\varrho_{l,n}| = 1$$

$$\overline{\lim_{n\to\infty}} \sum_{l=1}^{m_n} \varrho_{l,n}^2 = \infty\,.$$

Remark. A consequence of Proposition 4 is that either $\mathsf{P}_n \lhd \mathsf{Q}_n$ or $\mathsf{P}_n \,\Delta\, \mathsf{Q}_n$. This fact is true in general for all sequences P_n, Q_n of Gaussian distributions [1].

3. PROOFS

Proofs of Proposition 1. Let $0 \leqq a_i \leqq 1$, $1 \leqq i \leqq k$, be real numbers. The following inequalities are well-known:

$$1 - \prod_{i=1}^{k} a_i \leqq \sum_{i=1}^{k} (1 - a_i)$$

$$\prod_{i=1}^{k} a_i = \exp\{\sum_{i=1}^{k} \ln a_i\} \leqq \exp\{-\sum_{i=1}^{k} (1 - a_i)\}.$$

These inequalities imply the first relation in Proposition 1. In order to prove the second statement we make use of a general result [7] often applied in what follows.

Let W be a non-negative $\mathfrak{F}_k$ measurable random variable. Then

$$\mathsf{E}_{\mathsf{P}}(W \mid \mathfrak{F}_{k-1}) = \mathsf{E}_{\mathsf{R}}(U_k W \mid \mathfrak{F}_{k-1}) \quad \mathsf{P}\text{-a.s.} \tag{18}$$

$$\mathsf{E}_{\mathsf{Q}}(W \mid \mathfrak{F}_{k-1}) = \mathsf{E}_{\mathsf{R}}(V_k W \mid \mathfrak{F}_{k-1}) \quad \mathsf{Q}\text{-a.s.} \tag{19}$$

Hence, both (18) and (19) hold R-a.s. on $A_{k-1} := \{X_{k-1} > 0, Y_{k-1} > 0\}$. Put $W = U_l V_l^{\oplus}$ and assume $0 < s_1 < s_2 < 1$. Then by Hölder's inequality

$$H_{s_1,l} = \mathsf{E}_{\mathsf{R}}(U_l^{s_1} V_l^{1-s_1} \mid \mathfrak{F}_{l-1}) = \mathsf{E}_{\mathsf{Q}}(W^{s_1} \mid \mathfrak{F}_{l-1}) \leqq$$

$$\leqq [\mathsf{E}_{\mathsf{Q}}(W^{s_2} \mid \mathfrak{F}_{l-1})]^{s_1/s_2} = H_{s_2,l}^{s_1/s_2} \quad \mathsf{R}\text{-a.s.} \quad \text{on} \quad A_{l-1}.$$

Note that $H_{s,l} = 0$ R-a.s. on A_{l-1}^c for every $0 < s < 1$ by (2). Hence

$$H_{s_1,l} \leqq H_{s_2,l}^{s_1/s_2} \quad \mathsf{R}\text{-a.s.}$$

Taking the product up to k on both sides we get the right hand inequality in the second statement of Proposition 1. The inequality on the left hand side may be obtained similarly.

Proof of Theorem 1. Put

$$Z_{s,0} = 1, \quad Z_{s,k} = \frac{X_{k \wedge T}^s Y_{k \wedge T}^{1-s}}{G_{s,k \wedge T}}, \quad k \geqq 1.$$

Then

$$X_k^s Y_k^{1-s} = Z_{s,k} G_{s,k} \tag{20}$$

on the set $\{T \geqq k\}$. Put $\xi_k = X_k^s Y_k^{1-s}$. The inclusions (1) show that (20) is also fulfilled on the set $\{T < k\}$ as both $X_k^s Y_k^{1-s}$ and $G_{s,k}$ vanish on $\{T < k\}$. It follows that

$$\mathsf{E}_{\mathsf{R}}(Z_{s,k} - Z_{s,k-1} \mid \mathfrak{F}_{k-1}) =$$

$$= Z_{s,k-1} \, \mathsf{E}_{\mathsf{R}}\left(I(T \geqq k)\left[\frac{X_k^s Y_k^{1-s}}{\mathsf{E}_{\mathsf{R}}(X_k^s Y_k^{1-s} \mid \mathfrak{F}_{k-1})} - 1\right] \Big| \, \mathfrak{F}_{k-1}\right) = 0 \quad \mathsf{R}\text{-a.s.}$$

where we used the fact that by (1) and (2) and the definition of $G_{s,k}$

$$\mathsf{E}_{\mathsf{R}}(X_k^s Y_k^{1-s} \mid \mathfrak{F}_{k-1}) > 0 \quad \text{on} \quad \{T \geqq k\}.$$

Consequently, $Z_{s,k}$ is a martingale. As $Z_{s,0} = 1$ we see that

(21) $$\mathsf{E}_{\mathsf{R}} Z_{s,k} = 1\,, \quad k = 0, 1, \ldots$$

Relation (20) gives R-a.s.

(22) $$X_k^s Y_k^{1-s} = Z_{s_2,k}^{s/s_2}\left(G_{s_2,k}^{s/s_2} Y_k^{(s_2-s)/s_2}\right)$$

for $0 < s < s_2 < 1$ and

(23) $$Z_{s_2,k} = \left[X_{k \wedge T}^s Y_{k \wedge T}^{1-s}\right]^{s_2/s} \,.\, G_{s_2,k \wedge T}^{-1} Y_{k \wedge T}^{(s-s_2)/s}$$

for any $0 < s_2 < s < 1$.

Let S be a stopping time and N a natural number. Since $Z_{s,k}$ is a martingale we obtain from (21)

$$\mathsf{E}_{\mathsf{R}} Z_{s,N \wedge S \wedge T} = 1\,.$$

Suppose now $0 < s_2 < s < 1$ and substitute k by $N \wedge S$ in (23). Then by Hölder's inequality

(24) $$1 \leqq \left[\mathsf{E}_{\mathsf{R}} X_{N \wedge S \wedge T}^s Y_{N \wedge S \wedge T}^{1-s}\right]^{s_2/s} \left[\mathsf{E}_{\mathsf{R}} Y_{N \wedge S \wedge T} G_{s_2,N \wedge S \wedge T}^{-s/(s-s_2)}\right]^{(s-s_2)/s} = $$
$$= \left[\mathsf{E}_{\mathsf{R}} X_{N \wedge S \wedge T}^s Y_{N \wedge S \wedge T}^{1-s}\right]^{s_2/s} \left[\mathsf{E}_{\mathsf{Q}} G_{s_2,N \wedge S \wedge T}^{-s/(s-s_2)}\right]^{(s-s_2)/s}$$

X_k and Y_k are uniformly integrable martingales. This fact implies that

$$X_{N \wedge S \wedge T}^s Y_{N \wedge S \wedge T}^{1-s} \leqq s X_{N \wedge S \wedge T} + (1-s)\, Y_{N \wedge S \wedge T}\,, \quad N = 1, 2, \ldots,$$

is uniformly integrable, too. Moreover,

$$X_{N \wedge S \wedge T}^s Y_{N \wedge S \wedge T}^{1-s} \to X_{S \wedge T}^s Y_{S \wedge T}^{1-s} \quad \text{R-a.s.} \quad \text{as} \quad N \to \infty$$

where $X_\infty = X$, $Y_\infty = Y$. Because of $0 \leqq H_{s,k} \leqq 1$

$$G_{s_2,N \wedge S \wedge T}^{-s/(s-s_2)} \uparrow G_{s_2,S \wedge T}^{-s/(s-s_2)} \quad \text{as} \quad N \to \infty\,.$$

Therefore expectation and limit can be exchanged in (24). Hence, by the second statement in Proposition 1

$$\mathsf{E}_{\mathsf{R}} X_{S \wedge T}^s Y_{S \wedge T}^{1-s} \geqq \left[\mathsf{E}_{\mathsf{Q}} G_{s_2,S \wedge T}^{-s/(s-s_2)}\right]^{-(s-s_2)/s_2} \geqq \left[\mathsf{E}_{\mathsf{Q}} G_{s,S \wedge T}^{\alpha_2}\right]^{\beta_2}.$$

If $\mathsf{Q}(S > T) > 0$ then by our conventions $\left[\mathsf{E}_{\mathsf{Q}} G_{s,S}^{\alpha_2}\right]^{\beta_2} = 0$ so that

(25) $$\mathsf{E}_{\mathsf{R}} X_S^s Y_S^{1-s} \geqq \left[\mathsf{E}_{\mathsf{Q}} G_{s,S}^{\alpha_2}\right]^{\beta_2}.$$

If $\mathsf{Q}(S > T) = 0$ then $G_{s,S} = G_{s,S \wedge T}$ Q-a.s.. As $\{S > T\} \in \mathfrak{F}_{S \wedge T}$ and $Y_{S \wedge T}$ is the density of $\mathsf{Q}_{S \wedge T}$ with respect to $\mathsf{R}_{S \wedge T}$ we get from $\mathsf{Q}(S > T) = 0$ that R-a.s. $X_S^s Y_S^{1-s} = X_{S \wedge T}^s Y_{S \wedge T}^{1-s} = 0$ on the set $\{S > T\}$. Hence $\mathsf{E}_{\mathsf{R}} X_{S \wedge T}^s Y_{S \wedge T}^{1-s} = \mathsf{E}_{\mathsf{R}} X_S^s Y_S^{1-s}$ and (25) is proved. Furthermore by Hölder's inequality and (25)

$$H_s(\mathsf{P}, \mathsf{Q}) = \mathsf{E}_{\mathsf{R}} X_S^s Y_S^{1-s} + \mathsf{E}_{\mathsf{R}}(X^s Y^{1-s} - X_S^s Y_S^{1-s}) \geqq$$
$$\geqq \mathsf{E}_{\mathsf{R}} X_S^s Y_S^{1-s} - \mathsf{E}_{\mathsf{R}} I(S < \infty)\, X_S^s Y_S^{1-s} \geqq \left[\mathsf{E}_{\mathsf{Q}} G_{s,S}^{\alpha_2}\right]^{\beta_2} - \mathsf{P}^s(S < \infty)\, \mathsf{Q}^{1-s}(S < \infty)\,.$$

Analogously one proves for $0 < s < s_1 < 1$

$$H_s(\mathsf{P}, \mathsf{Q}) \geqq [\mathsf{E}_\mathsf{P} G_{s,S}^{\alpha_1}]^{\beta_1} - \mathsf{P}^s(S < \infty)\, \mathsf{Q}^{1-s}(S < \infty)\,.$$

Taking the maximum of the right hand sides we get (4).

Now we prove the second statement in Theorem 1. $Z_{s,k}$ is a non-negative martingale. Consequently $Z_{s,\infty} := \lim_{k\to\infty} Z_{s,k}$ exists R-a.s. Observe $X = \lim_{k\to\infty} X_k$, $Y = \lim_{k\to\infty} Y_k$ R-a.s. Hence (22) remains true for $k = \infty$. Note that by Fatou's lemma

$$\mathsf{E}_\mathsf{R} Z_{s,\infty} \leqq 1\,.$$

Suppose $0 < s < s_2 < 1$ and apply Hölder's inequality to (22). Then by the second statement in Proposition 1

$$\mathscr{H}_{s,\mathsf{P},\mathsf{Q}}(A) \leqq [\mathsf{E}_\mathsf{R}\, Z_{s_2}]^{s/s_2}\, [\mathsf{E}_\mathsf{R}\, Y G_{s_2}^{s/(s_2-s)}\, I(A)]^{(s_2-s)/s_2} \leqq [\mathsf{E}_\mathsf{Q} G_s^{\alpha_2} I(A)]^{\beta_2}\,.$$

Using similar arguments one concludes for $0 < s_1 < s < 1$

$$\mathscr{H}_{s,\mathsf{P},\mathsf{Q}}(A) \leqq [\mathsf{E}_\mathsf{P} G_s^{\alpha_1}\, I(A)]^{\beta_1}\,.$$

These inequalities yield (5). □

Proof of the Corollary. Because of $G_{s,0} = 1$ the set $\{k\colon G_{s,k} \geqq \gamma\}$ is non-empty for $0 < \gamma < 1$. Put $S = \sup\{k\colon G_{s,k} \geqq \gamma\}$. Then S is a stopping time. As $G_{s,S} \geqq \gamma$ inequality (7) is an immediate consequence of (4) and $\{S < \infty\} = \{G_{s,\infty} < \gamma\}$. For proving (6) we set $A = \{G_{s,\infty} \leqq \gamma\}$. Then by (5)

$$H_s(\mathsf{P}, \mathsf{Q}) = \mathscr{H}_{s,\mathsf{P},\mathsf{Q}}(A) + \mathscr{H}_{s,\mathsf{P},\mathsf{Q}}(A^c) \leqq \gamma^{\alpha_1\beta_1} \wedge \gamma^{\alpha_2\beta_2} + \mathscr{H}_{s,\mathsf{P},\mathsf{Q}}(G_{s,\infty} > \gamma)\,.$$

Since

$$\alpha_1\beta_1 = \frac{s_1(1-s)}{s(1-s_1)} \to 1 \quad \text{as} \quad s_1 \uparrow s$$

$$\alpha_2\beta_2 = \frac{(1-s_2)\,s}{(1-s)\,s_2} \to 1 \quad \text{as} \quad s_2 \downarrow s\,.$$

we get (6).

In order to prove Theorem 3 we need auxiliary results which will be established in the following lemmas. Put

$$L_k = \begin{cases} \infty & U_k > 0,\ V_k = 0 \\ U_k/V_k & V_k > 0 \\ 0 & U_k = 0,\ V_k = 0\,. \end{cases}$$

Introduce the stopping times S_c, T_c by

$$S_c = \inf\{k\colon J_k > c\}\,, \quad T_c = \inf\{k\colon L_k > c\}\,.$$

Given $A \in \mathfrak{F}$ and a random variable ξ taking values in $[0, \infty]$ the expression $(I(A)\,.\,\xi$ is understood to be ξ on A and 0 on A^c.

Lemma 1. Suppose $1 > s > \frac{1}{2}$ and $c > 1$. Then there exists a constant $d(c)$ depending only on c such that P-a.s.

$$J_{s,\infty} \leqq I(T_c \wedge S_c = \infty)\, d(c)\,(1-s) + I(S_c \wedge T_c < \infty)\, 2J_\infty + W(c)$$

where

$$W(c) = \sum_{k=1}^{\infty} \mathsf{E}_{\mathsf{P}}(I(T_c = k) \mid \mathfrak{F}_{k-1})\,.$$

Proof. The proof is divided into several steps.

1. Set

$$f(s,x) = sx + (1-s) - x^s$$

An easy consideration shows that there exists a constant $e(c)$ so that

$$f(s,x) \leqq e(c)\,(1-s)\, f(\tfrac{1}{2}, x)\,, \quad 0 \leqq x \leqq c\,, \quad \tfrac{1}{2} < s < 1\,.$$

Hence

$$su + (1-s)\,v - u^s v^{1-s} \leqq (1-s)\, e(c)\,(\tfrac{1}{2}u + \tfrac{1}{2}v - u^{1/2}v^{1/2}) \tag{26}$$

for every $0 \leqq u \leqq cv$.

Let $0 < x < \infty$ be fixed. Then $f(s,x)$ is a non-negative concave function on s. Hence, if $0 < s < 1$ then

$$f(\tfrac{1}{2}, x) = f(\tfrac{1}{2}s + \tfrac{1}{2}(1-s), x) \geqq \tfrac{1}{2} f(s,x)$$

or

$$f(s,x) \leqq 2 f(\tfrac{1}{2}, x)\,. \tag{27}$$

Before estimating $J_{s,\infty}$ we remark that

$$U_k \leqq cV_k \quad \text{on} \quad \{T_c > k\} \quad \text{and} \quad V_k \leqq c^{-1}U_k \leqq U_k \quad \text{on} \quad \{T_c = k\}\,.$$

Consequently, by (26) and (27)

$$\begin{aligned}
&\mathsf{E}_{\mathsf{R}}(sU_k + (1-s)\,V_k - U_k^s V_k^{1-s} \mid \mathfrak{F}_{k-1}) \leqq \\
&\leqq \mathsf{E}_{\mathsf{R}}(I(T_c > k)\,(sU_k + (1-s)\,V_k - U_k^s V_k^{1-s}) \mid \mathfrak{F}_{k-1}) + \\
&+ \mathsf{E}_{\mathsf{R}}(I(T_c = k)\,(sU_k + (1-s)\,V_k - U_k^s V_k^{1-s}) \mid \mathfrak{F}_{k-1}) + \\
&+ I(T_c \leqq k-1)\, \mathsf{E}_{\mathsf{R}}(sU_k + (1-s)\,V_k - U_k^s V_k^{1-s} \mid \mathfrak{F}_{k-1}) \leqq \\
&\leqq e(c)\,(1-s)\, \mathsf{E}_{\mathsf{R}}(\tfrac{1}{2}U_k + \tfrac{1}{2}V_k - U_k^{1/2}V_k^{1/2} \mid \mathfrak{F}_{k-1}) + \mathsf{E}_{\mathsf{R}}(I(T_c = k)\, U_k \mid \mathfrak{F}_{k-1}) \\
&+ I(T_c < \infty)\, 2\, \mathsf{E}_{\mathsf{R}}(\tfrac{1}{2}U_k + \tfrac{1}{2}V_k - U_k^{1/2}V_k^{1/2} \mid \mathfrak{F}_{k-1})\,.
\end{aligned}$$

It holds R-a.s.

$$1 - H_{s,k} = \mathsf{E}_{\mathsf{R}}(sU_k + (1-s)\,V_k - U_k^s V_k^{1-s} \mid \mathfrak{F}_{k-1})$$

on $\{T \geqq k\}$. Furthermore, by (18)

$$W(c) = \sum_{k=1}^{\infty} \mathsf{E}_{\mathsf{R}}(I(T_c = k)\, U_k \mid \mathfrak{F}_{k-1}) \quad \text{P-a.s.}$$

Inserting these relations into the above inequality we get P-a.s.

$$J_{s,T} \leqq e(c)\,(1-s)\, J_\infty + W(c) + 2I(T_c < \infty)\, J_\infty\,. \tag{28}$$

Obviously,

$$\mathsf{P}(\bigcap_{k=0}^{\infty} \{U_k > 0\}) = 1\,.$$

This means $\mathsf{P}(T_c = \infty, T < \infty) = 0$, and in view of (28)

$$J_{s,\infty} I(S_c \wedge T_c = \infty) \leqq d(c)(1-s) I(S_c \wedge T_c = \infty) + W(c)$$

where $d(c) = ce(c)$. In order to complete the proof we note that $J_{s,\infty} \leqq 2J_\infty$ by (27) and apply this inequality on the set $\{S_c \wedge T_c < \infty\}$. □

Lemma 2. There exists a function $c(s)$, $\frac{1}{2} < s < 1$, such that $\lim_{s \uparrow 1} c(s) = \infty$ and

$$G_{s,\infty} \leqq \exp\{-(\tfrac{1}{4}W(c(s)) + (1-s) J_\infty\} \quad \text{P-a.s.} \quad \text{for} \quad \tfrac{1}{2} < s < 1\,. \tag{29}$$

Proof. There exists a function $c(s)$, $\frac{1}{2} < s < 1$, which tends to infinity as s tends to 1 such that

$$s + (1-s)x - x^{1-s} \geqq \tfrac{1}{2} \quad \text{on} \quad 0 < x < 1/c(s)\,.$$

Hence

$$su + (1-s)v - u^s v^{1-s} \geqq \tfrac{1}{2}u \quad \text{for} \quad 0 < c(s)\, v < u\,.$$

Because of $\mathsf{E}_\mathsf{R}(U_k \mid \mathfrak{F}_{k-1}) \leqq 1$ and $\mathsf{E}_\mathsf{R}(V_k \mid \mathfrak{F}_{k-1}) \leqq 1$ we get

$$1 - H_{s,k} \geqq \mathsf{E}_\mathsf{R}(sU_k + (1-s)V_k - U_k^s V_k^{1-s} \mid \mathfrak{F}_{k-1}) \geqq$$
$$\geqq \mathsf{E}_\mathsf{R}(I(T_{c(s)} = k)\, \tfrac{1}{2} U_k \mid \mathfrak{F}_{k-1})\,.$$

This leads to $J_{s,\infty} \geqq \frac{1}{2}W(c(s))$ P-a.s. and by Proposition 1

$$G_{s,\infty} \leqq \exp\{-\tfrac{1}{2}W(c(s))\} \quad \text{P-a.s.}$$

Furthermore by the same Proposition for $\frac{1}{2} < s < 1$

$$G_{s,\infty} \leqq G_{1/2,\infty}^{2(1-s)} \leqq \exp\{-2(1-s)J_\infty\}$$

and

$$G_{s,\infty} \leqq (\exp\{-\tfrac{1}{2}W(c(s))\}) \wedge (\exp\{-2(1-s)J_\infty\}) \leqq$$
$$\leqq \exp\{-(\tfrac{1}{4}W(c(s)) + (1-s)J_\infty)\} \quad \text{P-a.s.}$$ □

Proof of Theorem 2. The equivalence of (8) and (9) has been established in [5]. Put $s_1 = s^2$. Then by (5)

$$H_s(\mathsf{P}_n, \mathsf{Q}_n) \leqq [\mathsf{E}_{\mathsf{P}_n} G_{s,\infty,n}]^{s/1+s}\,.$$

This proves (9) → (10). The implication (10) → (9) is an immediate consequence of (7).

Let us now prove (10) → (11). To do this we use Lemma 2 and obtain

$$\mathsf{E}_{\mathsf{P}_n} G_{s,\infty,n} \leqq \mathsf{E}_{\mathsf{P}_n} \exp\{-(\tfrac{1}{4}W_n(c(s)) + (1-s)J_{\infty,n})\}\,.$$

Taking $n \to \infty$ and then $s \uparrow 1$ we get

$$\overline{\lim_{s \uparrow 1}}\ \overline{\lim_{n \to \infty}}\ \mathsf{P}_n(\tfrac{1}{4}W_n(c(s)) + (1-s)J_{\infty,n} > d) = 0$$

for every $d > 0$. Consequently,

$$\overline{\lim_{s \uparrow 1}}\ \overline{\lim_{n \to \infty}}\ \mathsf{P}_n(J_{\infty,n} > d/(1-s)) = 0$$

and because of $0 \leqq W_n \leqq 1$ P_n-a.s.

$$\overline{\lim_{s\uparrow 1}}\ \overline{\lim_{n\to\infty}}\ \mathsf{E}_{\mathsf{P}_n} W_n(c(s)) = \overline{\lim_{s\uparrow 1}}\ \overline{\lim_{n\to\infty}}\ \mathsf{P}_n\big(\sup_{1\leqq k<\infty} L_{k,n} > c(s)\big) = 0\,.$$

Now (11) is assumed to be fulfilled. Inserting the inequality of Lemma 1 into the first inequality of Proposition 1 we get

(30) $(1 - G_{s,\infty,n}) \leqq I(T_{c,n} \wedge S_{c,n} \leqq \infty) + d(c)(1 - s)\, I(T_{c,n} \wedge S_{c,n} = \infty) + W_n(c)\,.$

As $\mathsf{E}_{\mathsf{P}_n} W_n(c) = \mathsf{P}_n(T_{c,n} < \infty) \leqq \mathsf{P}_n(S_{c,n} \wedge T_{c,n} < \infty)$ we obtain

$$\mathsf{E}_{\mathsf{P}_n}(1 - G_{s,\infty,n}) \leqq 2\mathsf{P}_n(T_{c,n} \wedge S_{c,n} < \infty) + d(c)(1 - s)\,.$$

Taking at first $n \to \infty$ then $s \uparrow 1$ and finally $c \to \infty$ we see that (10) holds. □

Proof of Theorem 3. The equivalence of (12) and (13) was shown in [5]. As $0 \leqq G_{s,\infty,n} \leqq 1$ R_n-a.s. and $\mathscr{H}_{s,\mathsf{P}_n,\mathsf{Q}_n} \ll \mathsf{R}_n$ we get

$$\int G_{s,\infty,n}\, \mathrm{d}\mathscr{H}_{s,\mathsf{P}_n,\mathsf{Q}_n} \leqq \mathscr{H}_{s,\mathsf{P}_n,\mathsf{Q}_n}(\Omega_n) = H_s(\mathsf{P}_n, \mathsf{Q}_n)$$

which proves (13) → (14). Alternatively, by (6)

$$H_s(\mathsf{P}_n, \mathsf{Q}_n) \leqq \gamma + \mathscr{H}_{s,\mathsf{P}_n,\mathsf{Q}_n}(G_{s,\infty,n} > \gamma) \leqq \gamma + \frac{1}{\gamma}\int G_{s,\infty,n}\, \mathrm{d}\mathscr{H}_{s,\mathsf{P}_n,\mathsf{Q}_n}$$

for every $0 < \gamma < 1$. Taking $n \to \infty$ and then $\gamma \to 0$ we see that (14) → (13).

By definition of $\mathscr{H}_{s,\mathsf{P}_n,\mathsf{Q}_n}$ and $\mathsf{P}_n \wedge \mathsf{Q}_n$

$$\int G_{s,\infty,n}\, \mathrm{d}(\mathsf{P}_n \wedge \mathsf{Q}_n) \leqq \int G_{s,\infty,n}\, \mathrm{d}\mathscr{H}_{s,\mathsf{P}_n,\mathsf{Q}_n}$$

which proves (14) → (15). For proving (15) → (14) we remark that

$$\int G_{s,\infty,n}\, \mathrm{d}\mathscr{H}_{s,\mathsf{P}_n,\mathsf{Q}_n} = \int G_{s,\infty,n} X_n^s Y_n^{1-s}\, \mathrm{d}\mathsf{R}_n =$$

$$= \int G_{s,\infty,n} I(X_n < Y_n)\, X_n^s Y_n^{1-s}\, \mathrm{d}\mathsf{R}_n + \int G_{s,\infty,n} I(X_n \geqq Y_n)\, X_n^s Y_n^{1-s}\, \mathrm{d}\mathsf{R}_n \leqq$$

$$\leqq \left[\int G_{s,\infty,n}\, \mathrm{d}(\mathsf{P}_n \wedge \mathsf{Q}_n)\right]^s + \left[\int G_{s,\infty,n}\, \mathrm{d}(\mathsf{P}_n \wedge \mathsf{Q}_n)\right]^{1-s}.$$

Corollary 1 results from condition (15) and the following inequality which holds for every non-negative measurable ξ:

$$\int \xi\, \mathrm{d}(\mathsf{P}_n \wedge \mathsf{Q}_n) \leqq (\mathsf{E}_{\mathsf{P}_n}\xi) \wedge (\mathsf{E}_{\mathsf{Q}_n}\xi)\,.$$

□

Proof of Corollary 2. The first part follows directly from Corollary 1. In order to prove the second statement it is enough to show (see the first inequality in Proposition 1)

$$\underline{\lim}_{c\to\infty}\ \underline{\lim}_{n\to\infty}\ \mathsf{P}_n(J_{\infty,n} < c) = 0\,.$$

For this sake we apply Lemma 1 and obtain from the first inequality in Proposition 1

$$(31)\quad \mathsf{P}_n(G_{s,\infty,n} < \gamma) \leqq \mathsf{P}_n(G_{s,\infty,n} < \gamma,\, S_{c,n} \wedge T_{c,n} = \infty) + \mathsf{P}_n(S_{c,n} \wedge T_{c,n} < \infty)$$
$$\leqq \mathsf{P}_n(W_n(c) \geqq 1 - \gamma - d(c)(1 - s)) + \mathsf{P}_n(S_{c,n} \wedge T_{c,n} < \infty)\,.$$

An easy consideration shows

$$\overline{\lim_{s\uparrow 1}}\ \overline{\lim_{n\to\infty}}\ \mathsf{P}_n(W_n(c) \geqq 1 - \gamma - d(c)(1 - s)) \leqq \overline{\lim_{n\to\infty}}\ \mathsf{P}_n(W_n(c) > 1 - \gamma^{1/2})\,.$$

Furthermore, in view of the additional assumption

$$\overline{\lim_{c\to\infty}}\ \overline{\lim_{n\to\infty}}\ \mathsf{P}_n(W_n(c) > \varepsilon) \leqq \varepsilon^{-1}\ \overline{\lim_{c\to\infty}}\ \overline{\lim_{n\to\infty}}\ \mathsf{E}_{\mathsf{P}_n} W_n(c) \leqq \varepsilon^{-1}\ \overline{\lim_{c\to\infty}}\ \overline{\lim_{n\to\infty}}\ \mathsf{P}_n(T_{c,n} < \infty) = 0$$

for every $\varepsilon > 0$. Inserting the last two relations into inequality (31) we get in accordance with the additional assumption

$$\overline{\lim_{\gamma\uparrow 1}}\ \overline{\lim_{s\uparrow 1}}\ \overline{\lim_{n\to\infty}}\ \mathsf{P}_n(G_{s,\infty,n} < \gamma) \leqq \overline{\lim_{c\to\infty}}\ \overline{\lim_{n\to\infty}}\ \mathsf{P}_n(S_{c,n} \wedge T_{c,n} < \infty) \leqq$$
$$\leqq \overline{\lim_{c\to\infty}}\ \overline{\lim_{n\to\infty}}\ \mathsf{P}_n(S_{c,n} < \infty)\,.$$

Applying this inequality to (7) we arrive to

$$\underline{\lim}_{s\uparrow 1}\ \underline{\lim}_{n\to\infty}\ H_s(\mathsf{P}_n, \mathsf{Q}_n) \geqq 1 - \overline{\lim_{c\to\infty}}\ \overline{\lim_{n\to\infty}}\ \mathsf{P}_n(S_{c,n} < \infty) = \underline{\lim}_{c\to\infty}\ \underline{\lim}_{n\to\infty}\ \mathsf{P}_n(J_{\infty,n} < c)\,.$$

In order to complete the proof it is sufficient to note that (13) implies (see the remark after Theorem 3)

$$\underline{\lim}_{s\uparrow 1}\ \underline{\lim}_{n\to\infty}\ H_s(\mathsf{P}_n, \mathsf{Q}_n) = 0\,. \qquad \square$$

Proof of Theorem 4. Put $s_2 = \psi(s)$, $S = \infty$ in inequality (4). Then by the assumption

$$\underline{\lim}_{s\uparrow 1}\ \underline{\lim}_{n\to\infty}\ H_s(\mathsf{P}_n, \mathsf{Q}_n) = 1$$

which is known to be necessary and sufficient for $\mathsf{P}_n \lhd \mathsf{Q}_n$ according to Theorem 3.

Proof of Theorem 5. As the restrictions of P_n and Q_n to $\mathfrak{F}_{k,n}$ are equivalent the functions $U_{k,n}$, $V_{k,n}$, $k = 1, 2, \ldots$, are positive with respect to P_n as well as to Q_n. Hence by (18) we have P_n-a.s. and Q_n-a.s.

$$H_{s,k,n} = \mathsf{E}_{\mathsf{R}_n}(U_{k,n}^s V_{k,n}^{1-s} \mid \mathfrak{F}_{k-1,n})$$
$$= \mathsf{E}_{\mathsf{P}_n}\left(\left(\frac{V_{k,n}}{U_{k,n}}\right)^{1-s} \middle|\ \mathfrak{F}_{k-1,n}\right) = \mathsf{E}_{\mathsf{P}_n}\left(\exp\left\{(1 - s)\ln\frac{V_{k,n}}{U_{k,n}}\right\} \middle|\ \mathfrak{F}_{k-1,n}\right)$$

and because of Jensen's inequality we may continue

$$\geqq \exp\left\{(1 - s)\,\mathsf{E}_{\mathsf{P}_n}\left(\ln\frac{V_{k,n}}{U_{k,n}} \middle|\ \mathfrak{F}_{k-1,n}\right)\right\} = \exp\left\{-(1 - s)\,\mathsf{E}_{\mathsf{R}_n}(\varphi(U_{k,n}, V_{k,n}) \mid \mathfrak{F}_{k-1,n})\right\}.$$

Hence

$$(32)\qquad G_{s,\infty,n} \geqq \exp\{-(1 - s)\, I_{\infty,n}\}\,.$$

Choose $a > 1$ such that $(1 + a)/a < 1 + \varepsilon$ and put $\psi(s) = s - a(1 - s)$ for $a/(1 + a) < s < 1$. Then $0 < \psi(s) < s$ and by (32) we have

$$[\mathsf{E}_{Q_n} G_{s,\infty,n}^{-(1-\psi(s))s/(1-s)(s-\psi(s))}]^{-(s-\psi(s))/\psi(s)} \geqq$$

$$\geqq \left[\mathsf{E}_{Q_n} \exp\left\{\frac{(1+a)s}{a} I_{\infty,n}\right\}\right]^{-a(1-s)/(s-a(1-s))} \geqq [\mathsf{E}_{Q_n} \exp\{(1 + \varepsilon) I_{\infty,n}\}]^{-a(1-s)/(s-a(1-s))}$$

The statement of Theorem 5 is a consequence of this inequality and Theorem 4. □

Proof of Proposition 4. Denote by $N(a, \sigma^2)$ the normal distribution with expectation a and variance σ^2. An easy calculation shows

$$H_s(N(a_1, \sigma_1^2), N(a_2, \sigma_2^2)) = \left[\frac{b_1^s b_2^{1-s}}{sb_1 + (1 - s) b_2}\right]^{1/2} \exp\left\{-\frac{s(1 - s)(a_1 - a_2)^2}{2\left(s\frac{1}{b_2} + (1 - s)\frac{1}{b_1}\right)}\right\}$$

where $b_i = \sigma_i^{-2}$. If $0 \leqq |\varrho| < 1$ then

$$H_s(N(\varrho x, 1 - \varrho^2), N(0, 1)) = \left[\frac{(1 - \varrho^2)^{1-s}}{1 - (1 - s)\varrho^2}\right]^{1/2} \exp\left\{-\frac{s(1 - s)\varrho^2 x^2}{2(1 - (1 - s)\varrho^2)}\right\}.$$

Put $s = \frac{1}{2}$. Then

$$\delta_{1/2}(x) := H_{1/2}(N(\varrho x, 1 - \varrho^2), N(0, 1)) =$$

$$= \left[\frac{(1 - \varrho^2)^{1/2}}{1 - \frac{1}{2}\varrho^2}\right]^{1/2} \exp\left\{-\frac{1}{8}\frac{\varrho^2 x^2}{1 - \frac{1}{2}\varrho^2}\right\} \leqq$$

$$\leqq \left[\frac{(1 - \varrho^2)^{1/2}}{1 - \frac{1}{2}\varrho^2}\right]^{1/2} \exp\{-\tfrac{1}{8}\varrho^2 x^2\}.$$

Consequently,

$$\Delta(\tfrac{1}{2}) = \int \delta_{1/2}(x) N(0, 1)(\mathrm{d}x) \leqq \left[\frac{1 - \varrho^2)^{1/2}}{1 - \frac{1}{2}\varrho^2}\right]^{1/2} [1 + \tfrac{1}{4}\varrho^2]^{-1/2}$$

and

$$\prod_{l=1}^{m_n} \Delta_{l,n}(\tfrac{1}{2}) \leqq \prod_{l=1}^{m_n} \left[\frac{(1 - \varrho_{l,n}^2)^{1/2}}{1 - \frac{1}{2}\varrho_{l,n}^2}\right]^{1/2} [1 + \tfrac{1}{4}\varrho_{l,n}^2]^{-1/2}.$$

Suppose $\overline{\lim}_{n\to\infty}(\sup_{1\leqq l\leqq m_n} |\varrho_{l,n}|) = 1$. Then

$$\underline{\lim}_{n\to\infty} \prod_{l=1}^{m_n} \Delta_{l,n} \leqq \underline{\lim}_{n\to\infty} (4 \inf_{1\leqq l\leqq m_n} (1 - \varrho_{l,n}^2))^{1/4} = 0.$$

Suppose now

$$\overline{\lim}_{n\to\infty} \sum_{l=1}^{m_n} \varrho_{l,n}^2 = \infty.$$

Then because of $-\frac{1}{2}\ln(1 + \frac{1}{4}x) \leqq -\frac{2}{5}x$, $0 \leqq x \leqq 1$, we get

$$\lim_{n\to\infty} \sum_{l=1}^{m_n} \Delta_{l,n}(\tfrac{1}{2}) \leqq \lim_{n\to\infty} \exp\{-\tfrac{2}{5}\sum_{l=1}^{m_n} \varrho_{l,n}^2\} = 0.$$

In order to complete the proof of Proposition 4 we show that the first and second condition imply $\mathsf{P}_n \lhd \mathsf{Q}_n$.

Denote by p the Radon-Nikodym derivative of $N(\alpha, \sigma^2)$ with respect to $N(0, 1)$. Then

$$\int p \ln p \,\mathrm{d}N(0, 1) = \tfrac{1}{2}(-\ln \sigma^2 + \sigma^2 - 1) + \tfrac{1}{2}\alpha^2 .$$

Hence

$$\gamma_{l+1,n}(x_l) = \tfrac{1}{2}(-\ln(1 - \varrho_{l,n}^2) - \varrho_{l,n}^2) + \tfrac{1}{2}\varrho_{l,n}^2 x_l^2$$

and

$$\chi_{l+1,n}(a) = [\exp\{\tfrac{1}{2}(-\ln(1 - \varrho_{l,n}^2) - \varrho_{l,n}^2)\}]^a . \tag{33}$$

$$. \int \exp\{a\, \tfrac{1}{2}\varrho_{l,n}^2 x_l^2\}\, N(0), 1)\,(\mathrm{d}x_l) =$$

$$= (1 - a\varrho_{l,n}^2)^{-1/2} \exp\{\tfrac{1}{2}a(-\ln(1 - \varrho_{l,n}^2) - \varrho_{l,n}^2)\}$$

for $a\, \varrho_{l,n}^2 < 1$.

Given a real number $0 < \delta < 1$ there exists a constant $C(\delta)$ depending only on δ such that

$$-\tfrac{1}{2}\ln\left(1 - \frac{1}{1-\delta}x^2\right) \leqq C(\delta)\, x^2 \tag{34}$$

$$-\frac{1}{1-\delta}\,\tfrac{1}{2}(\ln(1 - x^2) + x^2) \leqq C(\delta)\, x^2$$

for $0 \leqq |x| \leqq 1 - \delta$.

We suppose that both the first and second condition hold. Then there exists numbers n_0, $0 < \delta < 1, 0 < d < \infty$, so that

$$\varrho_{l,n}^2 \leqq (1 - \delta)^2 , \quad \sum_{l=1}^{m_n} \varrho_{l,n}^2 \leqq d \tag{35}$$

for every $n \geqq n_0$. Put $\varepsilon = (1 - \delta)^{-1} - 1$, $a = 1 + \varepsilon$. Then $a > 1$, $\varepsilon > 0$ and by (33), (34), (35)

$$\prod_{l=1}^{m_n} \chi_{l,n}(1 + \varepsilon) \leqq \exp\{C(\delta) . d\} \quad \text{for} \quad n \geqq n_0 .$$

Hence

$$\overline{\lim_{n\to\infty}} \prod_{l=1}^{m_n} \chi_{l,n}(1 + \varepsilon) < \infty$$

and we obtain $\mathsf{P}_n \lhd \mathsf{Q}_n$ in view of Proposition 3.

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