KYBERNETIKA — VOLUME 22 (1986), NUMBER 3

SPECTRAL DECOMPOSITION OF LOCALLY STATIONARY RANDOM PROCESSES

JIŘÍ MICHÁLEK

This paper deals with locally stationary random processes introduced by Silverman in [1]. The spectral representation of such processes is obtained; the results generalize those of Silverman.

The notion of a locally stationary process is introduced by Silverman in [1]. This is a new kind of a random process generalizing the notion of a weakly stationary process. Let $\{x(t)\}$, $t \in \mathbb{R}_1$ be a random process, generally complex, with vanishing mean value and finite covariance function $R(s, t) = \mathsf{E}\{x(s)\,\bar{x}(t)\}$ on $\mathbb{R}_1 \times \mathbb{R}_1$, where $\bar{x}(t)$ is the complex conjugate to $x(t)$. The author of [1] says that the random process $\{x(t)\}$, $t \in \mathbb{R}_1$, is locally stationary in the wide sense, or has a locally stationary covariance, if its covariance can be written as

$$R(s, t) = R_1\left(\frac{s + t}{2}\right) R_2(s - t) \quad \text{for every pair} \quad s, t \in \mathbb{R}_1\,,$$

where $R_1 \geqq 0$ and R_2 is a stationary covariance function. We can put $R_1(0) = 1$, $R_2(0) = 1$ without loss of generality. In case $R_1 = \text{const} \neq 0$ we obtain a weakly stationary covariance function. Some examples of locally stationary processes are exhibited in [1], too.

We need the following facts about the harmonic analysis of nonstationary random processes. Following [2], we say that $x(t)$ is harmonizable if $x(t)$ can be expressed in the form

$$x(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}t\lambda}\, \mathrm{d}\xi(\lambda)\,,$$

where the integral is meant in the quadratic mean and $\xi(\lambda)$ is a second order random process with zero mean value and covariance function $\gamma(\lambda, \mu) = \mathsf{E}\{\xi(\lambda)\,\bar{\xi}(\mu)\}$ of bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$. It is proved in [2] that $x(t)$ is harmonizable if and only if its covariance function $R(s, t)$ has the spectral representation

$$R(s, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} \, dd\gamma(\lambda, \mu)$$

where $\gamma(\lambda, \mu)$ is a covariance function of bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$. The last integral is understood in the Riemann-Stieltjes sense. In such a case we shall say that $R(s, t)$ is harmonizable, too. When

$$dd\gamma(\lambda, \mu) = f(\lambda, \mu) \, d\lambda \, d\mu ,$$

then $f(\lambda, \mu)$ is called the spectral density function of $x(t)$. If $x(t)$ is locally stationary and harmonizable with spectral density function then, as it is proved in [1], its covariance function has a spectral representation

$$R(s, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} f(\lambda, \mu) \, d\lambda \, d\mu ,$$

where $f(\lambda, \mu)$ is a locally stationary covariance, too, i.e.

$$f(\lambda, \mu) = f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu) ,$$

$f_1 \geqq 0$ and f_2 is a stationary covariance.

This relation can be understood as a generalization of the famous Wiener-Khintchine relation for the case of locally stationary random process.

The following Theorem 1 and Theorem 2 are opposite assertions with respect to the generalized Wiener-Khintchine relation.

Theorem 1. Let $x(t)$ be a harmonizable random process with the spectral density function $f(\lambda, \mu)$ of the form

$$f(\lambda, \mu) = f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu) ,$$

where $f_1 \geqq 0$ and f_2 is a stationary covariance function. Then $x(t)$ is a locally stationary random process.

Proof. We assume that $x(t)$ is harmonizable having a spectral density, i.e. its covariance function $R(s, t)$ has representation

$$(1) \qquad R(s, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu) \, d\lambda \, d\mu ,$$

where $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f_1((\lambda + \mu)/2) f_2(\lambda - \mu)| \, d\lambda \, d\mu$ exists.

Let us consider the transformation $T(\lambda, \mu) = (u, v)$, where $u = (\lambda + \mu)/2$, $v = \lambda - \mu$. Using this transformation, the integral (1) can be expressed as

$$R(s, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{iu(s-t)} \, e^{iv\left(\frac{s+t}{2}\right)} f_1(u) f_2(v) \, du \, dv ,$$

hence

$$R(s, t) = R_1\left(\frac{s + t}{2}\right) R_2(s - t)$$

where

$$R_1(x) = \int_{-\infty}^{+\infty} e^{iux} f_2(u)\, du\,, \quad R_2(y) = \int_{-\infty}^{+\infty} e^{ivy} f_1(v)\, dv\,.$$

As $f_1 \geqq 0$, R_2 is a stationary covariance and as f_2 is a stationary covariance, it is $R_1 \geqq 0$. It means that $R(s, t)$ is a locally stationary covariance function. □

We need the following Lemma 1 and Lemma 2 for proofs of further results.

Lemma 1. Let f, g be complex functions on $(-\infty, +\infty)$, let $\xi(\cdot), \eta(\cdot)$ be second order stochastic processes with $\mathsf{E}\{\xi(s)\,\bar{\eta}(t)\} = \Gamma_{\xi\eta}(s, t)$. If $\int_{-\infty}^{+\infty} f\, d\xi$, $\int_{-\infty}^{+\infty} g\, d\eta$ exist in the quadratic mean then

$$\mathsf{E}\left\{\int_{-\infty}^{+\infty} f\, d\xi \int_{-\infty}^{+\infty} \bar{g}\, d\bar{\eta}\right\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f\bar{g}\, dd\Gamma_{\xi\eta}\,.$$

Proof. As $\int_{-\infty}^{+\infty} f\, d\xi = \underset{\substack{A\to-\infty\\ B\to+\infty}}{\text{l.i.m.}} \int_A^B f\, d\xi$ it is sufficient to prove Lemma 1 for bounded intervals only, i.e.

$$\mathsf{E}\left\{\int_A^B f\, d\xi \int_A^B \bar{g}\, d\bar{\eta}\right\} = \int_A^B\int_A^B f\bar{g}\, dd\Gamma_{\xi\eta}\,.$$

According to the definition of the stochastic integrals $\int_A^B f\, d\xi$, $\int_A^B g\, d\eta$ there exists $\delta > 0$ for every $\varepsilon > 0$ such that for every subdivisions $\mathscr{D}_1$, $\mathscr{D}_2$ of $[A, B]$ with the norms $\|\mathscr{D}_1\| < \delta$, $\|\mathscr{D}_2\| < \delta$

$$\mathsf{E}\left\{\left|\int_A^B f\, d\xi - \sum_i f(t_i^*)\, \Delta\xi(t_i)\right|^2\right\} < \varepsilon^2$$

$$\mathsf{E}\left\{\left|\int_A^B g\, d\eta - \sum_j g(s_j^*)\, \Delta\eta(s_j)\right|^2\right\} < \varepsilon^2\,.$$

This fact gives, further, that

$$\left|\mathsf{E}\left\{\int_A^B f\, d\xi \int_A^B \bar{g}\, d\bar{\eta}\right\} - \sum_i f(t_i^*)\, \Delta\xi(t_i) \sum_j \bar{g}(s_j^*)\, \Delta\bar{\eta}(s_j)\right| \leqq$$

$$\leqq \left[(\mathsf{E}|\sum_j \bar{g}(s_j^*)\, \Delta\bar{\eta}(s_j)|^2)^{1/2} + \left(\mathsf{E}\left|\int_A^B f\, d\xi\right|^2\right)^{1/2}\right]\varepsilon$$

for every $\mathscr{D}_1$, $\mathscr{D}_2$ with $\|\mathscr{D}_1\| < \delta$, $\|\mathscr{D}_2\| < \delta$. That proves firstly the existence of the Riemann-Stieltjes integral $\int_A^B \int_A^B f\bar{g}\, dd\Gamma_{\xi\eta}$ and secondly the equality

$$\mathsf{E}\left\{\int_A^B f\, d\xi \int_A^B \bar{g}\, d\bar{\eta}\right\} = \int_A^B\int_A^B f\bar{g}\, dd\Gamma_{\xi\eta}\,.$$

□

Lemma 2. Let $\xi(t)$ be a second order stochastic process having the derivative ξ'

in the quadratic mean with continuous covariance function. If $\int_{-\infty}^{+\infty} f\,\mathrm{d}\xi$ exists in the quadratic mean then

$$\int_{-\infty}^{+\infty} f(t)\,\mathrm{d}\xi(t) = \int_{-\infty}^{+\infty} f(t)\,\xi'(t)\,\mathrm{d}t\,.$$

Proof. Let Γ be the covariance function of $\xi(\cdot)$. The existence of $\xi'(\cdot)$ implies the existence of the second derivative $\partial^2\Gamma(s,t)/\partial s\,\partial t$ because $\mathsf{E}\{\xi'(s)\,\bar{\xi}'(t)\} = \partial^2\Gamma(s,t)/\partial s\,\partial t$. Now, let $\int_{-\infty}^{+\infty} f\,\mathrm{d}\xi$ exist. Then the integral $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f\bar{f}\,\mathrm{d}\mathrm{d}\Gamma$ exists as the limit of $\int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} f\bar{f}\,\mathrm{d}\mathrm{d}\Gamma$ when $A \to -\infty$, $\mathrm{B} \to +\infty$. This integral can be approximated by sums of the form

$$\sum_i \sum_j f(s_i^*)\,\bar{f}(t_j^*)\,\Delta\Delta\Gamma(s_i, t_j)$$

where $s_i \leqq s_i^* \leqq s_{i+1}$, $t_j \leqq t_j^* \leqq t_{j+1}$. The existence and continuity of $\partial^2\Gamma(s,t)/\partial s\,\partial t$ yield that $\Delta\Delta\Gamma(s_i,t_j) = (\partial^2\Gamma(s,t)/\partial s\,\partial t|_{\substack{s=s_i+\theta_1\Delta s_i\\ t=t_j+\theta_2\Delta t_j}})\,\Delta s_i\,\Delta t_j$ with $0 < \theta_1 < 1$, $0 < \theta_2 < 1$. That proves the existence of the integral

$$\int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} f(s)\bar{f}(t)\,\frac{\partial^2\Gamma(s,t)}{\partial s\,\partial t}\,\mathrm{d}s\,\mathrm{d}t$$

and at the same time the equality

$$\int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} f(s)\bar{f}(t)\,\mathrm{d}\mathrm{d}\Gamma(s,t) = \int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} f(s)\bar{f}(t)\,\frac{\partial^2\Gamma(s,t)}{\partial s\,\partial t}\,\mathrm{d}s\,\mathrm{d}t\,.$$

As $\partial^2\Gamma(s,t)/\partial s\,\partial t$ is the covariance function of ξ' the existence of

$$\int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} f(s)\bar{f}(t)\,\frac{\partial^2\Gamma(s,t)}{\partial s\,\partial t}\,\mathrm{d}s\,\mathrm{d}t$$

insures the existence of the stochastic integral $\int_{\mathrm{A}}^{\mathrm{B}} f(t)\,\xi'(t)\,\mathrm{d}t$. It remains to prove the equality $\int_{\mathrm{A}}^{\mathrm{B}} f(t)\,\mathrm{d}\xi(t) = \int_{\mathrm{A}}^{\mathrm{B}} f(t)\,\xi'(t)\,\mathrm{d}t$. Let us prove that $\mathsf{E}\left|\int_{\mathrm{A}}^{\mathrm{B}} f\,\mathrm{d}\xi - \int_{\mathrm{A}}^{\mathrm{B}} f\xi'\right|^2 = 0$. We have

$$\mathsf{E}\left\{\left|\int_{\mathrm{A}}^{\mathrm{B}} f\,\mathrm{d}\xi - \int_{\mathrm{A}}^{\mathrm{B}} f\xi'\right|^2\right\} = \int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} f\bar{f}\,\mathrm{d}\mathrm{d}\Gamma - \mathsf{E}\left\{\int_{\mathrm{A}}^{\mathrm{B}} f\,\mathrm{d}\xi \int_{\mathrm{A}}^{\mathrm{B}} \bar{f}\bar{\xi}'\right\} -$$

$$- \mathsf{E}\left\{\int_{\mathrm{A}}^{\mathrm{B}} f\xi' \int_{\mathrm{A}}^{\mathrm{B}} \bar{f}\,\mathrm{d}\bar{\xi}\right\} + \int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} f\bar{f}\,\frac{\partial^2\Gamma}{\partial s\,\partial t}\,.$$

According to Lemma 1

$$\mathsf{E}\left\{\int_{\mathrm{A}}^{\mathrm{B}} f\,\mathrm{d}\xi \int_{\mathrm{A}}^{\mathrm{B}} \bar{f}\bar{\xi}'\right\} = \int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} f(s)\bar{f}(t)\,\mathsf{E}\{\mathrm{d}\xi(s)\,\bar{\xi}'(t)\,\mathrm{d}t\} =$$

$$= \int_{\mathrm{A}}^{\mathrm{B}}\int_{\mathrm{A}}^{\mathrm{B}} f(s)\bar{f}(t)\,\frac{\partial^2\Gamma(s,t)}{\partial s\,\partial t}\,\mathrm{d}s\,\mathrm{d}t \quad \text{because } \mathsf{E}\{\xi(s)\,\bar{\xi}'(t)\} = \frac{\partial\Gamma(s,t)}{\partial t}\,.$$

In this way we obtained that $\int_A^B f \, d\xi = \int_A^B f\xi'$. This equality holding for every bounded interval $[A, B]$ gives immediately that $\int_{-\infty}^{+\infty} f(t) \, d\xi(t) = \int_{-\infty}^{+\infty} f(t) \, \xi'(t) \, dt$. □

Theorem 2. Let $x(t)$ be a harmonizable random process with spectral density function $f(\lambda, \mu)$ of the form

$$f(\lambda, \mu) = f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu)$$

where f_1 is continuous and nonnegative, f_2 is a continuous stationary covariance. Then $x(t)$ has the following spectral representation

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, z'(\lambda) \, d\lambda \,,$$

where $z(\lambda)$ is a locally stationary random process, too.

Proof. Theorem 1 says that $x(t)$ is locally stationary. Being harmonizable $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda) \,,$$

where $\xi(\lambda)$ is a second order random process with covariance function $\gamma(\lambda, \mu)$. We assume the existence of spectral density function of $x(t)$, i.e.

(2) $$dd\gamma(\lambda, \mu) = f(\lambda, \mu) \, d\lambda \, d\mu = f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu) \, d\lambda \, d\mu \,.$$

It follows from the existence of $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(\lambda, \mu)| \, d\lambda \, d\mu$ that

$$\gamma(\lambda, \mu) = \int_{-\infty}^{\lambda} \int_{-\infty}^{\mu} f_1\left(\frac{u + v}{2}\right) f_2(u - v) \, du \, dv \,.$$

Let us prove that $\xi(\lambda)$ has the derivative in the quadratic mean. As familiarly known, such a derivative exists if and only if $\gamma(\lambda, \mu)$ has the generalized second derivative on the diagonal (λ, λ)

$$\lim_{h \to 0, h' \to 0} \frac{\Delta_h \Delta_{h'} \gamma(\lambda, \lambda)}{hh'} = \lim_{h \to 0, h' \to 0} \frac{1}{hh'} \int_{\lambda}^{\lambda + h} \int_{\lambda}^{\lambda + h'} f_1\left(\frac{u + v}{2}\right) f_2(u - v) \, du \, dv \,.$$

As f_1 is assumed to be continuous there exists the quadratic mean derivative $\xi'(\lambda)$ of $\xi(\lambda)$ and its covariance function

(3) $$E\{\xi'(\lambda) \, \bar{\xi}'(\mu)\} = f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu)$$

is, as we see, locally stationary. Now, we can apply Lemma 2 and we immediately obtain

(4) $$x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} \, \xi'(\lambda) \, d\lambda \,.$$ □

The following Theorem 3 expresses the inverse formula to the formula (4).

Theorem 3. Let $z(\lambda)$ be a locally stationary random process with the continuous covariance function $f(\lambda, \mu) = f_1((\lambda + \mu)/2) f_2(\lambda - \mu)$. If $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(\lambda, \mu)| < \infty$ then there exists

$$(5) \qquad x(t) = \int_{-\infty}^{+\infty} e^{it\lambda} z'(\lambda) \, d\lambda$$

in the quadratic mean sense and $x(t)$ is locally stationary, too. If $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |E\{x(s) . \bar{x}(t)\}| \, ds \, dt < \infty$ then

$$z(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda t} x(t) \, dt .$$

Proof. The integral (5) exists if and only if the integral $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} f(\lambda, \mu) . d\lambda \, d\mu$ exists, where $f(\lambda, \mu)$ is the covariance function of $z(\cdot)$. As we assume the existence of $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |f(\lambda, \mu)| \, d\lambda \, d\mu$ and $|e^{it\lambda}| \leqq 1$ then the integral (5) exists in the quadratic mean sense. It follows that $x(t)$ is harmonizable with covariance function

$$R(s, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s\lambda - t\mu)} f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu) \, d\lambda \, d\mu$$

according to Lemma 1. By use of transformation $T(\lambda, \mu) = (u, v)$, $u = (\lambda + \mu)/2$, $v = \lambda - \mu$ we can write

$$R(s, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s-t)u} f_1(u) \, e^{i\left(\frac{s+t}{2}\right)v} f_2(v) \, dv = R_1\left(\frac{s + t}{2}\right) R_2(s - t) .$$

Surely, $R_1 \geqq 0$ and R_2 is a stationary covariance. It means that $x(t)$ is a locally stationary process, too. We have proved that

$$R_1(x) = \int_{-\infty}^{+\infty} e^{ixv} f_2(v) \, dv , \quad R_2(y) = \int_{-\infty}^{+\infty} e^{iyu} f_1(u) \, du .$$

The integral

$$w(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\lambda t} x(t) \, dt$$

exists in the quadratic mean sense because according to Lemma 1

$$E\{w(\lambda) \, \bar{w}(\mu)\} = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-is\lambda} e^{it\mu} E\{x(s) \, \bar{x}(t)\} \, ds \, dt =$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(s\lambda - t\mu)} R_1\left(\frac{s + t}{2}\right) R_2(s - t) \, ds \, dt =$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i(\lambda - \mu)u} R_1(u) \, e^{-i\left(\frac{\lambda+\mu}{2}\right)} R_2(v) \, du \, dv =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(\lambda - \mu)u} R_1(u) \, du \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\left(\frac{\lambda+\mu}{2}\right)v} R_2(v) \, dv = f_1\left(\frac{\lambda + \mu}{2}\right) f_2(\lambda - \mu)$$

exists for every pair (λ, μ). Now, we must prove that $w(\lambda) = z(\lambda)$ for every $\lambda \in \mathbb{R}_1$. For this reason we estimate

$$\mathsf{E}\left\{\left|z(\lambda) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-it\lambda}\, x(t)\, dt\right|^2\right\}.$$

Surely, $z(\lambda) = \underset{h \to 0}{\text{l.i.m.}}\, (1/h) \int_\lambda^{\lambda+h} z(u)\, du$, where the integral is understood in the quadratic mean sense. The integral $\int_\lambda^{\lambda+h} z(u)\, du$ exists in the quadratic mean because the process $z(\cdot)$ has the continuous covariance function and therefore the integral $\int_\lambda^{\lambda+h} \int_\lambda^{\lambda+h} \mathsf{E}\{z(u)\, \bar{z}(v)\}\, du\, dv$ exists. We can express

$$\mathsf{E}\left\{\left|\frac{1}{h}\int_\lambda^{\lambda+h} z(u)\, du - z(\lambda)\right|^2\right\} = \mathsf{E}\left\{\left|\frac{1}{h}\int_\lambda^{\lambda+h} [z(u) - z(\lambda)]\, du\right|^2\right\} =$$

$$= \frac{1}{h^2}\int_\lambda^{\lambda+h}\int_\lambda^{\lambda+h} \mathsf{E}\{[z(u) - z(\lambda)] \,.\, [\bar{z}(v) - \bar{z}(\lambda)]\}\, du\, dv$$

according to Lemma 1. The continuity of $\mathsf{E}\{z(u)\, \bar{z}(v)\}$ at the point (λ, λ) implies that $\lim_{h \to 0} \mathsf{E}|(1/h) \int_\lambda^{\lambda+h} z(u)\, du - z(\lambda)|^2 = 0$. With respect to the inverse formula for harmonizable processes

$$\underset{\tau \to \infty}{\text{l.i.m.}}\, \frac{1}{2\pi}\int_{-\tau}^{+\tau} - \frac{e^{-i(\lambda+h)t} - e^{-i\lambda t}}{it}\, x(t)\, dt = \int_\lambda^{\lambda+h} z(u)\, du$$

for $h > 0$ and every $\lambda \in \mathbb{R}_1$. Then it is possible to write

$$\mathsf{E}\left\{\frac{1}{h}\int_\lambda^{\lambda+h} z(u)\, du\, \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it\lambda}\, \bar{x}(t)\, dt\right\} =$$

$$= \lim_{\tau \to \infty} \mathsf{E}\left\{\frac{1}{2\pi}\int_{-\tau}^{+\tau} \frac{e^{-i\lambda s}(e^{-ish} - 1)}{-ish}\, x(s)\, ds\, \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda t}\, \bar{x}(t)\, dt\right\} =$$

$$= \lim_{\tau \to \infty} \frac{1}{4\pi^2}\int_{-\tau}^{+\tau}\int_{-\infty}^{+\infty} \frac{e^{-i\lambda s}(e^{-ish} - 1)}{-ish}\, e^{i\lambda t}\, \mathsf{E}\{x(s)\, \bar{x}(t)\}\, ds\, dt =$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{-i\lambda s}(e^{-ish} - 1)}{-ish}\, e^{i\lambda t}\, R_1\left(\frac{s+t}{2}\right) R_2(s - t)\, ds\, dt$$

because $|(e^{-ish} - 1)/(-ish)| \leqq 1$ and $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |R(s, t)|\, ds\, dt$ exists. Now, we use the triangular inequality

$$(6) \qquad \left(\mathsf{E}\left|z(\lambda) - \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\lambda t}\, x(t)\, dt\right|^2\right)^{1/2} \leqq$$

$$\leqq \left(\mathsf{E}\left|z(\lambda) - \frac{1}{h}\int_\lambda^{\lambda+h} z(u)\, du\right|^2\right)^{1/2} + \left(\mathsf{E}\left|\frac{1}{h}\int_\lambda^{\lambda+h} z(u)\, du - \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-it\lambda}\, x(t)\, dt\right|^2\right)^{1/2}$$

which holds for every $h > 0$. The first term in (6) tends to zero for $h \to 0$, the second term can be calculated by aim of Lemma 1 as follows

$$\mathsf{E}\left\{\left|\frac{1}{h}\int_{\lambda}^{\lambda+h} z(u)\,\mathrm{d}u - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-it\lambda}\, x(t)\,\mathrm{d}t\right|^2\right\} = \frac{1}{h^2}\int_{\lambda}^{\lambda+h}\int_{\lambda}^{\lambda+h} f_1\left(\frac{u+v}{2}\right) f_2(u-v)\,\mathrm{d}u\,\mathrm{d}v +$$

$$+ \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}(s\lambda - t\lambda)}\, R_1\left(\frac{s+t}{2}\right) R_2(s-t)\,\mathrm{d}s\,\mathrm{d}t -$$

$$- \mathsf{E}\left\{\frac{1}{h}\int_{\lambda}^{\lambda+h} z(u)\,\mathrm{d}u\, \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{it\lambda}\, \bar{x}(t)\,\mathrm{d}t\right\} - \mathsf{E}\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\lambda t}\, x(t)\,\mathrm{d}t\, \frac{1}{h}\int_{\lambda}^{\lambda+h} \bar{z}(u)\,\mathrm{d}u\right\} =$$

$$= \frac{1}{h^2}\int_{\lambda}^{\lambda+h}\int_{\lambda}^{\lambda+h} f_1\left(\frac{u+v}{2}\right) f_2(u-v)\,\mathrm{d}u\,\mathrm{d}v + f_1(\lambda)\, f_2(0) -$$

$$- \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\lambda s}\,\frac{(\mathrm{e}^{-\mathrm{i}sh} - 1)}{-\mathrm{i}sh}\, \mathrm{e}^{\mathrm{i}\lambda t}\, R_1\left(\frac{s+t}{2}\right) R_2(s-t)\,\mathrm{d}s\,\mathrm{d}t -$$

$$- \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\lambda(t-s)}\,\frac{(\mathrm{e}^{\mathrm{i}sh} - 1)}{\mathrm{i}sh}\, R_1\left(\frac{s+t}{2}\right) R_2(t-s)\,\mathrm{d}t\,\mathrm{d}s\,.$$

As $|(\mathrm{e}^{\mathrm{i}sh} - 1)/(\mathrm{i}sh)| \leqq 1$ and $\lim_{h\to 0}(\mathrm{e}^{\mathrm{i}sh} - 1)/(\mathrm{i}sh) = 1$, it is

$$\lim_{h\to 0} \mathsf{E}\left\{\left|\frac{1}{h}\int_{\lambda}^{\lambda+h} z(u)\,\mathrm{d}u - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-it\lambda}\, x(t)\,\mathrm{d}t\right|^2\right\} = 0\,,$$

which proves that

$$z(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\lambda t}\, x(t)\,\mathrm{d}t\,. \qquad \square$$

We have so far assumed the existence of the spectral density function of $x(t)$. At this moment we omit this assumption and let $x(t)$ be generally locally stationary and harmonizable. It means that there exists a spectral decomposition

$$x(t) = \int_{-\infty}^{+\infty} \mathrm{e}^{it\lambda}\,\mathrm{d}\xi(\lambda)\,,$$

where $\mathsf{E}\{\mathrm{d}\xi(\lambda)\,\mathrm{d}\bar{\xi}(\mu)\} = \mathrm{d}\mathrm{d}\gamma(\lambda, \mu)$ and $\gamma(\lambda, \mu)$ is a covariance function with bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$. The covariance function of $x(t)$ $R(s, t)$ can be expressed as

$$(7) \qquad R(s, t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(s\lambda - t\mu)}\,\mathrm{d}\mathrm{d}\gamma(\lambda, \mu) = R_1\left(\frac{s+t}{2}\right) R_2(s-t)$$

where $R_1 \geqq 0$ and R_2 is a stationary covariance function. When we put $s = t$, then $R(s, s) = R_1(s)\,.\,R_2(0) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}s(\lambda-\mu)}\,\mathrm{d}\mathrm{d}\gamma(\lambda, \mu)$, similarly when $s = -t$, then

$$R\left(\frac{s}{2}, -\frac{s}{2}\right) = R_1(0)\, R_2(s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}s\left(\frac{\lambda+\mu}{2}\right)}\,\mathrm{d}\mathrm{d}\gamma(\lambda, \mu)\,.$$

These relations imply that

$$R(u, v) = R_1\left(\frac{u+v}{2}\right) R_2(u - v) = R\left(\frac{u-v}{2}, \frac{v-u}{2}\right) R\left(\frac{u+v}{2}, \frac{u+v}{2}\right)$$

for every pair (u, v). Now, we shall put

$$\frac{s+t}{2} = x\,, \quad s - t = y$$

into (7). Then $s = x + y/2$, $t = x - y/2$ and

$$R_1(x)\, R_2(y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\lambda\left(x+\frac{y}{2}\right)}\, e^{-i\mu\left(x-\frac{y}{2}\right)}\, dd\gamma(\lambda, \mu) =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{ix(\lambda-\mu)}\, e^{iy\left(\frac{\lambda+\mu}{2}\right)}\, dd\gamma(\lambda, \mu)\,.$$

We again apply the transformation $T\colon u = (\lambda + \mu)/2$, $v = \lambda - \mu$ which implies a new measure $r(u, v)$ in $\mathbb{R}_1 \times \mathbb{R}_1$ by the relation

$$\iint_{E\times F} dd r(u, v) = \iint_{T^{-1}(E\times F)} dd\gamma(\lambda, \mu)\,,$$

where $E \times F$ is a measurable rectangle in $\mathbb{R}_1 \times \mathbb{R}_1$. In terms of $r(u, v)$ we obtain

$$R_1(x)\, R_2(y) = \int_{-\infty}^{+\infty}\int e^{ixv}\, e^{iyu}\, dd r(u, v)\,.$$

If we put $y = 0$ then

$$R_1(x) = \int_{-\infty}^{+\infty}\int e^{ixv}\, dd r(u, v) = \int_{-\infty}^{+\infty} e^{ixv}\, dr_2(v)$$

where

$$dr_2(v) = \int_{-\infty}^{+\infty} dd r(u, v)\,,$$

similarly for $x = 0$ we get

$$R_2(y) = \int_{-\infty}^{+\infty} e^{iyu}\, dr_1(u)$$

where

$$dr_1(u) = \int_{-\infty}^{+\infty} dd r(u, v)\,.$$

But it means, together, that

$$\int_{-\infty}^{+\infty} e^{ixv}\, dr_2(v) \int_{-\infty}^{+\infty} e^{iyu}\, dr_1(u) = \int_{-\infty}^{+\infty} e^{i(xv+yu)}\, dd r(u, v)$$

which yields that

$$dd r(u, v) = dr_1(v)\, dr_2(u)\,.$$

The process $x(t)$ is locally stationary, it means that $R_1 \geqq 0$ and R_2 is a stationary covariance; hence R_2 can be written as

$$R_2(y) = \int_{-\infty}^{+\infty} e^{iyu} \, dF_1(u) ,$$

where F_1 is a probability distribution function because we put $R_2(0) = 1$.

This fact implies that

$$dr_1(u) = dF_1(u) .$$

As $R_1 \geqq 0$ and hence R_1 is real, then

$$R_1(x) = \int_{-\infty}^{+\infty} e^{ixv} \, dr_2(v) = \bar{R}_1(x) = \int_{-\infty}^{+\infty} e^{-ixv} \, d\bar{r}_2(v) = \int_{-\infty}^{+\infty} e^{ixv} \, d\bar{r}_2(-v) ;$$

It gives $dr_2(v) = d\bar{r}_2(-v)$, which means that d Re $r_2(\cdot)$ is symmetric, i.e.,

$$\int_A^B d \operatorname{Re} r_2(v) = \int_{-B}^{-A} d \operatorname{Re} r_2(v)$$

and d Im $r_2(v)$ is antisymmetric, i.e.,

$$\int_A^B d \operatorname{Im} r_2(v) = -\int_{-A}^{-B} d \operatorname{Im} r_2(v) .$$

Summarizing these facts we obtain conclusion that the transformation T associated with definition of locally stationary processes decomposes the induced measure $r(u, v)$ into two independent parts $r_1(v)$, $r_2(v)$, where $r_1(\cdot)$ is a probability distribution function and the Fourier transform of $r_2(v)$ is nonnegative. On the contrary, if $\xi(\lambda)$ is a random process whose covariance function $\gamma(\lambda, \mu)$ has bounded variation on $\mathbb{R}_1 \times \mathbb{R}_1$ and if under the transformation T the induced measure γT^{-1}; i.e.

$$dd\gamma T^{-1}(u, v) = ddr(u, v) ,$$

is decomposable into two independent parts

$$ddr(u, v) = dr_1(v) \, dr_2(u)$$

where $r_1(v)$ is a probability distribution function and the Fourier transform of $r_2(u)$ is nonnegative, then the Fourier transform of $\xi(\lambda)$ (in the quadratic mean sense)

$$\int_{-\infty}^{+\infty} e^{it\lambda} \, d\xi(\lambda)$$

is a locally stationary random process.

The following Theorem 4 gives the answer when a locally stationary random process is harmonizable.

Theorem 4. Let $x(t)$ be a locally stationary random process such that

$$R_1(x) = \int_{-\infty}^{+\infty} e^{ix\lambda} \, dF_2(\lambda) ,$$

where F_2 is generally a complex measure with bounded variation. Then $x(t)$ is harmonizable.

Proof. We know that $\mathsf{E}\{x(s)\,\bar{x}(t)\} = R_1((s+t)/2)\,R_2(s-t) = R(s,t)$ where $R_2(y) = \int_{-\infty}^{+\infty} e^{iyu}\,dF_1(u)$ with a probability distribution function F_1. This yields

$$R(s,t) = \int_{-\infty}^{+\infty} e^{i\left(\frac{s+t}{2}\right)\lambda}\,dF_2(\lambda)\int_{-\infty}^{+\infty} e^{i(s-t)\mu}\,dF_1(\mu) =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{is\left(\mu+\frac{\lambda}{2}\right)}\,e^{-it\left(\mu-\frac{\lambda}{2}\right)}\,dd F_1(\mu)\,F_2(\lambda)\,.$$

Now, let us consider the transformation S: $u = \mu + \lambda/2$, $v = \mu - \lambda/2$. Then $R(s,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(su-tv)}\,dd\gamma(u,v)$ where $\gamma(u,v)$ is induced from $F_1(\mu)\,F_2(\lambda)$ by the transformation S. As the function $e^{i(su-tv)}$ is continuous we can assume that $\gamma(u,v)$ is normalized as usually assumed in the harmonic analysis. At this moment we must prove that $\gamma(u,v)$ is covariance function belonging to a random process $\xi(u)$. $F_1(\mu)\,F_2(\lambda)$ is of bounded variation so $\gamma(u,v)$ has bounded variation, too. It is easy to prove that at every point (u,v) and for every pair $h, h' > 0$ there exists the limit

$$\lim_{\tau\to\infty}\int_{-\tau}^{+\tau}\int_{-\tau}^{+\tau}\frac{e^{-iut}(e^{-ith}-1)\,e^{+ivu}(e^{ish'}-1)}{(-it)\,(is)}\,R(t,s)\,dt\,ds$$

equal to $\Delta_h\Delta_{h'}\gamma(u,v)$. Using this fact we immediately see that the sequence

$$\left\{\int_{-\tau}^{+\tau}\frac{e^{-iut}(e^{-iht}-1)}{-it}\,x(t)\,dt\right\}_{\tau\nearrow\infty}$$

is fundamental in the quadratic mean and hence there exists a random variable

$$z_h(u) = \underset{\tau\to\infty}{\text{l.i.m.}}\int_{-\tau}^{+\tau}\frac{e^{-iut}(e^{-ith}-1)}{-it}\,x(t)\,dt \tag{8}$$

(for every $u \in \mathbb{R}_1$ and every $h > 0$). Surely,

$$\mathsf{E}\{z_h(u)\,\bar{z}_{h'}(u')\} = \Delta_h\Delta_{h'}\gamma(u,u')\,.$$

From the assumption of bounded variation of $\gamma(u,v)$ it is possible to put $\gamma(-\infty,-\infty) = 0$. Formula (8) gives by elementary calculation the additive property of $z_h(u)$ in the following sense

$$z_{u+t}(-t) = \sum_{j=0}^{n} z_h(u_j) \quad\text{where}\quad -t < u\,,\quad h = \frac{u+t}{n}\,,\quad u_j = \frac{u+t}{n}\,j\,;$$

the last equality holds for every subdivision of $[-t, \lambda)$. Let us prove that there exists the limit

$$\underset{t\uparrow\infty}{\text{l.i.m.}}\, z_{u+t}(-t) = \xi(u)\,;$$

it follows from that

$$\mathsf{E}\{|z_{u+t}(-t) - z_{u+\tau+t}(-(t+\tau))|^2\} = \mathsf{E}\{|z_\tau(-(t+\tau))|^2\} =$$
$$= \Delta_\tau \Delta_t \gamma(-(t+\tau), -(t+\tau)) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for every} \quad \tau > 0 .$$

The continuity of scalar product with respect to convergence in the quadratic mean yields that

$$\operatorname{cov}(\xi(u), \xi(v)) = \mathsf{E}\{\xi(u)\, \bar{\xi}(v)\} = \lim_{t \uparrow \infty} \mathsf{E}\{z_{u+t}(-t)\, \bar{z}_{v+t}(-t)\} =$$
$$= \lim_{t \uparrow \infty} \Delta_{u+t} \Delta_{v+t} \gamma(-t, -t) = \gamma(u, v) .$$

This assertion says that $\gamma(u, v)$ must be a covariance function. Using the theory of harmonizable random processes presented in [2] we obtain from these facts that $x(t)$ is harmonizable and hence $x(t)$ has a representation of the form

$$x(t) = \int_{-\infty}^{+\infty} e^{itu}\, d\xi(u) . \qquad \square$$

(Received May 31, 1985.)

REFERENCES

[1] R. A. Silverman: Locally stationary random processes. IRE Trans. Inform. Theory *IT-3* (1957), 3, 182–187.

[2] M. Loève: Probability Theory. D. van Nostrand, Toronto–New York–London 1955.

RNDr. Jiří Michálek, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.