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ERGODIC PROPERTIES OF LOCALLY STATIONARY PROCESSES

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This article deals with asymptotic properties of locally stationary processes. Necessary and sufficient conditions under whose validity a locally stationary process is ergodic are given.

The spectral theory of locally stationary processes is studied in [1]. This paper deals with locally stationary processes from the point of view of their asymptotic behaviour.

Let $x(t)$, $t \in \mathbb{R}_1$, be a locally stationary process with $\mathsf{E}\{x(t)\} = 0$ and with covariance function

$$R(s, t) = R_1\left(\frac{s + t}{2}\right) R_2(s - t),$$

where $R_1 \geqq 0$ and R_2 is a stationary covariance function. Let us suppose that $x(t)$ is harmonizable, i.e. $x(t)$ can be expressed in the form of a stochastic integral understood in the quadratic mean, see [2],

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda)$$

where $\xi(\lambda)$ is a second order stochastic process having covariance function $\gamma(\lambda, \mu)$ with bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$.

We shall need the following lemma for the proof of Theorem 1.

Lemma 1. Let $x(t)$, $t \in \mathbb{R}_1$, be a harmonizable process, let $T > 0$. Then

$$\int_{-T}^{T} x(t) \, dt = \int_{-\infty}^{+\infty} \left\{\int_{-T}^{T} e^{it\lambda} \, dt\right\} d\xi(\lambda) \quad \text{a.s.}$$

if

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda).$$

Proof. The process $x(t)$ is harmonizable, i.e. $x(t) = \int_{-\infty}^{+\infty} e^{it\lambda}\, d\xi(\lambda)$ where $\int\int |dd\gamma(\lambda, \mu)| < \infty$, $\gamma(\lambda, \mu) = \mathsf{E}\{\xi(\lambda)\, \overline{\xi(\mu)}\}$. The stochastic integral $\int_{-\infty}^{+\infty} e^{it\lambda}\, d\xi(\lambda)$ is understood as a limit in the quadratic mean

$$x(t) = \underset{\substack{A \to -\infty \\ B \to +\infty}}{\text{l.i.m.}} \int_A^B e^{it\lambda}\, d\xi(\lambda)\,.$$

At first, we prove the possibility of order change between $\int_{-T}^{T}$ and $\int_A^B$. As $e^{it\lambda}$ is continuous and bounded $\int_{-T}^{T} \mathsf{E}\{|\int_A^B e^{it\lambda}\, d\xi(\lambda)|^2\}\, dt$ is finite for every pair A, B. By the Fubini theorem then

$$\int_{-T}^{T} \mathsf{E}\left\{\left|\int_A^B e^{it\lambda}\, d\lambda(\lambda)\right|^2\right\} dt = \mathsf{E}\left\{\int_{-T}^{T} \left|\int_A^B e^{it\lambda}\, d\xi(\lambda)\right|^2 dt\right\}$$

and this implies the existence of $\int_{-T}^{T} \{\int_A^B e^{it\lambda}\, d\xi(\lambda)\}\, dt$ a.s.. On the other hand, $\int_{-T}^{T} e^{it\lambda}\, dt = 2T(\sin T\lambda)/(T\lambda))$ is continuous and bounded on every interval (A, B) and hence the stochastic integral

$$\int_A^B \left\{\int_{-T}^{T} e^{it\lambda}\, dt\right\} d\xi(\lambda)$$

exists too.

By the definition of the stochastic integral in the quadratic mean

$$\int_A^B e^{it\lambda}\, d\xi(\lambda) = \underset{\|\mathscr{D}\| \to 0}{\text{l.i.m.}} \sum_j \exp(it\lambda_j)\, \Delta\xi(\lambda_j)\,,$$

where $\mathscr{D}$ is a partition of (A, B). Now

$$\mathsf{E}\left\{\left|\int_{-T}^{T} \left\{\int_A^B e^{it\lambda}\, d\xi(\lambda)\right\} dt - \int_{-T}^{T} \{\sum_j \exp(it\lambda_j)\, \Delta\xi(\lambda_j)\}\, dt\right|\right\} \leqq$$

$$\leqq \mathsf{E}\left\{\int_{-T}^{T} \left|\int_A^B e^{it\lambda}\, d\xi(\lambda) - \sum_j \exp(it\lambda_j)\, \Delta\xi(\lambda_j)\right| dt\right\} =$$

$$= \int_{-T}^{T} \mathsf{E}\left\{\left|\int_A^B e^{it\lambda}\, d\xi(\lambda) - \sum_j \exp(it\lambda_j)\, \Delta\xi(\lambda_j)\right|\right\} dt \to 0 \quad \text{as} \quad \|\mathscr{D}\| \to 0$$

because

$$\mathsf{E}\left\{\left|\int_A^B e^{it\lambda}\, d\xi(\lambda) - \sum_j \exp(it\lambda_j)\, \Delta\xi(\lambda_j)\right|^2\right\} \to 0 \quad \text{as} \quad \|\mathscr{D}\| \to 0$$

for every $t \in [-T, T]$ and

$$\left\{\mathsf{E}\left\{\left|\int_A^B e^{it\lambda}\, d\xi(\lambda) - \sum_j \exp(it\lambda_j)\, \Delta\xi(\lambda_j)\right|\right\}\right\}_{\mathscr{D}}$$

is bounded in t on $[-T, T]$. But

$$\int_{-T}^{T} \sum_j \exp(it\lambda_j)\, \Delta\xi(\lambda_j)\, dt = \sum_j \left\{\int_{-T}^{T} \exp(it\lambda_j)\, dt\right\} \Delta\xi(\lambda_j) \to$$

$$= \sum_j 2T \frac{\sin(T\lambda_j)}{T\lambda_j} \Delta\xi(\lambda_j) \to \int_A^B \left\{ \int_{-T}^{T} e^{it\lambda} \, dt \right\} d\xi(\lambda) \quad \text{as } \|\mathscr{D}\| \to 0$$

in the quadratic mean. As the limit in the quadratic mean is defined unambiguously a.s. we proved

$$\int_{-T}^{T} \left\{ \int_A^B e^{it\lambda} \, d\xi(\lambda) \right\} dt = \int_A^B \left\{ \int_{-T}^{T} e^{it\lambda} \, dt \right\} d\xi(\lambda) \quad \text{a.s.}$$

In a similar way as before we can prove the existence of the integrals

$$\int_{-\infty}^{+\infty} \left\{ \int_{-T}^{T} e^{it\lambda} \, dt \right\} d\xi(\lambda), \quad \int_{-T}^{T} \left\{ \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda) \right\} dt.$$

As

$$\int_{-\infty}^{+\infty} \left\{ \int_{-T}^{T} e^{it\lambda} \, dt \right\} d\xi(\lambda) = \underset{\substack{A \to -\infty \\ B \to +\infty}}{\text{l.i.m.}} \int_A^B \left\{ \int_{-T}^{T} e^{it\lambda} \, dt \right\} d\xi(\lambda)$$

the previous results give immediately

$$\int_{-\infty}^{+\infty} \left\{ \int_{-T}^{T} e^{it\lambda} \, dt \right\} d\xi(\lambda) = \int_{-T}^{T} \left\{ \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda) \right\} dt \quad \text{a.s.} \qquad \square$$

Theorem 1. Let $x(t)$, $t \in \mathbb{R}_1$, be a harmonizable random process, $\mathsf{E}\{x(t)\} = 0$, then as $T \to \infty$

$$\frac{1}{2T} \int_{-T}^{+T} x(t) \, dt \to \xi(\langle 0 \rangle) \quad \text{in the quadratic mean}$$

where $\xi(\langle 0 \rangle)$ is the jump at 0 of $\xi(\cdot)$.

Proof. We suppose that $x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda)$ with $\mathsf{E}\{d\xi(\lambda) \, \overline{d\xi(\mu)}\} = dd\gamma(\lambda, \mu)$. Then by use of the previous Lemma 1

$$\frac{1}{2T} \int_{-T}^{+T} x(t) \, dt = \frac{1}{2T} \int_{-T}^{+T} \left[\int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda) \right] dt = \frac{1}{2T} \int_{-\infty}^{+\infty} \frac{e^{iT\lambda} - e^{-iT\lambda}}{i\lambda} \, d\xi(\lambda).$$

Now

$$\mathsf{E}\left\{ \left| \frac{1}{2T} \int_{-\infty}^{+\infty} \frac{e^{iT\lambda} - e^{-iT\lambda}}{i\lambda} \, d\xi(\lambda) \right|^2 \right\} = \frac{1}{4T^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(e^{iT\lambda} - e^{-iT\mu})(e^{-iT\mu} - e^{iT\mu})}{(i\lambda)(-i\mu)} \, dd\gamma(\lambda, \mu$$

according to Lemma 1 in [1].

Let $\varepsilon > 0$ and let us consider a partition of $\mathbb{R}_1 \times \mathbb{R}_1$ in the following form

$$\mathbb{R}_1 \times \mathbb{R}_1 = [-\varepsilon, +\varepsilon] \times [-\varepsilon, +\varepsilon] \cup [-\varepsilon, +\varepsilon] + \mathbb{R}_1 \cup \mathbb{R}_1 \times \times [-\varepsilon, +\varepsilon] \cup \{(\lambda, \mu) \in \mathbb{R}_1 \times \mathbb{R}_1 : |\lambda| > \varepsilon, |\mu| > \varepsilon\}.$$

Using this partition we obtain

$$\left| \frac{1}{4T^2} \int_{|\lambda| > \varepsilon} \int_{|\mu| > \varepsilon} \frac{(e^{iT\lambda} - e^{-iT\lambda})(e^{iT\mu} - e^{-iT\mu})}{(+i\lambda)(-i\mu)} \, dd\gamma(\lambda, \mu) \right| \leqq$$

$$\leqq \frac{1}{4T^2}\int_{|\lambda|>\varepsilon}\int_{|\mu|>\varepsilon}\frac{4}{\varepsilon^2}\,|\mathrm{dd}\gamma(\lambda,\mu)| \to 0 \quad \text{as} \quad T\to\infty$$

because $\gamma(\lambda, \mu)$ is of bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$. Further, as $\varepsilon \downarrow 0$

$$(1)\qquad \frac{1}{4}\int_{-\varepsilon}^{+\varepsilon}\int_{-\varepsilon}^{+\varepsilon}\frac{(e^{iT\lambda}-e^{-iT\lambda})(e^{-iT\mu}-e^{iT\mu})}{(+iT\lambda)(-iT\mu)}\,\mathrm{dd}\gamma(\lambda,\mu)\to \mathrm{dd}\gamma(0,0) =$$

$$= \lim_{\varepsilon\downarrow 0}\,[\gamma(\varepsilon,\varepsilon)-\gamma(\varepsilon,-\varepsilon)-\gamma(-\varepsilon,\varepsilon)+\gamma(-\varepsilon,-\varepsilon))] \quad \text{uniformly in } T.$$

Surely, we can write (let us put $f(T, \lambda, \mu) = \dfrac{(e^{iT\lambda}-e^{-iT\lambda})(e^{-iT\mu}-e^{iT\mu})}{(-4iT\lambda\, iT\mu)}$ for simplicity)

$$\int_{-\varepsilon}^{+\varepsilon}\int_{-\varepsilon}^{+\varepsilon} f(T,\lambda,\mu)\,\mathrm{dd}\gamma(\lambda,\mu) - \mathrm{dd}\gamma(0,0) =$$

$$= \int_{-1}^{+1}\int_{-1}^{+1}(\psi_\varepsilon(\lambda,\mu)\,f(T,\lambda,\mu) - \delta_{(0,0)}(\lambda,\mu))\,\mathrm{dd}\gamma(\lambda,\mu)$$

if $0 < \varepsilon < 1$, $\psi_\varepsilon(\lambda, \mu) = 1$ for $(\lambda, \mu) \in [-\varepsilon, +\varepsilon] \times [-\varepsilon, +\varepsilon]$, $\psi_\varepsilon(\lambda, \mu) = 0$ otherwise, $\delta_{(0,0)}(0, 0) = 1$ and $\delta_{(0,0)}(\lambda, \mu) = 0$ otherwise. As $\{\psi_\varepsilon(\lambda, \mu) f(T, \lambda, \mu) - \delta_{(0,0)}(\lambda, \mu)\}_{0<\varepsilon<1}$ is a nonincreasing net tending to 0, hence tending uniformly in $[-1, +1] \times [-1, +1]$, if $\varepsilon \downarrow 0$, then (1) holds. In a similar way, $\psi_\varepsilon(\lambda, \mu) - \delta_{(0,0)}(\lambda, \mu) \to 0$ uniformly as $\varepsilon \downarrow 0$ and the following inequality

$$|\psi_\varepsilon(\lambda,\mu)f(T,\lambda,\mu) - \delta_{(0,0)}(\lambda,\mu)| \leqq \psi_\varepsilon(\lambda,\mu) - \delta_{(0,0)}(\lambda,\mu)$$

holds for every (λ, μ) because $|f(T, \lambda, \mu)| \leqq 1$ and $f(T, 0, 0) = 1$. These facts prove the uniform convergence of (1) with respect to T.

Similarly, we can estimate

$$\left|\frac{1}{4T^2}\int_{-\varepsilon}^{+\varepsilon}\int_{|\mu|>\varepsilon}\frac{(e^{iT\lambda}-e^{-iT\lambda})(e^{-iT\mu}-e^{iT\mu})}{(i\lambda)(-i\mu)}\,\mathrm{dd}\gamma(\lambda,\mu)\right| \leqq$$

$$\leqq \frac{1}{4T^2}\int_{-\varepsilon}^{+\varepsilon}\int_{|\mu|>\varepsilon}|\mathrm{dd}\gamma(\lambda,\mu)| \to 0 \quad \text{as} \quad T\to\infty \text{ for every } \varepsilon > 0\,.$$

Analogously,

$$\frac{1}{4T^2}\int_{|\lambda|>\varepsilon}\int_{-\varepsilon}^{+\varepsilon}\frac{(e^{iT\lambda}-e^{-iT\lambda})(e^{-iT\mu}-e^{iT\mu})}{(i\lambda)(-i\mu)}\,\mathrm{dd}\gamma(\lambda,\mu)\to 0 \quad \text{as} \quad T\to\infty$$

for every $\varepsilon > 0$. Summarizing these results we obtain that

$$\mathsf{E}\left|\frac{1}{2T}\int_{-T}^{+T}x(t)\,\mathrm{d}t - \xi(\langle 0\rangle)\right|^2 \to 0\,, \quad \text{as} \quad T\to\infty$$

where $\xi(\langle 0\rangle)$ is the jump of the random process $\xi(.)$ at 0. □

If $x(t)$ has a spectral density function, i.e. $\mathrm{dd}\gamma(\lambda, \mu) = f(\lambda, \mu)\,\mathrm{d}\lambda\,\mathrm{d}\mu$, then $\xi(\langle 0\rangle) =$

$= 0$ and $x(t)$ is ergodic because

$$\frac{1}{2T}\int_{-T}^{+T} x(t)\,\mathrm{d}t \to \mathsf{E}\{x(0)\} = 0 \quad \text{as} \quad T \to \infty\,.$$

In case the existence of spectral density is not supposed, we must calculate

$$\mathsf{E}|\xi(+\varepsilon) - \xi(-\varepsilon)|^2 =$$

$$= \gamma(+\varepsilon, +\varepsilon) - \gamma(-\varepsilon, +\varepsilon) - \gamma(+\varepsilon, -\varepsilon) + \gamma(-\varepsilon, -\varepsilon) = \int_{-\varepsilon}^{+\varepsilon}\int_{-\varepsilon}^{+\varepsilon} \mathrm{dd}\gamma(\lambda, \mu)\,.$$

At this moment we use the results of [1]; under assumption of local stationarity the following relation

$$\int_{-\varepsilon}^{+\varepsilon}\int_{-\varepsilon}^{+\varepsilon} \mathrm{dd}\gamma(\lambda, \mu) = \iint_{A(\varepsilon)} \mathrm{d}F_1(u)\,\mathrm{d}F_2(v)$$

holds, where $A(\varepsilon) = \{(u, v) \in \mathbb{R}_1 \times \mathbb{R}_1 : -\varepsilon < u + (v/2) < +\varepsilon,\ -\varepsilon < u - (v/2) < +\varepsilon\}$. Now

$$\iint_{A(\varepsilon)} \mathrm{d}F_1(u)\,\mathrm{d}F_2(v) = \int_0^{\varepsilon}\int_{-2\varepsilon+2u}^{2\varepsilon-2u} \mathrm{d}F_1(u)\,\mathrm{d}F_2(v) + \int_{-\varepsilon}^{0}\int_{-2\varepsilon-2u}^{2u+2\varepsilon} \mathrm{d}F_1(u)\,\mathrm{d}F_2(v) =$$

$$= \int_0^{\varepsilon} [F_2(2\varepsilon - 2u) - F_2(-(2\varepsilon - 2u))]\,\mathrm{d}F_1(u) +$$

$$+ \int_{-\varepsilon}^{0} [F_2(2u + 2\varepsilon) - F_2(-(2u + 2\varepsilon))]\,\mathrm{d}F_1(u)\,.$$

This equality yields immediately the following conclusion.

Theorem 2. A locally stationary harmonizable random process is ergodic if and only if at least one of its spectral measures is continuous at 0.

A very simple sufficient condition ensuring ergodicity of a second order random process is that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R(s, t)\,\mathrm{d}s\,\mathrm{d}t < \infty\,,$$

because then

$$\mathsf{E}\left|\frac{1}{2T}\int_{-T}^{+T} x(t)\,\mathrm{d}t\right|^2 = \frac{1}{4T^2}\int_{-T}^{+T}\int_{-T}^{+T} R(s, t)\,\mathrm{d}s\,\mathrm{d}t \to 0 \quad \text{as} \quad T \to \infty\,.$$

In the case of locally stationary processes this condition can be weakened. Thanks to the decomposition $R(s, t) = R_1((s + t)/2)\,R_2(s - t)$ we can write

$$\int_{-T}^{+T}\int_{-T}^{+T} R(s, t)\,\mathrm{d}s\,\mathrm{d}t = \int_{-T}^{+T}\int_{-T}^{+T} R_1\left(\frac{s + t}{2}\right) R_2(s - t)\,\mathrm{d}s\,\mathrm{d}t =$$

$$= \int_0^{T}\int_{2u-2T}^{2T-2u} R_1(u)\,R_2(v)\,\mathrm{d}u\,\mathrm{d}v + \int_{-T}^{0}\int_{-2T-2u}^{2T+2u} R_1(u)\,R_2(v)\,\mathrm{d}u\,\mathrm{d}v\,.$$

As R_2 is a stationary covariance function, it is $|R_2(v)| \leqq R_2(0)$, then

$$\frac{1}{4T^2}\int_{-T}^{+T}\int_{-T}^{+T} R(s,t)\,\mathrm{d}s\,\mathrm{d}t \leqq \frac{R_2(0)}{T^2}\left[\int_0^T (T-u)\,R_1(u)\,\mathrm{d}u + \int_{-T}^{0}(T+u)\,R_1(u)\,\mathrm{d}u\right] =$$

$$= \frac{R_2(0)}{T}\left[\int_{-T}^{+T}\left(1-\frac{|u|}{T}\right)R_1(u)\,\mathrm{d}u\right].$$

This inequality gives, of course, that the integrability of R_1 is a sufficient condition for ergodicity of the locally stationary process $x(t)$.

Now, let us study a possible estimate of covariance function of a locally stationary process. As we need the fourth moments, let us consider a Gaussian case. The basic property of covariance function of a locally stationary process gives that

$$\mathsf{E}\{X_{u+v/2}\overline{X}_{u-v/2}\} = R_1(u)\,R_2(v) \quad \text{for every pair} \quad u, v\,.$$

It is reasonable to study the asymptotic behaviour of integrals

$$\int_{-T}^{+T} X_{u+v/2}\overline{X}_{u-v/2}\,\mathrm{d}u\,, \qquad \int_{-T}^{+T} X_{u+v/2}\overline{X}_{u-v/2}\,\mathrm{d}v\,,$$

when these integrals are considered in the quadratic mean sense. It holds according to Lemma 1 in [1]

$$\mathsf{E}\left\{\int_{-T}^{+T} X_{u+v/2}\overline{X}_{u-v/2}\,\mathrm{d}u\right\} = R_2(v)\int_{-T}^{+T} R_1(u)\,\mathrm{d}u$$

and similarly

$$\mathsf{E}\left\{\int_{-T}^{+T} X_{u+v/2}\overline{X}_{u-v/2}\,\mathrm{d}v\right\} = R_1(u)\int_{-T}^{+T} R_2(v)\,\mathrm{d}v\,.$$

Let us prove that the sequence

$$\left\{\int_{-T}^{+T} X_{u+v/2}\overline{X}_{u-v/2}\,\mathrm{d}v\right\},\ T > 0\,,$$

is fundamental in the quadratic mean. If $T \geqq S$, then

$$\mathsf{E}\left|\int_{-T}^{+T} X_{u+v/2}\overline{X}_{u-v/2}\,\mathrm{d}v - \int_{-S}^{+S} X_{u+v/2}\overline{X}_{u-v/2}\,\mathrm{d}v\right|^2 = \mathsf{E}\left|\int_{-T}^{-S} + \int_S^T\right|^2 =$$

$$= 4\mathsf{E}\left|\mathrm{Re}\int_S^T X_{u+v/2}\overline{X}_{u-v/2}\,\mathrm{d}v\right|^2$$

because $S \leqq v \leqq T$ if and only if $-T \leqq -v \leqq -S$ and $X_{u+v/2}\overline{X}_{u-v/2} = \overline{\overline{X}_{u-v/2}X_{u+v/2}}$.

For the sake of simplicity we shall consider, further, a real case. Then

$$\mathsf{E}\left|\int_S^T X_{u+v/2}X_{u-v/2}\,\mathrm{d}v\right|^2 = \int_S^T\int_S^T \mathsf{E}\{X_{u+v/2}X_{u-v/2}X_{u-z/2}X_{u+z/2}\}\,\mathrm{d}v\,\mathrm{d}z =$$

$$= \int_S^T\int_S^T R_1^2(u)\,R_2(v)\,R_2(z)\,\mathrm{d}v\,\mathrm{d}z +$$

$$+\int_S^T\int_S^T R_1\left(u+\frac{v+z}{4}\right)R_2\left(\frac{v-z}{2}\right)R_1\left(u-\frac{v+z}{4}\right)R_2\left(\frac{z-v}{2}\right)\mathrm{d}v\,\mathrm{d}z\,+$$

$$+\int_S^T\int_S^T R_1\left(u+\frac{v-z}{4}\right)R_2\left(\frac{v+z}{2}\right)R_1\left(u+\frac{z-v}{4}\right)R_2\left(-\frac{v+z}{2}\right)\mathrm{d}v\,\mathrm{d}z=$$

$$=R_1^2(u)\left(\int_S^T R_2(z)\,\mathrm{d}z\right)^2+4\iint_{\substack{\{S\leqq 2x+y\leqq T\}\\\{S\leqq 2x-y\leqq T\}}}R_1(u+x)R_2(y)R_1(u-x)R_2(-y)\,\mathrm{d}x\,\mathrm{d}y+$$

$$+4\iint_{\substack{\{S\leqq 2x+y\leqq T\}\\\{S\leqq y-2x\leqq T\}}}R_1(u+x)R_2(y)R_1(u-x)R_2(-y)\,\mathrm{d}x\,\mathrm{d}y=$$

$$=R_1^2(u)\left(\int_S^T R_2(z)\,\mathrm{d}z\right)^2+\int_{S/2}^{(S+T)/4}\int_{S-2x}^{2x-S}R_1(u+x)R_2^2(y)R_1(u-x)\,\mathrm{d}x\,\mathrm{d}y\,+$$

$$+4\int_{(S+T)/4}^{T/2}\int_{2x-T}^{T-2x}R_1(u+x)R_2^2(y)R_1(u-x)\,\mathrm{d}x\,\mathrm{d}y\,+$$

$$+4\int_{S/2}^{(T+S)/4}\int_{(S-y)/2}^{(y-S)/2}R_1(u+x)R_2^2(y)R_1(u-x)\,\mathrm{d}x\,\mathrm{d}y\,+$$

$$+4\int_{(S+T)/4}^{T/2}\int_{(y-T)/2}^{(T-y)/2}R_1(u+x)R_1(u-x)R_2^2(y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

Now, using $R_2(y)=R_2(-y)$ and $R_1(u+(-x))\,R_1(u-(-x))=R_1(u+x)\,.$ $.\,R_1(u-x)$ one can write

$$\int_{(T+S)/4}^{T/2}\int_{S-2x}^{2x-S}R_1(u+x)R_1(u-x)R_2^2(y)\,\mathrm{d}x\,\mathrm{d}y=$$

$$=2\int_{(T+S)/4}^{T/2}R_1(u+x)R_1(u-x)\int_0^{2x-S}R_2^2(y)\,\mathrm{d}y\,\mathrm{d}x\,.$$

(Similarly for the other integrals.) These results yield the following Theorem 3.

Theorem 3. Let $\int_{-\infty}^{+\infty}R_2(y)\,\mathrm{d}y<\infty$, $\int_{-\infty}^{+\infty}R_1^2(x)\,\mathrm{d}x<\infty$, then there exists the limit in the quadratic mean

$$\underset{T\to\infty}{\mathrm{l.i.m.}}\int_{-T}^{+T}X_{u+v/2}X_{u-v/2}\,\mathrm{d}v=\int_{-\infty}^{+\infty}X_{u+v/2}X_{u-v/2}\,\mathrm{d}v$$

and

$$\mathsf{E}\left\{\int_{-\infty}^{+\infty}X_{u+v/2}X_{u-v/2}\,\mathrm{d}v\right\}=R_1(u)\int_{-\infty}^{+\infty}R_2(v)\,\mathrm{d}v$$

for every $u\in\mathbb{R}_1$.

Proof. We have proved that

$$\mathsf{E}\left|\int_{-T}^{+T}X_{u+v/2}X_{u-v/2}\,\mathrm{d}v-\int_{-S}^{+S}X_{u+v/2}X_{u-v/2}\,\mathrm{d}v\right|^2=4R_1^2(u)\left(\int_S^T R_2(v)\,\mathrm{d}v\right)^2+$$

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