KYBERNETIKA — VOLUME 22 (1986), NUMBER 5

CHARACTERIZATION OF A DENSITY BY MINIMIZING THE LOGARITHMIC INFORMATION OF DEGREE q

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We define the logarithmic information of degree q of a random variable X, the probability density f of which is given. In a class of probability densities having some particular properties, we look for that one minimizing this information. We first obtain an extension of an inequality of Nagy [3], which gives the expected result in a particular case.

1. INTRODUCTION

We consider the probability space $(\mathbb{R}, \mathscr{B}, P)$ where $\mathscr{B}$ is the Borel σ-algebra of the subsets of $\mathbb{R}$ and P is the probability law of a random variable X which we suppose absolutely continuous with regard to Lebesgue measure. We denote by f the probability density of P with respect to this measure.

In [2] Bouchon and Pessoa have defined the logarithmic information of degree q of a random variable and obtained non-additivity relations linking this information with the classic Fisher information and the mean Fisher's information of degree q. We shall give later the definition of the logarithmic information of degree q and, in a class of densities we will look for that one minimizing this information.

2. FUNDAMENTAL INEQUALITY

Let $h(x)$ be a function defined and continuous on $\mathbb{R}$. We denote by $E(h)$ the set of points where the function admits a maximum or a minimum. In the case where the function h is differentiable in every point of $\mathbb{R} - E(h)$, $E(h)$ being supposed finite, we prove an inequality concerning the function h and the set $E(h)$. This inequality is actually an extension of an inequality of Nagy (cf. [1] page 167). We first prove the following lemma:

Lemma 1.1. We suppose that the function h is strictly positive, that the set $E(h)$

has a finite number k of elements and that $\int_{-\infty}^{+\infty} h(x)\,\mathrm{d}x < \infty$. Then k is necessarily an odd number.

Proof. We denote by $x_1 < \ldots < x_k$ the points of $E(h)$. Then x_1 and x_k are necessarily points where the function admits a maximum, since otherwise the integral would not be finite. The function being continuous by hypothesis, the points x_i, $i = 1, \ldots, k$ are alternatively the points of maximum and minimum respectively. We can conclude that k is necessarily an odd number. □

Theorem 1.2. We suppose that $h(x) > 0$ for every $x \in \mathbb{R}$ and that $E(h) = \{x_1, \ldots, x_k\} \subset \mathbb{R}$, where k is a positive integer. If the function h is differentiable on $\mathbb{R} - E(h)$ and if the integrals

$$\int_{-\infty}^{+\infty} h(x)^q\,\mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{+\infty} |h'(x)|^p\,\mathrm{d}x$$

exist for $q > 0$ and $p \geqq 1$, then

$$(1) \qquad \left(\int_{-\infty}^{+\infty} h(x)^q\,\mathrm{d}x\right)^{(p-1)/p} \left(\int_{-\infty}^{+\infty} |h'(x)|^p\,\mathrm{d}x\right)^{1/p} \geqq \frac{2}{r} \sum_{i=1}^{k} (-1)^{i+1} h(x_i)^r$$

where $r = (p-1)\,q/p + 1$, the equality holding if and only if h^q and $|h'|^p$ are positively proportional almost everywhere.

Proof. From Holder's inequality we have

$$(2) \qquad \int_{-\infty}^{+\infty} h(x)^{(p-1)q/p}\,|h'(x)|\,\mathrm{d}x \leqq \left(\int_{-\infty}^{+\infty} h(x)^q\,\mathrm{d}x\right)^{(p-1)/p} \left(\int_{-\infty}^{+\infty} |h'(x)|^p\,\mathrm{d}x\right)^{1/p}$$

We denote

$$(3) \qquad J(h, p, q) = \int_{-\infty}^{+\infty} h(x)^{(p-1)q/p}\,|h'(x)|\,\mathrm{d}x$$

and suppose that $x_1 < \ldots < x_k$. Then, taking into account the sign of derivative h' in each of the considered intervals, we have

$$J(h, p, q) = \int_{-\infty}^{x_1} h(x)^{(p-1)q/p}\,h'(x)\,\mathrm{d}x + \sum_{i=1}^{k-1}(-1)^i \int_{x_i}^{x_{i+1}} h(x)^{(p-1)q/p}\,h'(x)\,\mathrm{d}x +$$

$$+ \int_{x_k}^{+\infty} h(x)^{(p-1)q/p}\,h'(x)\,\mathrm{d}x = \frac{p}{(p-1)\,q + p}\,\{[h(x)^{1+(p-1)q/p}]_{-\infty}^{x_1} +$$

$$+ \sum_{i=1}^{k-1}(-1)^i\,[h(x)^{1+(p-1)q/p}]_{x_i}^{x_{i+1}} + [h(x)^{1+(p-1)q/p}]_{x_k}^{+\infty}\} =$$

$$= \frac{p}{(p-1)\,q + p}\,\{h(x_1)^{1+(p-1)q/p} + \sum_{i=1}^{k}(-1)^i\,[h(x_{i+1})^{1+(p-1)q/p} - h(x_i)^{1+(p-1)q/p}]\} =$$

$$= \frac{2p}{(p-1)\,q + p} \sum_{i=1}^{k}(-1)^{i+1}\,h(x_i)^{1+(p-1)q/p}$$

Then

$$J(h, p, q) = \frac{2}{r} \sum_{i=1}^{k} (-1)^{i+1} h(x_i)^r \tag{4}$$

where

$$r = \frac{(p-1)\,q}{p} + 1$$

If we replace (4) in (2) and (3) we obtain the expected result. The case where the equality holds is an immediate consequence of Holder's inequality. □

In the sequel we study the particular case where $p = q$, $q \geqq 1$.
In this case we have

$$\left(\int_{-\infty}^{+\infty} h(x)^q \, dx\right)^{(q-1)/q} \left(\int_{-\infty}^{+\infty} |h'(x)|^q \, dx\right)^{1/q} \geqq \frac{2}{q} \sum_{i=1}^{k} (-1)^{i+1} h(x_i)^q \tag{5}$$

the equality holding in (5) if and only if

$$|h'(x)| = c\, h(x)$$

almost everywhere on $\mathbb{R}$, where c is a positive constant.

Then, we have a.e.

$$\begin{aligned} h'(x) &= c\, h(x) \quad \text{if } x \in \left]-\infty, x_1\right[\cup \Big(\bigcup_{n=1}^{(k-1)/2} [x_{2n}, x_{2n+1}[\Big) \\ h'(x) &= -c\, h(x) \quad \text{if } x \in \Big(\bigcup_{n=1}^{(k-1)/2} [x_{2n-1}, x_{2n}[\Big) \cup [x_k, +\infty[\end{aligned} \tag{6}$$

By solving these equations and denoting by 1_A the characteristic function of a set $A \subset \mathbb{R}$, we obtain

$$\begin{aligned} h(x) = {} & \Big(\alpha\, 1_{]-\infty, x_1[}(x) + \sum_{n=1}^{(k-1)/2} \beta_n\, 1_{[x_{2n}, x_{2n+1}[}(x)\Big)\, e^{cx} + \\ & + \Big(\sum_{n=1}^{(k-1)/2} \gamma_n\, 1_{[x_{2n-1}, x_{2n}[}(x) + \delta\, 1_{[x_k, +\infty[}(x)\Big)\, e^{-cx} \end{aligned} \tag{7}$$

where α, β_n, γ_n and δ are arbitrary constants linked together by the following relation, which is a consequence of the continuity of h:

$$\begin{aligned} \gamma_1 &= \alpha\, e^{2cx_1} = \beta_1\, e^{2cx_2} \\ \gamma_n &= \beta_n\, e^{2cx_{2n}} = \beta_{n-1}\, e^{2cx_{2n-1}} \qquad 2 \leqq n \leqq (k-1)/2 \\ \delta &= \beta_{(k-1)/2}\, e^{2cx_k} \end{aligned} \tag{8}$$

Therefore

$$\begin{aligned} \alpha &= \beta_1\, e^{2c(x_2 - x_1)} \\ \beta_{n-1} &= \beta_n\, e^{2c(x_{2n} - x_{2n-1})} \qquad 2 \leqq n \leqq (k-1)/2 \\ \beta_{(k-1)/2} &= \delta\, e^{-2cx_k} \end{aligned} \tag{9}$$

From this recurrent relation we get

$$\alpha = \delta \exp\left(2c \sum_{i=1}^{k} (-1)^i x_i\right) \tag{10}$$

$$\beta_n = \delta \exp\left(2c \sum_{i=2n+1}^{k} (-1)^i x_i\right), \quad n = 1, \ldots, (k-1)/2$$

if we substitute (10) in relation (8) we have

$$\gamma_n = \delta \exp\left(2c \sum_{i=2n}^{k} (-1)^i x_i\right), \quad n = 1, \ldots, (k-1)/2 \tag{11}$$

and by replacing (10) and (11) in (7) we obtain

$$h(x) = \delta\left(\exp\left(2c \sum_{i=1}^{k} (-1)^i x_i\right) 1_{]-\infty, x_1[}(x) + \sum_{n=1}^{(k-1)/2} \exp\left(2c \sum_{i=2n+1}^{k} (-1)^i x_i\right) . \right. \tag{12}$$

$$\left. . \, 1_{[x_{2n}, x_{2n+1}[}(x)\right) e^{cx} + \delta\left(\sum_{n=1}^{(k-1)/2} \exp\left(2c \sum_{i=2n}^{k} (-1)^i x_i\right) 1_{[x_{2n-1}, x_{2n}[}(x) + 1_{[x_k, +\infty[}(x)\right) e^{-cx}$$

Thus, in the case where $p = q$ we get the functions h satisfying the equality in (1).

3. MINIMIZATION OF THE LOGARITHMIC INFORMATION OF DEGREE q

In this part we define the logarithmic information of degree q of a random variable, the probability law of which is absolutely continuous with respect to Lebesgue's measure.

By using the result obtained in the previous part we get a density minimizing this information in a class of densities having a given mode.

Definition 1.3. Let q be a real number, $q > 1$. The logaritmic information of degree q of a random variable X with a probability density $f(x)$ is defined by

$$H^q(f) = \int_{-\infty}^{+\infty} \left|\frac{d \log f(x)}{dx}\right|^q f(x) \, dx \tag{13}$$

We suppose that the set $E(f)$ of points where the density admits a maximum or a minimum is finite, and we note

$$E(f) = \{a_1, \ldots, a_k\}, \quad k \in \mathbb{N}^*$$

We further suppose that f is positive and continuous for every $x \in \mathbb{R}$ and that it is differentiable on $\mathbb{R} - E(f)$.

We note, for every $x \in \mathbb{R}$,

$$h(x) = q\, f(x)^{1/q} \tag{14}$$

From inequality (5) we obtain

$$\int_{-\infty}^{+\infty} \left|\frac{d \log f(x)}{dx}\right|^q f(x) \, dx \geqq 2^q\left(\sum_{i=1}^{k} (-1)^{i+1} f(x_i)^q\right) \tag{15}$$

the equality holding, from (12), if and only if

$$f(x) = \left(\frac{\delta}{q}\right)^q \{[\exp(2qc\sum_{i=1}^{k}(-1)^i a_i)\, 1_{]-\infty,a_1[}(x) + \tag{16}$$

$$+ \sum_{n=1}^{(k-1)/2} \exp(2qc \sum_{i=2n+1}^{k} (-1)^i a_i)\, 1_{[a_{2n},a_{2n+1}[}(x)]\, e^{qcx} +$$

$$+ [\sum_{n=1}^{(k-1)/2} \exp(2qc \sum_{i=2n}^{k} (-1)^i a_i)\, 1_{[a_{2n-1},a_{2n}[}(x) + 1_{[a_k,+\infty[}(x)]\, e^{-qcx}\}$$

where δ and c are arbitrary positive constants.

Since $\int_{-\infty}^{+\infty} f(x)\,dx = 1$, we get

$$\left(\frac{\delta}{q}\right)^q = \frac{qc}{2\sum_{i=1}^{k}(-1)^{i+1}\exp((-1)^{i+1} qca_i)\,.\,\exp(2qc\sum_{j=i}^{k}(-1)^j a_j)}$$

Writing $qc = \lambda$ the equality holds in (15) if and only if

$$f(x) = \frac{\lambda}{A} \{[\exp(2\lambda\sum_{i=1}^{k}(-1)^i a_i)\, 1_{]-\infty,a_1[}(x) + \tag{17}$$

$$+ \sum_{n=1}^{(k-1)/2} \exp(2\lambda \sum_{i=2n+1}^{k} (-1)^i a_i)\, 1_{[a_{2n},a_{2n+1}[}(x)]\, e^{\lambda x} +$$

$$+ [\sum_{n=1}^{(k-1)/2} \exp(2\lambda \sum_{i=2n}^{k} (-1)^i a_i)\, 1_{[a_{2n-1},a_{2n}[}(x) + 1_{[a_k,+\infty[}(x)]\, e^{-\lambda x}\}$$

where λ is a positive constant and

$$A = \frac{1}{2\sum_{i=1}^{k}(-1)^{i+1}\exp((-1)^{i+1}\lambda a_i + 2\lambda\sum_{j=i}^{k}(-1)^j a_j)} \tag{18}$$

Let k be an odd number, $a_1 < \ldots < a_k$ real numbers and $\alpha_1, \ldots, \alpha_k$ positive real numbers such that, for every $i = 1, \ldots, (k-1)/2$, $\alpha_{2i} < \min(\alpha_{2i-1}, \alpha_{2i+1})$. We denote by $\mathscr{F}\{a_1, \ldots, a_k, \alpha_1, \ldots, \alpha_k\}$ the set of probability densities such that, if f belongs to this set, then f is positive and continuous on $\mathbb{R}$ differentiable on $\mathbb{R} - E(f)$, with $E(f) = \{a_1, \ldots, a_k\}$ and $f(a_i) = \alpha_i$ for every $i = 1, \ldots, k$.

We have the following result:

Theorem 1.3. Let k be an odd number, $a_1 < \ldots < a_k$ real numbers and $\alpha_1, \ldots, \alpha_k$ positive real numbers such that, for every $i = 1, \ldots, (k-1)/2$, $\alpha_{2i} < \min(\alpha_{2i-1}, \alpha_{2i+1})$. In the set $\mathscr{F}\{a_1, \ldots, a_k, \alpha_1, \ldots, \alpha_k\}$, the probability density given by formulas (17) and (18) by taking $\lambda = 2\sum_{i=1}^{k}(-1)^{i+1}\alpha_i$, minimizes the logarithmic information of degree q.

Proof. From the previous results we have only to prove that $\lambda = 2\sum_{i=1}^{k}(-1)^{i+1}\alpha_i$.

The density f being continuous, it is easy to see that, for every $i = 1, \ldots, k$,

$$\alpha_i = \frac{\lambda}{A} \exp\left((-1)^{i+1} \lambda a_i + 2\lambda \sum_{j=i}^{k} (-1)^j a_j\right)$$

Therefore, from the value of A given by (18), we get

$$\lambda = \sum_{i=1}^{k} (-1)^{i+1} \alpha_i . \qquad \square$$

Particularly we have

Corollary 3.1. In the class of unimodal positive and continuous densities having a given mode m, differentiable in every point $x \neq m$, and the value of which equals α at the point m, the density

$$f(x) = \alpha \exp\left(-2\alpha|x - m|\right)$$

minimizes the logarithmic information of degree q.

(Received March 26, 1985.)

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