DEADBEAT PERFORMANCE UNDER MEASUREMENT DYNAMICS

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The possibility of achieving a deadbeat regulation and tracking in linear systems independently of their initial conditions is investigated. The attention is focused on single-input single-output control systems incorporating a dynamical sensor in the loop. A necessary and sufficient condition for a deadbeat controller to exist is established. All deadbeat controllers are described in parametric form and the one which yields the shortest transient is identified.

1. INTRODUCTION

One of the fundamental problems associated with the discrete-time control of linear (either discrete or continuous) systems is that of driving some signal to zero in finite time and holding it there for all discrete (sampling) times thereafter.

The problem is simple if it is the system *state* that is to be driven to zero. The state deadbeat controller is independent of the initial state of the system and results in a nilpotent state trasition matrix.

A more difficult problem is that of deadbeat regulation or tracking, when one wishes the system *output* either to be zeroed or to track a reference signal in a deadbeat fashion. Such a deadbeat controller is a dynamical system which may depend on the initial states of the system to be controlled and reference generator. Moreover the question of causality and stability arises for the closed loop.

The possibility of achieving a deadbeat regulation and tracking independently of the initial conditions was investigated by Kučera [2], Wolovich [4], Eichstaedt [1] and Kučera and Šebek [3]. The last paper provides a necessary and sufficient condition for the existence of deadbeat controllers under the constraint of loop causality and stability.

The present paper studies the effect of *measurement dynamics* upon the deadbeat performance. Such a situation arises when the response of the sensor is not instantaneous but contains some transients that are not negligible with respect to those of the system to be controlled. This additional dynamics within the feedback loop makes the problem much more difficult. We propose here a simple solution for the case of single-input single-output systems which is based on polynomial algebra.

2. FORMULATION

Consider the system to be controlled called hereafter *plant*

$$x_{t+1} = A_p x_t + B_p u_t \qquad (1)$$
$$y_t = C_p x_t + D_p u_t$$

another system called *sensor*

$$v_{t+1} = A_s v_t + B_s y_t \qquad (2)$$
$$z_t = C_s v_t + D_s y_t$$

and the *reference* generator

$$w_{t+1} = A_r w_t \qquad (3)$$
$$r_t = C_r w_t$$

for $t = 0, 1, 2, \ldots$. Here u_t is the control, y_t is the output, z_t is the measurement and r_t is the reference at time t, all of them being scalar quantities.

The *deadbeat controller* is defined as a system of the form

$$s_{t+1} = A_c s_t + B_{1c} z_t + B_{2c} r_t \qquad (4)$$
$$u_t = C_c s_t + D_{1c} z_t + D_{2c} r_t$$

which makes y_t follow r_t exactly after a finite time t, independently of x_0, v_0, w_0 and s_0. As an additional requirement, the composite system (1)–(4) shown in Fig. 1 is to be causal and stable, i.e. its free motion should start no earlier than at $t = 0$ and should converge to zero when $t \to \infty$.

Write x for the sequence $\{x_t\}_{t=-\infty}^{\infty}$ and define the unit delay operator d: $x_t \to x_{t+1}$.

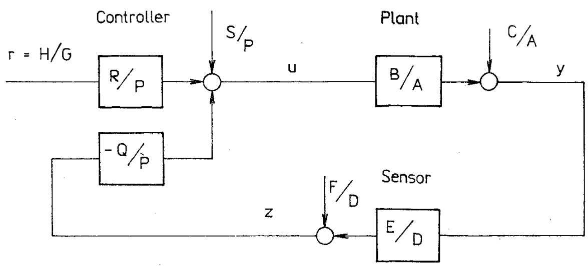

Fig. 1. Closed loop system.

Then (1)–(3) give

$$A(d)\,y = B(d)\,u + C(d)$$
$$D(d)\,z = E(d)\,y + F(d)$$
$$G(d)\,r = H(d)$$

where A through H are polynomials in d given by

(5) $$D_p + C_p(I - A_p d)^{-1} B_p d = \frac{B(d)}{A(d)}$$

(6) $$C_p(I - A_p d)^{-1} x_0 = \frac{C(d)}{A(d)}$$

(7) $$D_s + C_s(I - A_s d)^{-1} B_s d = \frac{E(d)}{D(d)}$$

(8) $$C_s(I - A_s d)^{-1} v_0 = \frac{F(d)}{D(d)}$$

and

(9) $$C_r(I - A_r d)^{-1} w_0 = \frac{H(d)}{G(d)}.$$

The polynomial pairs A, C and D, F and H, G are assumed to be coprime for at least one x_0, v_0 and w_0.

Accordingly the controller will be sought in the form

$$P(d)\,u = -Q(d)\,z + R(d)\,r + S(d)$$

where

(10) $$D_{1c} + C_c(I - A_c d)^{-1} B_{1c} d = -\frac{Q(d)}{P(d)}$$

(11) $$D_{2c} + C_c(I - A_c d)^{-1} B_{2c} d = \frac{R(d)}{P(d)}$$

(12) $$C_c(I - A_c d)^{-1} s_0 = \frac{S(d)}{P(d)}.$$

We say that polynomial $A(d)$ is *causal* (or *stable*) if the sequence obtained by expanding $1/A$ into ascending powers of d is zero for negative powers of d (or converges to zero). Note that A, D, G and P are all causal polynomials. Denote by B_0 and D_0 the greatest causal stable factors of B and D and write

(13) $$B(d) = B_0(d)\,B'(d)$$
$$D(d) = D_0(d)\,D'(d)\,.$$

3. DEADBEAT CONTROLLERS

The deadbeat regulation and tracking requires that the *error*

$$e = r - y$$

be a finite sequence for every x_0, v_0, w_0 and s_0, i.e., for every polynomial C, F, H and S. A simple algebra yields

$$e = -\frac{DP}{ADP + BEQ} C + \frac{BQ}{ADP + BEQ} F - \tag{14}$$
$$- \frac{BD}{ADP + BEQ} S + \left(1 - \frac{BDR}{ADP + BEQ}\right)\frac{H}{G}$$

where the explicit dependence upon the argument d has been suppressed for the sake of brevity.

The existence of deadbeat controllers will now be investigated.

Theorem 1. There exists a deadbeat controller which makes the closed loop system (1)–(4) causal and stable if and only if
1) the uncontrollable and unconstructible parts of plant (1) and sensor (2) are stable,
2) polynomials AD' and $B'E$ are coprime, and
3) polynomials $B'D'$ and G are coprime.

Proof. The necessity of 1) is clear. As for 2), suppose that e is a finite sequence for every C, F, H and S. Then (14) implies that

$$\frac{BD}{ADP + BEQ}$$

is a polynomial, i.e. $ADP + BEQ$ divides BD. Moreover $ADP + BEQ$ is causal and stable for a causal and stable closed loop. Hence it actually divides B_0D_0, i.e. there exists a polynomial T_1 such that

$$(ADP + BEQ)\, T_1 = B_0 D_0 \,. \tag{15}$$

This means that the gratest common factor of AD and BE divides B_0D_0. Then 2) follows on using (13). To prove the necessity of 3) suppose that e is a finite sequence for every C, F, H and S. Then

$$\left(1 - \frac{BDR}{ADP + BEQ}\right)\frac{1}{G}$$

is a polynomial. Using (15), G is seen to divide $1 - B'D'RT_1$, i.e.

$$1 - B'D'RT_1 = GT_2$$

for some polynomial T_2. Hence the claim follows.

The sufficiency of 1)–3) will be proved by construction. Let

$$P = B_0 P' \,, \quad Q = D_0 Q' \tag{16}$$

and let P', Q' and R, T be any polynomials satisfying the equations

$$AD'P' + B'EQ' = 1 \tag{17}$$

$$B'D'R + GT = 1\,. \tag{18}$$

These equations are solvable whenever 2) and 3) hold. Moreover P' can always be taken causal. Then (14) gives

$$e = -D'P'C + B'Q'F - B'D'S + TH \tag{19}$$

so that P, Q and R define a deadbeat controller. The closed loop system is causal and stable by 1) and by causality and stability of

$$ADP + BEQ = B_0 D_0\,. \qquad \square$$

Theorem 1 admits a simple intuitive interpretation. In addition to 1), condition 2) requires stability of the uncontrollable part of the tandem plant-sensor. In other words, only stable common factors between A and B, B and D, and D and E are allowed. Otherwise the closed loop system cannot be stabilized. Moreover 2) also requires complete constructibility of the tandem plant-sensor. This rules out any common factor between A and E; otherwise the deadbeat performance is out of reach. Finally condition 3) rules out unstable common factors between G and BD. Thus deadbeat tracking is possible only if the reference generator and sensor have no unstable poles in common and if the unstable poles of the reference generator are disjoint with the unstable zeros of the plant.

The sufficiency part of the proof provides a simple constructive procedure to find a deadbeat controller. There are good many other deadbeat controllers, however. To delineate all of them let B_x and D_x be any, not necessarily the greatest, *causal stable factors* of B and D, respectively, such that B' and D', defined by

$$B = B_x B'\,, \quad D = D_x D' \tag{20}$$

satisfy the hypotheses of Theorem 1. The family of deadbeat controllers is then given by

$$\begin{aligned} P &= B_x \bar{P}' + BEV \\ Q &= D_x \bar{Q}' - ADV \\ R &= \bar{R} + GW \end{aligned} \tag{21}$$

where $\bar{P}'$, $\bar{Q}'$ is a particular solution of equation (17) and $\bar{R}$ is a particular solution of equation (18). The polynomials V and W are *free parameters* but such that P is causal.

It is to be noticed that the deadbeat controller makes but e finite. All other signals in the composite system shown in Figure 1 are infinite sequences in general. The free motion of the closed loop system contains, in addition to finite modes, the infinite modes associated with $B_x D_x$ corresponding to pole-zero cancellations in the interconnection of sensor-controller-plant and, of course, all uncontrollable and unconstructible modes of the plant, sensor and controller.

It is sometimes of interest to have the error sequence not only finite but as *short* as possible. Then (19) indicates that we simply have to take the least degree solutions P', Q' of (17) and T of (18) and use (16) to define the triple P, Q and R. Thus the controller has a uniquely specified transfer function which must be minimally realized to keep the degree of S as low as possible.

4. EXAMPLE

Consider a simple illustrative example. The plant, sensor and reference generator are given by (1)–(3) where

$$\begin{aligned} A_p &= -\alpha & B_p &= \beta - \alpha \\ C_p &= 1 & D_p &= 1 \\ A_s &= 0 & B_s &= 1 \\ C_s &= 1 & D_s &= 0 \\ A_r &= 1 & & \\ C_r &= 1 & & \end{aligned}$$

where α and β are real numbers such that $\alpha \neq \beta$. Then (5)–(9) gives

$$\begin{aligned} A &= 1 + \alpha d & B &= 1 + \beta d & C &= x_0 \\ D &= 1 & E &= d & F &= v_0 \\ G &= 1 - d & H &= w_0 . \end{aligned}$$

Applying Theorem 1, deadbeat tracking is possible if and only if $\beta \neq -1$. To obtain the deadbeat controllers, set

$$\begin{aligned} B_x &= 1 & B' &= 1 + \beta d \\ D_x &= 1 & D' &= 1 \end{aligned}$$

and solve equations (17)–(18). This gives the controllers in the parametric form (21) as

$$P = 1 + \frac{\alpha\beta}{\alpha - \beta} d + (1 + \beta d) dV$$

$$Q = -\frac{\alpha^2}{\alpha - \beta} - (1 + \alpha d) V$$

$$R = \frac{1}{1 + \beta} + (1 - d) W$$

and

$$T = \frac{\beta}{1 + \beta} - (1 - \beta d) W$$

for any polynomial V and W. The shortest error sequence is obtained for $V = 0$, $W = 0$; namely, if the controller is realized as in (4) with

$$A_c = -\frac{\alpha\beta}{\alpha - \beta} \quad B_{1c} = -\alpha\beta\left(\frac{\alpha}{\alpha - \beta}\right)^2 \quad B_{2c} = -\frac{1}{1 + \beta}\frac{\alpha\beta}{\alpha - \beta}$$

$$C_c = 1 \qquad D_{1c} = \frac{\alpha^2}{\alpha - \beta} \qquad D_{2c} = \frac{1}{1 + \beta}$$

then (19) gives

$$e = -x_0\left(1 + \frac{\alpha\beta}{\alpha - \beta}d\right) - v_0\frac{\alpha^2}{\alpha - \beta}(1 + \beta d) - s_0(1 + \beta d) + w_0\frac{\beta}{1 + \beta}.$$

If $|\beta| < 1$, however, one can take

$$B_x = 1 + \beta d \qquad B' = 1$$
$$D_x = 1 \qquad D' = 1$$

and another solution is possible. Equations (17)–(18) now give the triple (21)

$$P = 1 + \beta d + (1 + \beta d)\, dV$$
$$Q = -\alpha - (1 + \alpha d)\, V$$
$$R = 1 + (1 - d)\, W$$

and

$$T = -W$$

for any polynomials V and W. The shortest error sequence is obtained again for $V = 0$, $W = 0$. If the controller is realized as in (4) with

$$A_c = -\beta \qquad B_{1c} = -\alpha\beta \qquad B_{2c} = -\beta$$
$$C_c = 1 \qquad D_{1c} = \alpha \qquad D_{2c} = 1$$

then

$$e = -x_0 - \alpha v_0 - s_0 .$$

The diference between the two cases is that the former closed loop system has three finite modes while the latter has just two plus one infinite mode associated with $1 + \beta d$, unobservable at the plant output. This makes the error sequence shorter but, of course, more sensitive to plant variations.

5. CONCLUSION

A necessary and sufficient condition for achieving a deadbeat regulation and tracking in single-input single-output linear systems has been established. The result has two distinct features: it provides a truly closed-loop solution as the controller is independent of the initial states and it allows for imperfect measurements modelled by a dynamical sensor in the feedback loop.

All deadbeat controllers have been identified in parametric form (21) in terms of two free polynomial parameters V and W for any choice of B_x and D_x in (20). Then it is a simple matter to single out those controllers that yield the shortest error sequence. They correspond to the least-degree solution of polynomial equations (17) and (18).

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