KYBERNETIKA — VOLUME 22 (1986), NUMBER 4

CONTROLLABILITY OF NONLINEAR SYSTEMS WITH DELAYS IN BOTH STATE AND CONTROL VARIABLES

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Relative controllability of nonlinear systems with distributed delays in state and control is considered. Using the Schauder fixed point theorem, sufficient conditions for relative controllability are obtained. These conditions extend some previous results by considering more general class of dynamical systems.

1. INTRODUCTION

Controllability of linear systems with lumped delays in state and control has been studied by Chyung [2] and Manitius and Olbrot [10]. Curakova [3] considered the controllability of linear systems with distributed delays in state and Klamka [7] for the systems with distributed delays in control. In [9], Manitius derived a determining equation and a sufficient condition for controllability of stationary systems with distributed delays in state and control. The results of [7] were extended to nonlinear systems by Klamka [8] and Balachandran [12]. An example of nonlinear system with distributed delays in control is given in [11]. Sufficient conditions were obtained by Dauer and Gahl [6] for nonlinear systems with distributed delays in state and lumped delay in control. In this paper we shall consider the controllability of nonlinear systems with distributed delays in both state and control variables. The approach we will use is to define the appropriate control and its corresponding solution by an integral equation. We then obtain the solution by applying the Schauder fixed point theorem.

2. BASIC NOTATIONS AND DEFINITION

Let $C_{n+m}[t_0, t_1]$ denote the Banach space of continuous $\mathbb{R}^n \times \mathbb{R}^m$ valued functions defined on the interval $J = [t_0, t_1]$ with the norm defined as follows; for $(z, v) \in$

$\in C_{n+m}[t_0, t_1]$,

$$\|[z, v]\| = \|z\| + \|v\| \quad \text{where}$$

$$\|z\| = \sup |z(t)| \quad \text{for} \quad t \in [t_0, t_1]$$

and

$$\|v\| = \sup |v(t)| \quad \text{for} \quad t \in [t_0, t_1].$$

Let $h > 0$ be a given real number. For functions $u: [t_0 - h, t_1] \to \mathbb{R}^m$ and $t \in [t_0, t_1]$, $(t_0 < t_1)$, we use the symbol u_t to denote the function on $[-h, 0)$, defined by $u_t(s) = u(t + s)$ for $s \in [-h, 0)$. In the sequel some integrals are in the Lebesgue-Stieltjes sense which is denoted by the symbol d_s.

We consider the controllability of nonlinear perturbations of linear delay system

$$\dot{x}(t) = L(x, u) \tag{1}$$

where the operator L is defined by

$$L(x, u) = A(t)\, x(t) + B(t)\, x(t - h) + \\ + \int_{-h}^{0} K(t, s)\, x(t + s)\, \mathrm{d}s + \int_{-h}^{0} \mathrm{d}_s\, H(t, s)\, u(t + s)\,.$$

We will show that, if the system (1) is relatively controllable, then the perturbed system

$$\dot{x}(t) = L(x, u) + f(t, x(t), x(t - h), u(t), u(t - h))\,, \quad t \in [t_0, t_1] \tag{2}$$
$$x(t) = \varphi(t)\,, \quad t \in [t_0 - h, t_0]$$

is relatively controllable provided the function f satisfies appropriate growth condition.

Here the vector function $x(t) \in \mathbb{R}^n$, $u(t)$ is an m-dimensional control vector and $u \in C_m[t_0 - h, t_1]$. The $n \times n$ matrix functions A, B are assumed to be continuous on J and the $n \times n$ matrix function $K(t, s)$ is continuous on $J \times [-h, 0]$ and $H(t, s)$ is an $n \times m$ matrix, continuous in t for fixed s and of bounded variation in s on $[-h, 0]$ for each $t \in J$. The n-dimensional vector function f is continuous in its arguments. The following definition of controllability [9,2] is assumed.

Definition. System (1) or (2) is said to be relatively controllable on $[t_0, t_1]$, if for every continuous function φ and initial control function u_{t_0} defined on $[t_0 - h, t_0]$ and every $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on $[t_0, t_1]$, such that the solution of system (1) or (2) satisfies $x(t_1) = x_1$.

3. PRELIMINARIES

Let φ be a continuous function on $[t_0 - h, t_0]$. Then there exist a unique solution of the system (1) on J satisfying $x(t) = \varphi(t)$ for $t \in [t_0 - h, t_0]$ and is given by

$$x_L(t) = X(t, t_0)\, \varphi(t_0) + \int_{t_0 - h}^{t_0} X(t, s + h)\, B(s + h)\, \varphi(s)\, \mathrm{d}s + \tag{3}$$

$$+ \int_{t_0-h}^{t_0} \int_{t_0}^{\omega+h} X(t, s)\, K(s, \omega - s)\, \varphi(\omega)\, ds\, d\omega +$$

$$+ \int_{t_0}^{t} X(t, s) \left[\int_{-h}^{0} d_\omega H(s, \omega)\, u(s + \omega) \right] ds$$

where $X(t, s)$ is an $n \times n$ matrix function satisfying

$$\partial X(t, s)/\partial t = A(t)\, X(t, s) + B(t)\, X(t - h, s) + \int_{-h}^{0} K(t, \omega)\, X(t + \omega, s)\, d\omega$$

for $t_0 \leqq s \leqq t \leqq t_1$, such that $X(t, t) = I$, the identity matrix and $X(t, s) = 0$ for $t < s$. Since $\partial X/\partial t$ exist, it is obvious that $X(t, s)$ is continuous in t for fixed s, where $s \leqq t$. It is easy to prove as in [6] that $X(t, s)$ is continuous in (t, s) in the compact region $t_0 \leqq s \leqq t \leqq t_1$.

The equation (3) can be written as

$$(4) \qquad x_L(t) = x_L(t, \varphi) + \int_{t_0}^{t} X(t, s) \left[\int_{-h}^{0} d_\omega H(s, \omega)\, u(s + \omega) \right] ds$$

where

$$x_L(t, \varphi) = X(t, t_0)\, \varphi(t_0) + \int_{t_0-h}^{t_0} X(t, s + h)\, B(s + h)\, \varphi(s)\, ds +$$

$$+ \int_{t_0-h}^{t_0} \int_{t_0}^{\omega+h} X(t, s)\, K(s, \omega - s)\, \varphi(\omega)\, ds\, d\omega\,.$$

The second term in the right-hand side of (4) contains the values of the control $u(t)$ for $t > t_0$ as well as for $t < t_0$. The values of the control $u(t)$ for $t \in [t_0 - h, t_0]$ enter into the definition. To separate them, the second term of (4) must be transformed by changing the order of integration. Using the unsymmetric Fubini theorem, we have the following equalities

$$(5) \qquad x_L(t) = x_L(t, \varphi) + \int_{-h}^{0} d_{H\omega} \left(\int_{t_0+\omega}^{t+\omega} X(t, s - \omega)\, H(s - \omega, \omega)\, u(s)\, ds \right) =$$

$$= x_L(t, \varphi) + \int_{-h}^{0} d_{H\omega} \left(\int_{t_0+\omega}^{t_0} X(t, s - \omega)\, H(s - \omega, \omega)\, u_{t_0}\, ds \right) +$$

$$+ \int_{t_0}^{t} \left[\int_{-h}^{0} X(t, s - \omega)\, d_\omega H_t(s - \omega, \omega) \right] u(s)\, ds$$

where

$$H_t(s, \omega) = \begin{cases} H(s, \omega) & \text{for } s \leqq t \\ 0 & \text{for } s > t \end{cases}$$

and the symbol $d_{H\omega}$ denotes that the integration is in the Lebsegue-Stieltjes sense with respect to the variable ω in $H(t, \omega)$. For brevity let us introduce the following notation

$$q(t, u_{t_0}) = \int_{-h}^{0} \mathrm{d}_{H\omega} \left(\int_{t_0+\omega}^{t_0} X(t, s-\omega)\, H(s-\omega, \omega)\, u_{t_0}\, \mathrm{d}s \right) \tag{6}$$

$$S(t, s) = \int_{-h}^{0} X(t, s-\omega)\, \mathrm{d}_\omega\, H_t(s-\omega, \omega) \tag{7}$$

and define the controllability matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} S(t_1, s)\, S'(t_1, s)\, \mathrm{d}s \tag{8}$$

where the prime indicates the matrix transpose. Hence the solution of the linear system (1) can be written as

$$x_L(t) = x_L(t, \varphi) + q(t, u_{t_0}) + \int_{t_0}^{t} S(t, s)\, u(s)\, \mathrm{d}s \tag{9}$$

and the solution of the perturbed system (2) is given by

$$x(t) = x_L(t) + \int_{t_0}^{t} X(t, s)\, f(s, x(s), x(s-h), u(s), u(s-h))\, \mathrm{d}s\,. \tag{10}$$

4. MAIN RESULTS

Theorem 1. The system (1) is relatively controllable on $[t_0, t_1]$ iff W is nonsingular.

Proof. Assume W is nonsingular. Let the control function u be defined on J as

$$u(t) = S'(t_1, t)\, W^{-1}[x_1 - x_L(t_1, \varphi) - q(t_1, u_{t_0})]\,. \tag{11}$$

Then from equation (9), it follows that

$$x_L(t_1) = x_L(t_1, \varphi) + q(t_1, u_{t_0}) + \int_{t_0}^{t_1} S(t_1, s)\, S'(t_1, s)\, W^{-1}\,.$$
$$.\, [x_1 - x_L(t_1, \varphi) - q(t_1, u_{t_0})]\, \mathrm{d}s = x_1\,.$$

Conversely, assume that W is singular. Then, there exists a vector $v \neq 0$, such that $v'Wv = 0$. It follows that

$$\int_{t_0}^{t_1} v'\, S(t_1, s)\, (v'\, S(t_1, s))'\, \mathrm{d}s = 0\,.$$

Therefore, $v'\, S(t_1, s) = 0$ for $s \in J$. Consider the zero initial function $\varphi = 0$ and $u_{t_0} = 0$ on $[t_0 - h, t_0]$ and the final point $x_1 = v$. Since the system is controllable there exists a control $u(t)$ on J that steers the response to $x_1 = v$ at $t = t_1$ that is, $x_L(t_1) = v$. From $\varphi = 0$, $x_L(t_1, \varphi) = 0$ and $v'v \neq 0$ for $v \neq 0$. On the other hand

$$v = x_L(t_1) = \int_{t_0}^{t_1} S(t_1, s)\, u(s)\, \mathrm{d}s$$

and hence

$$v'v = \int_{t_0}^{t_1} v' \, S(t_1, s) \, u(s) \, \mathrm{d}s = 0 \, .$$

This is a contradiction for $v \neq 0$. Hence W is nonsingular. □

Now we are able to prove our main result on the controllability of the nonlinear perturbed delay system (2). For this, we will take

$$p = (x, x', u, u') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$$

and let

$$|p| = |x| + |x'| + |u| + |u'| \, .$$

Theorem 2. If the continuous function f satisfies the condition

$$\lim_{|p| \to \infty} \frac{|f(t, p)|}{|p|} = 0$$

uniformly in $t \in J$ and if the system (1) is relatively controllable on J, then the system (2) is relatively controllable on J.

Proof. Let φ, u_{t_0} be continuous on $[t_0 - h, t_0]$ and let $x_1 \in \mathbb{R}^n$. Define

$$T \colon C_{n+m}[t_0 - h, t_1] \to C_{n+m}[t_0 - h, t_1]$$

by

$$T(x, u) = (y, v)$$

where

$$v(t) = S'(t_1, t) \, W^{-1} \Big[x_1 - x_L(t_1; \varphi) - q(t_1, u_{t_0}) - \\ - \int_{t_0}^{t_1} X(t_1, s) f(s, x(s), x(s - h), u(s), u(s - h)) \, \mathrm{d}s \Big] \quad \text{for} \quad t \in J \tag{12}$$

$v(t) = u_{t_0}(t)$ for $t \in [t_0 - h, t_0]$, and

$$y(t) = x_L(t; \varphi) + q(t, u_{t_0}) + \int_{t_0}^{t} S(t, s) \, v(s) \, \mathrm{d}s + \\ + \int_{t_0}^{t} X(t, s) f(s, x(s), x(s - h), u(s), u(s - h)) \, \mathrm{d}s \quad \text{for} \quad t \in J \tag{13}$$

$y(t) = \varphi(t)$ on $[t_0 - h, t_0]$.

Let

$$a_1 = \sup |S(t, s)| \quad \text{for} \quad t_0 \leqq s \leqq t \leqq t_1$$

$$a_2 = |W^{-1}(t_0, t_1)|$$

$$a_3 = \sup |x(t_1; \varphi| + |x_1| + |q(t_1, u_{t_0})| \quad \text{for} \quad t \in J$$

$$a_4 = \sup |X(t, s)| \quad \text{for} \quad (t, s) \in J \times J$$

$$b = \max \{(t_1 - t_0) \, a_1, 1\} \, , \quad c_1 = 8ba_1a_2a_4(t_1 - t_0)$$

$$c_2 = 8a_4(t_1 - t_0) \, , \quad d_1 = 8a_1a_2a_3b \, , \quad d_2 = 8a_3$$

$$c = \max\{c_1, c_2\}, \quad d = \max\{d_1, d_2\}$$

$$\sup |f| = \sup [|f(s, x(s), x(s-h), u(s), u(s-h))| : s \in J].$$

Then,

$$|v(t)| \leqq a_1 a_2 [a_3 + a_4(t_1 - t_0) \sup |f|] = \\ = \frac{d_1}{8b} + \frac{c_1}{8b} \sup |f| \leqq \\ \leqq \frac{1}{8b} [d + c \sup |f|]$$

and

$$|y(t)| \leqq a_3 + a_1(t_1 - t_0) \|v\| + a_4(t_1 - t_0) \sup |f| \leqq b\|v\| + \tfrac{1}{8}d + \tfrac{1}{8}c \sup |f|.$$

By Proposition 1 in [5], f satisfies the following condition: for each pair of positive constants c and d, there exists a positive constant r such that, if $|p| \leqq r$, then

$$(14) \qquad c|f(t, p)| + d \leqq r \quad \text{for all} \quad t \in J.$$

Also, for given c and d, if r is a constant such that the inequality (14) is satisfied, then any r_1 such that $r < r_1$ will also satisfy the inequality (14). Now, take c and d as given above, and let r be chosen so that the inequality (14) is satisfied and

$$\sup |\varphi(t)| \leqq \tfrac{1}{4}r, \quad \sup |u_{t_0}(t)| \leqq \tfrac{1}{4}r \quad \text{for} \quad t \in [t_0 - h, t_0].$$

Therefore, if $\|x\| \leqq \frac{1}{4}r$ and $\|u\| \leqq \frac{1}{4}r$ then

$$|x(s) + |x(s-h)| + |u(s)| + |u(s-h)| \leqq r \quad \text{for all} \quad s \in J.$$

It follows that

$$d + c \sup |f| \leqq r.$$

Therefore,

$$|v(t) \leqq r/8b \quad \text{for all} \quad t \in J$$

and hence

$$\|v\| \leqq r/8b.$$

It follows that

$$|y(t)| \leqq \tfrac{1}{8}r + \tfrac{1}{8}r \quad \text{for all} \quad t \in J,$$

and hence that

$$\|y\| \leqq \tfrac{1}{4}r.$$

Thus we have proved that, if

$$G = \{(x, u) \in C_{n+m}[t_0 - h, t_1] : \|x\| \leqq \tfrac{1}{4}r \text{ and } \|u\| \leqq \tfrac{1}{4}r\}$$

then T maps G into itself. Since f is continuous, it implies that the operator T is continuous. By using Arzela-Ascoli's theorem it is easy to verify that, T is completely continuous. Since G is closed, bounded and convex, the Schauder fixed point theorem guarantees that, T has a fixed point $(x, u) \in G$. It follows that

$$x(t) = x_L(t; \varphi) + q(t, u_{t_0}) + \int_{t_0}^{t} S(t, s)\, u(s)\, ds + \\ + \int_{t_0}^{t} X(t, s)\, f(s, x(s), x(s-h), u(s), u(s-h))\, ds$$

for $t \in J$ and $x(t) = \varphi(t)$ for $t \in [t_0 - h, t_0]$. Hence $x(t)$ is the solution of the system (2) and

$$x(t_1) = x_L(t_1; \varphi) + q(t_1, u_{t_0}) + \int_{t_0}^{t_1} S(t_1, s)\, S'(t_1, s)\, W^{-1} \cdot$$

$$\cdot \left[x_1 - x_L(t_1; \varphi) - q(t_1, u_{t_0}) - \int_{t_0}^{t_1} X(t_1, s) f(s, x(s), x(s - h), u(s), u(s - h)) \right] \mathrm{d}s +$$

$$+ \int_{t_0}^{t_1} X(t_1, s) f(s, x(s), x(s - h), u(s), u(s - h))\, \mathrm{d}s = x_1 \,.$$

Hence, the system (2) is relatively controllable on $[t_0, t_1]$. $\square$

Corollary. If the continuous function f is bounded on $J \times R^n \times R^n \times R^m \times R^m$ and if the system (1) is relatively controllable on J, then the system (2) is relatively controllable on J.

5. EXTENSIONS

The results can be directly extended to the general nonlinear perturbed delay system. For that, consider the general linear delay system

$$\dot{x}(t) = L(x, u) \tag{15}$$

where

$$L(x, u) = \int_{-h}^{0} \mathrm{d}_s A(t, s)\, x(t + s) + \int_{-h}^{0} \mathrm{d}_s H(t, s)\, u(t + s).$$

$A(t, s)$ is an $n \times n$ continuous matrix in t uniformly with respect to $s \in [-h, 0]$ and of bounded variation in s on $[-h, 0]$ for each $t \in J$, its corresponding perturbed nonlinear delay system is

$$\dot{x}(t) = L(x, u) + g(t, x(t), x(t - h), u(t), u(t - h)). \tag{16}$$

Let $x_L(t; \varphi)$ be the solution of the equation

$$\dot{x}(t) = \int_{-h}^{0} \mathrm{d}_s A(t, s)\, x(t + s)$$

with initial function $x(t) = \varphi(t)$ on $[t_0 - h, t_0]$ and let $Y(s, t)$ be the $n \times n$ matrix solution of

$$Y(s, t) + \int_{s}^{s + \alpha(s)} Y(\omega, t)\, A(\omega, s - \omega)\, \mathrm{d}\omega = I$$

where

$$\alpha(s) = \begin{cases} h & \text{if } t_0 \leqq s \leqq t - h \\ t - s & \text{if } t - h \leqq s \leqq t \end{cases}$$

and I is the identity matrix. Then there exists an absolutely continuous solution of

(15) and it can be written as

$$x_L(t) = x_L(t;\varphi) + \int_{-h}^{0} \mathrm{d}_{H\omega}\left(\int_{t_0+\omega}^{t_0} Y(s-\omega,t)\,H(s-\omega,\omega)\,u_{t_0}(s)\,\mathrm{d}s\right) +$$

$$+ \int_{t_0}^{t}\left[\int_{-h}^{0} Y(s-\omega,t)\,\mathrm{d}_\omega\,H_t(s-\omega,\omega)\right]u(s)\,\mathrm{d}s\,.$$

Define the following

$$S_1(s,t) = \int_{-h}^{0} Y(s-\omega,t)\,\mathrm{d}_\omega\,H_t(s-\omega,\omega)$$

and

$$W_1(t_0,t_1) = \int_{t_0}^{t_1} S_1(s,t_1)\,S_1'(s,t_1)\,\mathrm{d}s\,.$$

Hence the solution of the system (16) can be written as

$$x(t) = x_L(t) + \int_{t_0}^{t} Y(s,t)\,g(s,x(s),x(s-h),u(s),u(s-h))\,\mathrm{d}s\,.$$

The proofs of the following theorems are similar to the proofs of Theorem 1 and 2, and hence they are omitted.

Theorem 3. System (15) is relatively controllable on J iff W_1 is nonsingular.

Theorem 4. If the continuous function g satisfies the condition

$$\lim_{|p|\to\infty} \frac{|g(t,p)|}{|p|} = 0$$

uniformly in $t \in J$ and if the system (15) is relatively controllable on J then the system (16) is relatively controllable on J.

(Received June 28, 1984.)

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