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A SOLUTION OF THE CONTINUOUS LYAPUNOV EQUATION BY MEANS OF POWER SERIES

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The discrete/continuous, stationary/nonstationary Lyapunov equation is investigated. An effective algorithm is presented for the continuous equation, based on power series analogous to those for computing the matrix exponential and its integral.

1. INTRODUCTION

The equation dealt with in this paper bears the name of the famous stability scientist A. M. Lyapunov. It is a linear equation for an unknown n by n symmetric matrix and has two versions:

(1) $$X(t + 1) = A' X(t) A + B\,, \quad t = 0, 1, 2, \ldots$$

for discrete systems, and

(2) $$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = A' X(t) + X(t) A + B\,, \quad t \in (0, \infty)$$

for continuous systems. Here A is a given stable matrix, i.e. all its eigenvalues α_i satisfy $|\alpha_i| < 1$ or $\operatorname{Re} \alpha_i < 0$, respectively, and B is a given symmetric matrix. This difference/differential equation is to be solved with a given initial condition $X(0)$. Of special interest is a limit for $t \to \infty$; it is a solution of the stationary Lyapunov equation

(3) $$A'XA - X + B = 0$$

or

(4) $$A'X + XA + B = 0\,.$$

The latter equation is investigated in matrix algebra books [1], [2] as a special case of the non-symmetric equation

(5) $$A_1X + XA_2 + B = 0\,.$$

The structure of solution of (5) is exhibited using elementary divisors or Jordan forms of A_1, A_2, see also [3]. For (4), the solution always exists and is unique, due to stability of A. The same holds for (3). Moreover, if B is positive semi-definite then X has also that property.

The main application of the Lyapunov equation is in the stability theory. Furthermore, it has connections with the Riccati equation for the problem of a quadratically optimal linear control system.

As the equation contributes to the most fundamental problems in the linear system theory, its theory is well elaborated and plenty of numerical algorithms is known; for a survey, see [4], [5]. The straightforward solution of (3) or (4) as a set of $n(n+1)/2$ linear equations was described in [6], [8]. It suffers from enormous computational complexity: $\sim n^4$ storage, $\sim n^6$ operations. Several approaches have been suggested to avoid that problem, e.g. [9], [10], [11]; some of them reach $\sim n^4$ or $\sim n^3$ operations. Very attractive methods are those which use the characteristic polynomial of A, see [12], [13]. Another sort of methods uses eigenvectors or Jordan forms, see [14], [5], [7]. Besides all these finite methods for solving (3), (4), there are such which perform or imitate numerical summation/integration of (1), (2), see [15]. For finding the stationary solution, the main problem is the speed of convergence: the farther are the eigenvalues of A from the stability boundary, the better. A crucial trick here is a recurrent formula between $X(t)$ and $X(2t)$ which speeds up the convergence dramatically. In the present paper, the latter approach for the continuous equation is augmented by a power series method analogous to that for computing e^{At} and $\int_0^t e^{A\varkappa}\,d\varkappa$.

2. PRELIMINARY: THE VECTOR EQUATION

The Lyapunov equation (1) or (2) can be viewed as an analogue of the "common" linear equation for an unknown vector X

$$X(t+1) = A\,X(t) + B\,, \quad t = 0, 1, 2, \ldots \tag{6}$$

or

$$\frac{dX(t)}{dt} = A\,X(t) + B\,, \quad t \in (0, \infty) \tag{7}$$

whose solution is

$$X(t) = F(t)\,X(0) + G(t)\,, \tag{8}$$

$$F(t) = A^t\,, \tag{9}$$

$$G(t) = \sum_{k=0}^{t-1} A^k B = (A^t - I)(A - I)^{-1} B \tag{10}$$

or

$$F(t) = e^{At}\,, \tag{11}$$

$$G(t) = \int_0^t e^{A\varkappa} B\,d\varkappa = (e^{At} - I)\,A^{-1} B\,. \tag{12}$$

When solving the discrete equation (6) numerically, we can proceed recurrently:

$$F(1) = A\,, \qquad G(1) = B\,, \tag{13}$$

$$F(2t) = F^2(t)\,, \tag{14}$$

$$G(2t) = G(t) + F(t)\,G(t)\,. \tag{15}$$

This yields the solution for $t = 2^m$, $m = 0, 1, 2, \ldots$. When the solution for a given general t is required, the procedure gets slightly more complicated but works equally well. The algorithm reads:

a) express the number t in the binary system:

$$t = \sum_{i=0}^{L} d_i\, 2^i\,, \quad d_i = \left\langle \begin{matrix} 0 \\ 1 \end{matrix} \right. \qquad d_L = 1\,; \tag{16}$$

b) construct a sequence Q_i for $i = 0, \ldots, L$ according to the recurrent formula

$$Q_0 = A\,, \quad Q_{i+1} = Q_i^2\,; \tag{17}$$

c) construct a sequence R_i for $i = 0, \ldots, L$ according to the recurrent formula

$$R_0 = I\,, \quad R_{i+1} = \left\langle \begin{matrix} (I + Q_i)\,R_i & \ldots d_i = 0\,, \\ Q_i(I + Q_i)\,R_i & \ldots d_i = 1\,; \end{matrix} \right. \tag{18}$$

d) the result is:

$$F(t) = \prod_{\substack{i=0 \\ d_i=1}}^{L} Q_i\,, \quad G(t) = \Big(\sum_{\substack{i=0 \\ d_i=1}}^{L} R_i\Big)\,B\,. \tag{19}$$

E.g. for $t = 21 = 10\,101_2 = 1 + 4 + 16$, we compute successively A^2, A^4, A^8, A^{16} and

$$\begin{aligned} & I + A \\ & A(I + \ldots + A^3) = A(I + A)(I + A^2) \\ & A(I + \ldots + A^7) = A(I + A)(I + A^2)(I + A^4) \\ A\,.\,A^4 & (I + \ldots + A^{15}) = A\,.\,A^4(I + A)(I + A^2)(I + A^4)(I + A^8)\,. \end{aligned}$$

Then

$$\begin{aligned} F &= A^{21} = A\,.\,A^4\,.\,A^{16} \\ G &= [I + A + \ldots + A^{20}]\,B \\ &= [I + A(I + \ldots + A^3) + A\,.\,A^4(I + \ldots + A^{15})]\,B\,. \end{aligned}$$

By that, the discrete equation (6) is solved for a finite t. For $t \to \infty$ (we use only $t = 2^m$, $m \to \infty$) the process converges very rapidly to the solution of the stationary equation

$$(A - I)\,X + B = 0\,. \tag{20}$$

The speed of convergence in this case is termed quadratic; the error is squared in each step:

$$\delta_{m+1} \sim (\delta_m)^2\,, \quad \delta_m \sim (\delta_0)^{2^m}\,. \tag{21}$$

For the continuous equation (7), the recurrent formulae (14), (15) hold for all real $t > 0$. But the initial condition (13) does not apply. For a given t, to get the process started, we must have computed $F(\tau)$, $G(\tau)$ for some small τ, coupled with t by

(22) $$t = 2^m \tau\,, \quad m = 0, 1, 2, \ldots.$$

It can be done by means of the Taylor series

(23) $$F(\tau) = \sum_{k=0}^{\infty} \frac{(A\tau)^k}{k!}\,,$$

(24) $$G(\tau) = \tau \sum_{k=0}^{\infty} \frac{(A\tau)^k}{(k+1)!} B\,.$$

They are convergent for any A; the speed of convergence is $\sim 1/k!$ i.e. very rapid. But note that $1/k!$ is multiplied by $(A\tau)^k$. This factor may grow with k when $\|A\tau\| > 1$ what may cause numerical troubles even though the increase is finally overpowered by the factorial decrease. On that ground, we choose in (22) the smallest nonnegative m for which the matrix $A\tau$ is "normalized", i.e. $\|A\tau\| < 1$. Here $\|\cdot\|$ stands for any matrix norm suitable for calculation, say $\|A\| = \max\limits_i \sum\limits_{j=1}^{n} |A_{ij}|$.

Moreover, the required number of terms K for the desired precision ε can be estimated. The remainder of $F(\tau)$ is given by the Taylor theorem:

(25) $$R_K = \frac{e^{A\vartheta\tau} (A\vartheta\tau)^K}{K!} < \varepsilon\,.$$

With $\|A\tau\| < 2^{-h}$, $h = 0, 1, 2, \ldots$ and with the use of $\|e^A\| \leqq e^{\|A\|}$ we have the condition

(26) $$\frac{e}{K!\, 2^{Kh}} < \varepsilon\,.$$

With $\varepsilon = 2^{-L}$, where L is the word length of the computer, we do our best by taking the smallest K satisfying (26). The values of K and m for various $\|At\|$ are presented in Table 1 for $L = 24$ (the 4-byte floating point format) and for $L = 56$ (the 8-byte format).

When less than full precision is needed, some reserve should be added on the following ground. The exponentiation is followed by m-times squaring, according to (14). Each squaring may result in loss of $\frac{1}{2}$ (relative) precision, i.e. of 1 bit. This is a limitation of the method.

The series for $G(\tau)$ can be estimated similarly, one term less is needed. As

(27) $$F(\tau) = I + A\tau \sum_{k=0}^{\infty} \frac{(A\tau)^k}{(k+1)!}\,,$$

only one of the two series must be actually summed.

For computing the polynomial

$$G(\tau) = \sum_{k=0}^{K} \gamma_k (A\tau)^k B \tag{28}$$

Table 1.

$\|At\|$	$L = 24$		$L = 56$	
	K	m	K	m
0	0	0	0	0
0·0001	1	0	4	0
0·001	2	0	5	0
0·01	3	0	7	0
0·1	5	0	10	0
0·2	6	0	12	0
0·5	11	0	19	0
1	11	1	19	1
2	11	2	19	2
4	11	3	19	3
10	11	4	19	4
100	11	7	19	7
1 000	11	10	19	10
10 000	11	14	19	14

of the matrix $A\tau$, the Horner scheme is used:

$$G_{-1}(\tau) = 0 \tag{29}$$

$$G_i(\tau) = (A\tau)\, G_{i-1}(\tau) + \gamma_{K-i} B\,, \quad i = 0, 1, \ldots, K \tag{30}$$

$$G_K(\tau) = G(\tau)\,. \tag{31}$$

By that, the numerical solution of (7) is concluded. Instead of the Taylor polynomial another approximation could be used, e.g. that of Chebyshev. Note that the use of a polynomial to approximate e^{At} is essentially the same as the numerical integration of (7).

For $t \to \infty$, the solution converges quadratically to that of the stationary equation

$$AX + B = 0\,. \tag{32}$$

3. THE LYAPUNOV EQUATION

Now we shall modify the results of the previous chapter for the Lyapunov equation (1) or (2). The solution is

$$X(t) = F'(t)\, X(0)\, F(t) + G(t)\,, \tag{33}$$

$$F(t) = A^t\,, \tag{34}$$

$$G(t) = \sum_{k=0}^{t-1} A'^k B A^k, \quad t = 0, 1, 2, \ldots \tag{35}$$

or

(36) $$F(t) = \mathrm{e}^{At},$$

(37) $$G(t) = \int_0^t \mathrm{e}^{A'\varkappa} B\, \mathrm{e}^{A\varkappa}\, \mathrm{d}\varkappa, \quad t \in (0, \infty).$$

Unlike (10), (12), no closed expression exists for $G(t)$. Nevertheless, the numerical solution can be performed like that for the vector equation (6) or (7). The recurrent formula now reads

(38) $$F(1) = A, \qquad G(1) = B,$$

(39) $$F(2t) = F^2(t),$$

(40) $$G(2t) = G(t) + F'(t)\, G(t)\, F(t).$$

It yields the solution of the discrete equation (1).

For the continuous equation (2), we must compute the solution for some $\tau > 0$. Let us introduce an operator $\mathscr{A}$ in the space of symmetric matrices

(41) $$\mathscr{A}B = A'B + BA$$

(the Lyapunov operator). It is a special (symmetry preserving) tensor of rank 4. Its algebraic properties, i.e. its eigenvalues, elementary divisors and canonical forms were investigated in [3], [16]. For our purposes, note only the powers of $\mathscr{A}$

(42) $$\mathscr{A}^k B = \sum_{q=0}^{k} \binom{k}{q} A'^{k-q} B A^q$$

and the exponential of $\mathscr{A}$

(43) $$\mathrm{e}^{\mathscr{A}t} B = \mathrm{e}^{A't} B\, \mathrm{e}^{At}.$$

With the help of $\mathscr{A}$, the equation (2) and its solution can be written as

(44) $$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \mathscr{A}\, X(t) + B,$$

(45) $$X(t) = \mathscr{F}(t)\, X(0) + G(t),$$

(46) $$\mathscr{F}(t) = \mathrm{e}^{\mathscr{A}t}, \quad G(t) = \int_0^t \mathrm{e}^{\mathscr{A}\varkappa} B\, \mathrm{d}\varkappa$$

which is formally the same as (7), (8), (11), (12). The Taylor series (23), (24) carry over as well:

(47) $$\mathscr{F}(\tau) = \sum_{k=0}^{\infty} \frac{(\mathscr{A}\tau)^k}{k!},$$

(48) $$G(\tau) = \tau \sum_{k=0}^{\infty} \frac{(\mathscr{A}\tau)^k}{(k+1)!}\, B.$$

These two series can be used for computation. There is no need to resort to the tensorial form of $\mathscr{A}$: the polynomial of $\mathscr{A}$ applied to the matrix B

$$G(\tau) = \sum_{k=0}^{K} \gamma_k (\mathscr{A}\tau)^k B \tag{49}$$

can be computed by the Horner scheme

$$G_{-1}(\tau) = 0\,, \tag{50}$$

$$G_i(\tau) = (\mathscr{A}\tau)\, G_{i-1}(\tau) + \gamma_{K-i} B\,, \quad i = 0, 1, \ldots, K \tag{51}$$

$$G_K(\tau) = G(\tau)\,. \tag{52}$$

For computation, (51) reads

$$G_i(\tau) = (A'\tau)\, G_{i-1}(\tau) + G_{i-1}(\tau)\,(A\tau) + \gamma_{K-i} B\,. \tag{53}$$

As for the norm $\|\mathscr{A}\|$, needed for the choice of τ, we use

$$\|\mathscr{A}\| = \sup_B \frac{\|\mathscr{A}B\|}{\|B\|}\,. \tag{54}$$

But $\|\mathscr{A}B\| = \|A'B + BA\| \leqq 2\|A\| \,.\, \|B\|$ and we can take $\|\mathscr{A}\| = 2\|A\|$.

The operator $\mathscr{F}(\tau)$ is best computed by means of (43). Having obtained $\mathscr{F}(\tau)$, $G(\tau)$, we transit to $\mathscr{F}(t)$, $G(t)$ by applying m-times the formulae

$$\mathscr{F}(2t) = \mathscr{F}^2(t)\,, \tag{55}$$

$$G(2t) = G(t) + \mathscr{F}(t)\, G(t)\,. \tag{56}$$

Actually we use (39), (40) and (43).

Like in the case of vector equation, for $t \to \infty$, the solution converges quadratically to that of the stationary equation, namely (3) or (4). This process can be also performed numerically.

4. CONCLUSION

The recurrent formula for linear time-invariant systems is known as are the properties of the Lyapunov operator $\mathscr{A}$. What is claimed to be new is the computation of $e^{\mathscr{A}t}$ and $\int_0^t e^{\mathscr{A}\varkappa}\, d\varkappa$ including the utilization of $\|\mathscr{A}\|$ to control the computations. The chain of the three following steps:

- pre-normalization
- Taylor polynomial
- post-recurrent formula

is sufficient to get an effective algorithm for solving the continuous Lyapunov equation.

The algorithm was implemented on a computer and tested. The experience from matrix computation e^{At}, $\int_0^t e^{At}\, d\varkappa$ was utilized as well as the software developed during that research [17]. All cases discrete/continuous, stationary/nonstationary were solved. Of course the critical case occurs when an eigenvalue of A approaches the boundary of stability. The convergence is slowed down but due to the quadratic rate of convergence, such cases are still manageable: they result quickly either in settling or in overflow. Moreover, the results are very error-sensitive in this case, as can be seen from the following example:

$$A'X + XA + B = 0$$

$$A = \begin{bmatrix} -0{\cdot}01 & 1 & 0 \\ 0 & -0{\cdot}01 & 1 \\ 0 & 0 & -0{\cdot}01 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 150 & 7\,600 & 380\,050 \\ 7\,600 & 760\,150 & 57\,010\,100 \\ 380\,050 & 57\,010\,100 & 5\,701\,010\,150 \end{bmatrix}$$

If the computer precision is about 5 decimal digits then the result can be computed correctly up to all 5 digits but such a truncated result satisfies the equation very badly; the exact solution depends on higher digits cancellation. This is the property of the equation near a singularity, not of the numerical method for solving the equation.

To summarize, the method is reliable, simply to implement and the computational complexity remains reasonably low.

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