ON A SIMULATION OF THE OSCILLATION EXCITED BY A RANDOM FORCE

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We propose a numerical method for simulation of the weak solution of a system of stochastic differential equations. The essential features of the system are the discontinuity of the drift-vector and the singularity of the diffusion-matrix. Such systems could find applications when describing the behaviour of several interacting mechanical systems including external random forces. We restrict ourselves to one such example given in Fig. 1, and use the results of [1] for a rough comparison with our results.

1. FORMULATION OF THE PROBLEM

Our note is motivated by investigation of the oscillations of a two-mass-system in [1]. The system is depicted in Fig. 1 and describes the oscillations of masses M_1 and M_2 representing the wheel and the corresponding sprung part of the vehicle which moves on a pavement with random unevennesses. The unevennesses are represented by a stationary Gaussian process with non-smooth trajectories.

The interaction between M_1 and M_2 includes the dry friction and thus it leads

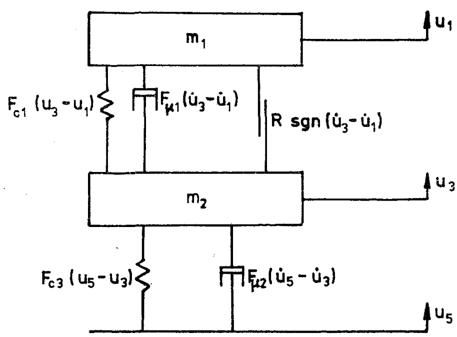

Fig. 1.

to the discontinuity of the coefficients of the system of equations describing the problem (see (1) below).

The author of [1] uses a difference method for ordinary differential equations to generate a trajectory of the solution of (1). The two mentioned peculiarities of the system are not considered. However the numerical results seem to be quite satisfactory.

Our first aim was to find some exact interpretation of (1). We shall interpret it as a system of stochastic differential equations (see (2), or (3) for a more general formulation) and we discuss the problem of the existence of the solution.

The main aim of our investigations is to find a simulation method which will respect discontinuity of the coefficients. We propose a procedure leading to such a method.

To make sure that the (numerical) method developed by our procedure leads to reasonable results, we use it for solution of the system (2) which represents a reformulation of (1). Then we compare the results of [1] with our results.

From this reason we introduce now a short description of the problem taken over from [1].

The coordinates corresponding to the centers of gravity of the masses M_1 and M_2 are denoted by u_1, u_3, respectively, and the coordinate describing the unevenness of the pavement is denoted by u_5. The values $u_1 = u_3 = u_5 = 0$ correspond to the equilibrium state of the system, and the coordinates thus describe the deviations from this state where the deviations upwards are positive . The dots represent some, not exactly specified, time derivatives.

The connection between the masses M_1, M_2 includes a viscous damping $(F_{\mu 1}(\dot{u}_3 - \dot{u}_1))$, a spring element $(F_{c1}(u_3 - u_1))$, and especially a dry friction $(R \,.\, \operatorname{sgn}(\dot{u}_3 - \dot{u}_1)$, $R \geqq 0$, $\operatorname{sgn}(x) = 1$ for $x > 0$, $\operatorname{sgn}(0) = 0$, $\operatorname{sgn}(x) = -1$ for $x < 0)$. The connection between the mass M_1, representing the wheel, and the pavement includes a viscous damping $(F_{\mu 2}(\dot{u}_5 - \dot{u}_3))$ and a spring element $(F_{c2}(u_5 - u_3))$.

The stationary Gaussian process u_5 is described by its spectral density function $\sigma^2(\alpha/\pi)(\omega^2 + \alpha^2)^{-1}$. Here α, and σ are some positive constants with corresponding physical dimensions. As usual, g denotes the acceleration of gravity. The forward speed of the vehicle is denoted by v_0.

The information about the system is summarized in [1] using the following system of equations:

$$
\begin{aligned}
&\dot{u}_1 = u_2 \\
&\dot{u}_2 = M_1^{-1}[F_{c1}(u_3 - u_1) + F_{\mu 1}(u_4 - u_2) + R \,.\, \operatorname{sgn}(u_4 - u_2)] - g \\
(1)\qquad &\dot{u}_3 = u_4 \\
&\dot{u}_4 = M_2^{-1}[F_{c2}(u_5 - u_3) + F_{\mu 2}(\dot{u}_5 - u_4) - F_{c1}(u_3 - u_1) - \\
&\qquad - F_{\mu 1}(u_4 - u_2) - R \,.\, \operatorname{sgn}(u_4 - u_2)] - g \\
&\dot{u}_5 = -\alpha v_0 u_5 + \sigma \sqrt{(2\alpha v_0)}\, \eta(t)\,,
\end{aligned}
$$

where η is the white noise.

2. A SYSTEM OF STOCHASTIC DIFFERENTIAL EQUATIONS

We are going now to rewrite (1) in the form of the system of stochastic differential equations. We shall suppose that $F_{\mu 2}(x) = \mu_2 . x$ for some positive constant μ_2. With respect to a remark in [1] (see p. 144, lines 10–19 from below) this assumption is reasonable. Let $z = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{R}^5$. We introduce the following denotations:

$$
\begin{aligned}
a_1(z) &= z_2 \\
a_2(z) &= M_1^{-1}[F_{c1}(z_3 - z_1) + F_{\mu 1}(z_4) + R \operatorname{sgn}(z_4)] - g \\
a_3(z) &= z_4 + z_2 \\
a_4(z) &= \{M_2^{-1}[F_{c2}(z_5 - z_3) + \mu_2(-\alpha v_0 z_5 - z_4 - z_2) - F_{c1}(z_3 - z_1) - \\
&\quad - F_{\mu 1}(z_4) - R \operatorname{sgn}(z_4)] - g\} - a_2(z) \\
a_5(z) &= -\alpha v_0 z_5 \\
b_1 &= b_2 = b_3 = 0 \\
b_4 &= \mu_2 M_2^{-1} \sigma \sqrt{(2\alpha v_0)} \\
b_5 &= \sigma \sqrt{(2\alpha v_0)} \\
a(z) &= (a_1(z), a_2(z), a_3(z), a_4(z), a_5(z)) \\
b &= (b_1, b_2, b_3, b_4, b_5)\,.
\end{aligned}
$$

Using the substitution $z_1 = u_1$, $z_2 = u_2$, $z_3 = u_3$, $z_4 = u_4 - u_2$, $z_5 = u_5$ we see that (1) can be rewritten as

$$\mathrm{d}z = a(z)\,\mathrm{d}t + b\,\mathrm{d}w(t)\,. \tag{2}$$

Here, w is the standard Wiener process in five-dimensional space, and (2) is thus a system of stochastic differential equations with discontinuous coefficients (the drift-coefficient a is discontinuous), and with the singular diffusion.

We summarize several known results or their slight generalizations in Theorems 1 and 2, and in Corollary in the last section of this note. Our main motivation which leads us to the derivation of the numerical method comes from the following corollaries of those results.

Although it is not the case in our problem, we shall suppose in all what follows that the coefficients a_i, $i = 1, 2, \ldots, 5$, are bounded functions from $\mathbb{R}^5$ to $\mathbb{R}^5$.

Proposition 1. For any $z^0 \in \mathbb{R}^5$ there is a continuous weak solution $z(t)$ of (2) on $[0, T]$.

Before we state Proposition 2 let us introduce some denotations.

Let $v: \mathbb{R}^5 \to \mathbb{R}$ be such that

(a) $v, (\partial/\partial z_i)\, v, (\partial^2/\partial z_i\, \partial z_j)\, v$ are bounded continuous functions on $\mathbb{R}^5$ if $i, j = 1, \ldots, 5, (i, j) \neq (4, 4)$.

(b) $(\partial^2/\partial z_4^2)\, v$ is bounded continuous in the half-spaces $H_- = \{z; z_4 < 0\}$ and

$H_+ = \{z; z_4 > 0\}$, and the limits $\lim_{\substack{z \to z^0 \\ z \in H_\pm}} (\partial^2/\partial z_4^2)\, v(z) \equiv (\partial^2/\partial z_4^2)\, v_\pm(z^0)$ exist for $z^0 = (z_1^0, z_2^0, z_3^0, 0, z_5^0)$.

(c) The function $\sum_{i=1}^{5} a_i(z)\,(\partial v/\partial z_i)\,(z) + (\frac{1}{2} \sum_{i,j=1}^{5} b_i b_j (\partial^2 v/\partial z_i\, \partial z_j)\,(z)$ is continuously extendable to $\mathbb{R}^5$, i.e. $(\partial^2/\partial z_4^2)\, v_+(z) - (\partial^2/\partial z_4^2)\, v_-(z) = 4b_4^{-2} R[M_2^{-1}(\partial v/\partial z_4)\,(z) + M_1^{-1}(\partial v/\partial z_4)\,(z) - M_1^{-1}(\partial v/\partial z_2)\,(z)$ for $z = (z_1, z_2, z_3, 0, z_5)$, and let us denote the extension by $\mathsf{D}v$.

We shall use the denotation $v \in \mathscr{V}$ for v: $\mathbb{R}^5 \to \mathbb{R}$ fulfilling (a), (b), and (c).

Proposition 2. Let $z(t)$ be a weak solution of (2) on $[0, T]$ with $z(0) = z^0$ a.e. Let $v \in \mathscr{V}$. Then

$$\lim_{h \to 0+} \mathsf{E}((v(z(h)) - v(z^0))/h) = \mathsf{D}v(z^0)\,.$$

Remark. Notice that $\mathsf{D}v$ is continuous in the case that $\partial v/\partial z_2 = \partial v/\partial z_4 = 0$ on $\{z; z_4 = 0\}$ and $v \in C^2(\mathbb{R}^5)$. We tired to obtain a larger class of functions v with $\mathsf{D}v$ continuous. It is obvious that there are no other functions from $C^2(\mathbb{R}^5)$ with this property. This is the reason for the use of the Itô formula with generalized derivatives in Section 5 which leads to the preceding Proposition 2.

3. DERIVATION OF A METHOD WORKING NEAR TO THE DISCONTINUITY

We shall look for probabilities P_{h,z^0} on $\mathbb{R}^5$ approximating the distribution of $z(h)$ for z being a solution with $z(0) = z^0$. Our main requirement on P_{h,z^0} is that

$$(*) \qquad \int (v(z) - v(z^0))\, P_{h,z^0}(\mathrm{d}z) - \mathsf{D}v(z^0) = o(h)\,, \quad h \to 0_+\,,$$

for $v \in \mathscr{V}$. We shall try to fulfil $(*)$ with Gaussian probabilities P_{h,z^0} such that

$$(**) \qquad \int (z_i - z_i^0)\, P_{h,z^0}(\mathrm{d}z) = O(h)\,, \quad h \to 0_+, i = 1, \ldots, 5, \quad \text{and}$$

$$\int (z_i - z_i^0)\,(z_j - z_j^0)\, P_{h,z^0}(\mathrm{d}z) = O(h)\,, \quad h \to 0_+\,, \quad i, j = 1, \ldots, 5\,.$$

Let $v \in \mathscr{V}$, and we shall express the integral in the left-hand side of $(*)$.

$$\int (v(z) - v(z^0))\, P_{h,z^0}(\mathrm{d}z) = \sum_{i=1}^{5} (\partial v/\partial z_i)\,(z^0) \int (z_i - z_i^0)\, P_{h,z^0}(\mathrm{d}z) +$$
$$+ \sum_{i,j=1}^{5} \int \int_0^1 (\partial^2 v/\partial z_i\, \partial z_j)\,(z^0 + \tau(z - z^0))\,(z_i - z_i^0)\,(z_j - z_j^0)\,(1 - \tau)\,\mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z) =$$
$$= \sum_{i=1}^{5} (\partial v/\partial z_i)\,(z^0) \int (z_i - z_i^0)\, P_{h,z^0}(\mathrm{d}z) +$$

$$+ \tfrac{1}{2} \sum_{i,j=1}^{5} (\partial^2 v/\partial z_i\, \partial z_j)(z^0) \int (z_i - z_i^0)(z_j - z_j^0)\, P_{h,z^0}(\mathrm{d}z) +$$
$$+ \sum_{i,j=1}^{5} \int\int_0^1 [(\partial^2 v/\partial z_i\, \partial z_j)(z^0 + \tau(z - z^0)) -$$
$$- (\partial^2 v/\partial z_i\, \partial z_j)(z^0)]\,(z_i - z_i^0)(z_j - z_j^0)(1 - \tau)\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z)\,.$$

Now we interrupt the development of the integral from $(*)$, and state a lemma which we shall use several times in what follows.

Lemma. Let $\varphi(\tau, z)$ be a bounded measurable function defined for $\tau \in [0, 1]$ and $z \in \mathbb{R}^5$. Let $\lim_{z \to z^0} \varphi(\tau, z) = 0$ uniformly in τ, and let P_{h,z^0} fulfil $(**)$. Then

$$\lim_{h \to 0+} (1/h) \int\int_0^1 \varphi(\tau, z)(z_i - z_i^0)(z_j - z_j^0)(1 - \tau)\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z) = 0\,.$$

Proof. For a given $\varepsilon > 0$ we find $\delta > 0$ such that for any $\tau \in [0, 1]$ and $|z - z^0| < \delta$ the inequality $|\varphi(\tau, z)| < \varepsilon$ holds. Further we can find $h > 0$ such that $\sqrt[3]{(h)} < \delta$. Then

$$\int_{|z - z^0| < h^{\frac{1}{3}}} \int_0^1 \varphi(\tau, z)(z_i - z_i^0)(z_j - z_j^0)(1 - \tau)\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z) \leqq$$
$$\leqq \int 2\varepsilon(|z_i - z_i^0|^2 + |z_j - z_j^0)\, P_{h,z^0}(\mathrm{d}z) = \varepsilon\,.\, O(h)\,,$$

we have used $(**)$ in the last equality, and

$$\left|\int_{|z - z^0| \geqq h^{\frac{1}{3}}} \int_0^1 \varphi(\tau, z)(z_i - z_i^0)(z_j - z_j^0)(1 - \tau)\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z)\right| \leqq$$
$$\leqq 2 \sup |\varphi(\tau, z)| \int_{|z - z^0| \geqq h^{\frac{1}{3}}} (|z_i - z_i^0|^2 + |z_j - z_j^0|^2)\, P_{h,z^0}(\mathrm{d}t) \leqq$$
$$\leqq 8 \sup |\varphi(\tau, z)| \left(\max_i \int_{|z - \mathsf{E}z| \geqq h^{\frac{1}{3}} - Lh} |z_i - \mathsf{E}z_i|^2\, P_{h,z^0}(\mathrm{d}z) + |\mathsf{E}z - z^0|^2\right) \leqq$$
$$\leqq 8 \sup |\varphi(\tau, z)| \max_i \int_{|z - \mathsf{E}z| \geqq Mh^{\frac{1}{3}}} |z_i - \mathsf{E}z_i|^2\, P_{h,z^0}(\mathrm{d}z) + o(h) = o(h)\,,$$

where L, M are constants and E denotes the mean value with respect to P_{h,z^0}. □

We return now to the estimation of $\int(v(z) - v(z^0))\, P_{h,z^0}(\mathrm{d}z)$. The term

$$\int\int_0^1 [(\partial^2 v/\partial z_i\, \partial z_j)(z^0 + \tau(z - z^0)) - (\partial^2 v/\partial z_i\, \partial z_j)(z^0)]\,.$$
$$.\,(z_i - z_i^0)(z_j - z_j^0)(1 - \tau)\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z) = o(h)$$

according to the lemma for $(i, j) \neq (4, 4)$. It suffices to put $\varphi(\tau, z) = [(\partial^2 v/\partial z_i\, \partial z_j)\,.\,(z^0 + \tau(z - z^0)) - (\partial^2 v/\partial z_i\, \partial z_j)(z^0)](1 - \tau)$ for this purpose.

For $i = j = 4$ the derivative $\partial^2 v/\partial z_4^2$ differs from a continuous function on the segment $\{z^0 + \tau(z - z^0);\ \tau \in [0, 1]\}$ by a function which equals zero for $\tau \in [0, \tau_0]$, and

$$(-\operatorname{sgn} z_4^0)\, 4b_4^{-2} R[M_2^{-1}(\partial v/\partial z_4)(z^0 + \tau(z - z^0)) + \\ + M_1^{-1}(\partial v/\partial z_4)(z^0 + \tau(z - z^0)) - M_1^{-1}(\partial v/\partial z_4)(z^0 + \tau(z - z^0))] \\ \text{on} \quad (\tau_0, 1],$$

where $\tau_0 = -z_4^0/(z_4 - z_4^0)$ if $z_4 \neq z_4^0$ and $-z_4^0/(z_4 - z_4^0) \in [0, 1]$. We put $\tau_0 = 1$ otherwise. Using the lemma with $\varphi(\tau, z) =$ [the continuous part of $\partial^2 v/\partial z_4^2$ in $z^0 + \tau(z - z^0)$ minus the same function in z^0] $(1 - \tau)$ we obtain that

$$\sum_{i,j=1}^{5} \int\int_0^1 [(\partial^2 v/\partial z_i\, \partial z_j)(z^0 + \tau(z - z^0)) - (\partial^2 v/\partial z_i\, \partial z_j)(z^0)] \,. \\ .\, (z_i - z_i^0)(z_j - z_j^0)(1 - \tau)\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z) = o(h) + (-\operatorname{sgn} z_4^0)\,. \\ .\int\int_{\tau_0}^1 4b_4^{-2} R[M_2^{-1}(\partial v/\partial z_4)(z^0 + \tau(z - z^0)) + M_1^{-1}(\partial v/\partial z_3)(z^0 + \tau(z - z^0)) - \\ - M_1^{-1}(\partial v/\partial z_4)(z^0 + \tau(z - z^0))](z_4 - z_4^0)^2 (1 - \tau)\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z)\,.$$

Since

$$\left|\int\int_{\tau_0}^1 4b_4^{-2} R\{[M_2^{-1}(\partial v/\partial z_4)(z^0 + \tau(z - z^0)) + \right. \\ + M_1^{-1}(\partial v/\partial z_4)(z^0 + \tau(z - z^0)) - M_1^{-1}(\partial v/\partial z_4)(z^0 + \tau(z - z^0))] - \\ - [M_2^{-1}(\partial v/\partial z_4)(z^0) + M_1^{-1}(\partial v/\partial z_4)(z^0) - M_1^{-1}(\partial v/\partial z_2)(z^0)]\}\,. \\ \left. .\, (z_4 - z_4^0)^2 (1 - \tau)\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z)\right| \leqq \\ \leqq \int\int_0^1 \varphi(\tau, z)(z_4 - z_4^0)^2\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z)\,, \quad \text{where} \quad \varphi(\tau, z) = \\ = |4b_4^{-2} R\{[\ldots] - [\ldots]\}|\,(1 - \tau)$$

is the absolute value of the corresponding part of the integrand, we use the lemma to obtain

$$\int\int_0^1 [(\partial^2 v/\partial z_4^2)(z^0 + \tau(z - z^0)) - (\partial^2 v/\partial z_4^2)(z^0)](z_4 - z_4^0)^2\,. \\ .\,(1 - \tau)\, \mathrm{d}\tau\, P_{h,z^0}(\mathrm{d}z) = o(h) + (-\operatorname{sgn} z_4^0)\, 4b_4^{-2} R[M_2^{-1}(\partial v/\partial z_4)(z^0) + \\ + M_1^{-1}(\partial v/\partial z_4)(z^0) - M_1^{-1}(\partial v/\partial z_2)(z^0)]\,.$$

We can compute that

$$\int\left((z_4 - z_4^0)^2 \int_{\tau_0}^1 (1 - \tau)\, \mathrm{d}\tau\right) P_{h,z^0}(\mathrm{d}z) = \frac{1}{2}\int ([z_4 \operatorname{sgn} z_4^0]^-)^2\, P_{h,z^0}(\mathrm{d}z)\,,$$

where $[a]^-$ stands for the non-positive part of a. We have just used that $(z_4 - z_4^0)^2 . \int_{\tau_0}^1 (1 - \tau)\, \mathrm{d}\tau = \frac{1}{2} z_4^2$ for $\tau_0 < 1$.

Summarizing the above we get

$$\int (v(z) - v(z^0))\, P_{h,z^0}(\mathrm{d}z) = \sum_{i=1}^{5} (\partial v/\partial z_i)(z^0) \int (z_i - z_i^0)\, P_{h,z^0}(\mathrm{d}z) +$$

$$+ \tfrac{1}{2} \sum_{i,j=4}^{5} (\partial^2 v/\partial z_i\, \partial z_j)(z^0) \int (z_i - z_i^0)(z_j - z_j^0)\, P_{h,z^0}(\mathrm{d}z) +$$

$$+ \frac{1}{2} \int ([z_4 \operatorname{sgn} z_4^0]^-)^2\, P_{h,z^0}(\mathrm{d}z)\, (-\operatorname{sgn} z_4^0)\, 4b_4^{-2}\, R[M_2^{-1}(\partial v/\partial z_4)(z^0) +$$

$$+ M_1^{-1}(\partial v/\partial z_4)(z^0) - M_1^{-1}(\partial v/\partial z_2)(z^0)] .$$

We use the following denotations for the mean values

$$\int (z_i - z_i^0)\, P_{h,z^0}(\mathrm{d}z) = m_i(h, z^0) = m_i ,$$

$$\int (z_i - z_i^0)(z_j - z_j^0)\, P_{h,z^0}(\mathrm{d}z) = m_{ij}(h, z^0) = m_{ij} .$$

To fulfil our requirement (*) we need

(a_1) $\quad m_1 = h a_1(z^0) + o(h),$

(a_2) $\quad m_{11} = o(h) ,$

(a_3) $\quad m_2 = h a_2(z^0) + 2b_4^{-2} R M_1^{-1} (-\operatorname{sgn} z_4^0) \int ([z_4 \operatorname{sgn} z_4^0]^{-2})\, P_{h,z^0}(\mathrm{d}z) + o(h) ,$

(a_4) $\quad m_{22} = o(h) ,$

(a_5) $\quad m_3 = h a_3(z^0) + o(h) ,$

(a_6) $\quad m_{33} = o(h) ,$

(a_7) $\quad m_4 = h a_4(z^0) - 2b_4^{-2} R (M_2^{-1} + M_1^{-1}) (-\operatorname{sgn} z_4^0) \int ([z_4 \operatorname{sgn} z_4^0]^-)^2 \,.$
$\quad\quad . \, P_{h,z^0}(\mathrm{d}z) + o(h) ,$

(a_8) $\quad m_{44} = h b_4^2 + o(h) ,$

(a_9) $\quad m_5 = h a_5(z^0) + o(h) ,$

(a_{10}) $\quad m_{55} = h b_5^2 + o(h) ,$

(a_{11}) $\quad m_{45} = h b_4 b_5 + o(h) ,$

(a_{12}) $\quad m_{ij} = o(h)$ for $\{i, j\} \not\subset \{4, 5\}$.

Let us choose for example

($\bar{a}$) $\quad m_1 = h a_1(z^0) , \quad m_{11} = 0 , \quad m_{22} = 0 , \quad m_3 = h a_3(z^0) , \quad m_{33} = 0 ,$
$\quad\quad m_{44} = h b_4^2 , \quad m_5 = h a_5(z^0) , \quad m_{55} = h b_5^2 , \quad m_{45} = h b_4 b_5 .$

We notice that the second moment of the non-negative part of a Gaussian variable ξ with the mean value m and the variance $\sigma^2 > 0$ equals $\mathsf{E}(\xi^+)^2 = (\sigma^2 + m^2) \, . \, \Phi(m/\sigma) + \sigma m\, \varphi(m/\sigma)$, where Φ is the distribution function and φ is the density of the standard normal distribution.

Similarly we have

$$\int([z_4 \operatorname{sgn} z_4^0]^-)^2 P_{h,z^0}(\mathrm{d}z) = ((z_4^0 + m_4)^2 + \sigma_4^2)\, \Phi((z_4^0 + m_4)/\sigma_4) + (z_4^0 + m_4)\, \sigma_4\, \varphi((z_4^0 + m_4)/\sigma_4)$$

where σ_4 is the variance of z_4 by P_{h,z^0}, and z_4^0 is supposed to be negative. For positive z_4^0 the values m_4 and z_4^0 should be interchanged by $-m_4$, $-z_4^0$ in the right-hand-side. For the value $z_4^0 = 0$ the integral equals zero.

We suppose that $z_4^0 < 0$ in the following for simplicity.

Let us notice that we can not fulfil the equations (a_3), (a_7) with $m_2 = O(h)$ and $m_4 = O(h)$. We shall consider only small $|z_4^0|$ in some sense depending on h. We denote $z_4^0 = \alpha \sqrt{(h)}$, and let us recall that we have chosen the case $\alpha < 0$.

Let us consider equations (a_3) and (a_7). We have

$$\int([z_4]^+)^2 P_{h,z^0}(\mathrm{d}z) = (\alpha^2 h + 2\alpha \sqrt{(h)}\, m_4 + m_4^2 + \sigma_4^2)\,.$$

$$.\, \Phi((\alpha \sqrt{(h)} + m_4)/\sigma_4) + (m_4 + \alpha \sqrt{(h)})\, \sigma_4\, \varphi((\alpha \sqrt{(h)} + m_4)/\sigma_4) =$$

$$= (\alpha^2 h + h b_4^2)\, \Phi(\alpha/|b_4|) + \alpha h b_4\, \varphi(\alpha/|b_4|) + o(h)$$

for $|\alpha|$ less than some constant which does not depend on h and $h \to 0_+$ according to the fact that the variance of z_4 fulfils the relation $\sigma_4 = m_{44} + O(h)$. We put

$$(\bar{a}_3) \qquad m_2 = h[a_2(z^0) + 2b_4^{-2} M_1^{-1} R\{(\alpha^2 + b_4^2)\, \Phi(\alpha/b_4) + \alpha b_4\, \varphi(\alpha/b_4)\}]$$

and

$$(\bar{a}_7) \qquad m_4 = h[a_4(z^0) - 2b_4^{-2} R(M_1^{-1} + M_2^{-1})\{(\alpha^2 + b_4^2)\, \Phi(\alpha/b_4) + \alpha b_4\, \varphi(\alpha/b_4)\}]\,.$$

Our choice of m_1, m_{ij} is thus complete, and (*), (**) are fulfilled with the limitation to $z_4^0 = \alpha \sqrt{(h)}$ with $|\alpha|$ less than some constant.

4. THE SIMULATION AND RESULTS

Let us recall that the preceding derivation leads to a choice of the probability P_{h,z^0} which should approximate the distribution of a solution at time $t + h$ given that $z(t) = z^0$ with $|z_4^0| < \alpha \sqrt{(h)}$ for $\alpha = O(1)$. In fact we deal with the solution as it would be a homogeneous Markov process. This seems to be connected with the problem of uniqueness of the solution of (2).

We shall now describe the method which we used for the simulation on the computer EC 1021. Let us mention that the program from IBM SSP was used for the generation of the normal distribution.

We choose some $h > 0$ and an integer N such that $Nh = T$ and some initial value z^0 near to the supposed mean values of the stationary initial distribution.

Having z^k for $0 \leqq k < N$ we obtain z^{k+1} in dependence on the value of z_4^k:

A. Let $|z_4^k| \leqq 2\sqrt{(h)}\,(3b_4\sqrt{(h)} \doteq 2\sqrt{(h)})$. Then we generate the Gaussian distribution derived in Section 3 ((ā), (ā$_3$), (ā$_7$) for z^k instead of z^0).

B. Let $|z_4^k| > 2\sqrt{(h)}$. We use the following expression for z^{k+1}:

$$z^{k+1} - z^k = (h/2)\,(a(z^k) + a(z^k + a(z^k)\,h + b\xi)) + b\xi\,,$$

where ξ is a Gaussian variable with mean zero and standard deviation $\sqrt{(h)}$. This is the so called Heun's method [2] and its speed of convergence for z^k to $z(kh)$ in L_2-norm is in the case of sufficiently regular coefficients the best possible one, i.e. the difference $z(kh) - z^k$ is of order $o(h)$ (cf. [3]).

The value z_5^{k+1} is generated according to its known distribution in case B (cf. [1]).

Since we suppose that the solution should be Markovian, all the generated distributions are generated "independently". The mean values are computed from one long trajectory.

The used numerical values for the method which lead to reasonable results were: $h = 0{\cdot}005$; $T = 10$; the interval used for the expression of the means was $[0{\cdot}5, 10]$.

The constants and functions in (1) were defined to be as close as possible to those used in [1].

$$
\begin{aligned}
M_1 &= 390\ [\mathrm{kg}]\\
M_2 &= 150\ [\mathrm{kg}]\\
g &= 9{\cdot}81\ [\mathrm{ms}^{-2}]\\
\alpha &= 0{\cdot}45\ [\mathrm{m}^{-1}]\\
v_0 &= 15\ [\mathrm{ms}^{-1}]\\
\sigma_y^2 &= 3{\cdot}42\ [\mathrm{cm}^2]\\
F_{c2}(a) &= c_2 a, \text{ where } c_2 = 392{\cdot}4\ [\mathrm{kNm}^{-1}]\\
F_{\mu 2}(a) &= \mu_2 a \text{ where } \mu_2 = 0{\cdot}981\ [\mathrm{kNsm}^{-1}]\\
F_{c1}(a) &= 166\,667a \text{ for } a \leqq 0{\cdot}120\\
&= 2\,000 + 90\,000(a - 0{\cdot}012) \text{ for } a \in (0{\cdot}012, 0{\cdot}052]\\
&= 5\,600 + 45\,455(a - 0{\cdot}052) \text{ for } a \in (0{\cdot}052, 0{\cdot}14]\\
&= 9\,600 + 66\,667(a - 0{\cdot}14) \text{ for } a \in (0{\cdot}14, 0{\cdot}2]\\
&= 13\,600 + 150\,000(a - 0{\cdot}2) \text{ for } a \in (0{\cdot}2, \infty)\\
F_{\mu 1}(a) &= -1\,280 + (a + 0{\cdot}28)\,.\,1\,000 \text{ for } a \leqq -0{\cdot}28\\
&= 4\,421a \text{ for } a \in (-0{\cdot}28, 0{\cdot}09]\\
&= 400 + 1\,000(a - 0{\cdot}09) \text{ for } a \in (0{\cdot}09, \infty).
\end{aligned}
$$

We have expressed only the following two values:

$$\sigma_{3,1} = \sigma_{z_3 - z_1} = \sigma_{u_3 - u_1}$$

the deviation of $z_3 - z_1 = u_3 - u_1$, and

$$\sigma_{5,3} = \sigma_{z_5 - z_3} = \sigma_{u_5 - u_3}$$

the deviation of $z_5 - z_3 = u_5 - u_3$ with respect to the stationary (initial) distribution. We compare now our results with the corresponding data reproduced again from [1].

values of R		0·00	286·9425	573·885	860·8275	1147·77
M 1	$\sigma_{3,1}$	2·68	2·14	1·73	1·41	1·18
	$\sigma_{5,3}$	1·05	0·91	0·82	0·86	1·00
M 2	$\sigma_{3,1}$	2·64	2·18	1·82	1·68	1·55
	$\sigma_{5,3}$	1·05	0·95	0·86	0·86	1·00
M 3	$\sigma_{3,1}$	2·51	2·58	1·89	2·095	1·84
	$\sigma_{5,3}$	1·13	1·08	0·998	1·07	1·07

Here M 1 denotes the method of equivalent linearization used in [1], M 2 the method of simulation used in [1], and M 3 the method which we described in this note.

One can notice that our results are systematically greater than the results given in [1]. The only exception of this appearance is for the value of $\sigma_{3,1}$ for $R = 0$. It is interesting to note that the same can be said about the simulation method (M 2) with respect to the linearization method (M 1) in [1]. The systematic deviation in our case (M 3) is approximately two times greater.

The decreasing behaviour of $\sigma_{3,1}$ in dependence on R is strongest in the case of M 1, and the weakest for M 3. The curvature of the behaviour of $\sigma_{5,3}$ is also most expressive in the case of the linearization and least expressive in the case of our method.

We can conclude that the results obtained by our method are comparable with those given in [1]. Since all discussed methods are only approximative one can hardly decide which of them gives results that are closest to the reality.

5. RESULTS USED IN SECTION 2

We use a known result for the definition of the solution of (1). For this purpose we introduce another denotation for the coefficients and the unknown process from (2):

$$\begin{aligned}
x_1 &= (x_{11}, x_{12}, x_{13}, x_{14}), \quad x = (x_1, x_2),\\
x_{11} &= z_1, \quad x_{12} = z_2, \quad x_{13} = z_3, \quad x_{14} = z_5, \quad x_2 = z_4,\\
\alpha_1(x) &= (\alpha_{11}(x), \ldots, \alpha_{14}(x)) = (a_1(z), \quad a_2(z), \quad a_3(z), \quad a_5(z)),\\
\alpha_2(z) &= a_4(z), \quad \text{where} \quad z = (x_{11}, x_{12}, x_{13}, x_2, x_{14}),
\end{aligned}$$

and

$$\beta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_5 \end{pmatrix}, \quad \beta_2 = (0, 0, 0, 0, b_4).$$

Let W denote the five-dimensional standard Wiener process. Then (2) is equivalent to

$$\text{(3)} \qquad \begin{aligned} \mathrm{d}x_1 &= \alpha_1(x_1, x_2)\,\mathrm{d}t + \beta_1\,\mathrm{d}W \\ \mathrm{d}x_2 &= \alpha_2(x_1, x_2)\,\mathrm{d}t + \beta_2\,\mathrm{d}W . \end{aligned}$$

The following Theorem 1 applies to our situation. It is formulated in [4] in our form and it can be proved following the procedures of the proof of Theorem 1 in [5] and the proof of Theorem 1 in [6], Chap. II, § 6.

Theorem 1. Let

$$\text{(4)} \qquad \begin{aligned} \mathrm{d}y_1 &= \delta_1(y_1, y_2)\,\mathrm{d}t + \sigma_1(y_1, y_2)\,\mathrm{d}V(t) \\ \mathrm{d}y_2 &= \delta_2(y_1, y_2)\,\mathrm{d}t + \sigma_2(y_1, y_2)\,\mathrm{d}V(t) \end{aligned}$$

be a system of stochastic differential equations where V is an $(m + n)$-dimensional Wiener process, y_1 and y_2 are processes from the interval $[0, T]$ to the m-, respectively n-dimensional Euclidean spaces, and the coefficients are bounded measurable functions between the corresponding spaces. Moreover, let all coefficients be continuous in y_1 and $(\sigma_2\sigma_2^*\lambda, \lambda) > \varepsilon|\lambda|^2$ for any $\lambda \in \mathbb{R}^n$ for some $\varepsilon > 0$ (σ_2^* is the adjoint operator to σ_2). Let $y_0 \in \mathbb{R}^{m+n}$.

Then there is a weak solution $y(t)$ on $[0, T]$ of (4) with $y(0) = y_0$ a.e.

The assumptions of Theorem 1 are supposed in all what follows.

We would like to use Itô formula for the functions with generalized derivatives following [6], Chap. II, § 10. The generalized derivatives are derivatives in the sense of distributions which are locally integrable functions (cf. [6], p. 68).

Given compact balls K, Γ in $\mathbb{R}^m$ or in $\mathbb{R}^n$, respectively, we define the norm $|||\cdot|||_{p,K,\Gamma}$ by $|||f(y_1, y_2)|||_{p,K,\Gamma} = (\int|\sup_{y_1 \in K} f(y_1, y_2)|^p\,\mathrm{d}y_2)^{1/p}$ for any bounded measurable f from $R^m \times \mathbb{R}^n$ to $\mathbb{R}$ such that $f(y_1, y_2)$ is continuous in y_1 for y_2 fixed (cf. [5]).

We define the norm $|||\cdot|||_{W^2(K\times\Gamma)}$ like in [6, p. 71] with $|||\cdot|||$ instead $\|\cdot\|$. The space of functions $\overline{W}^2(K \times \Gamma)$ is obtained from $C^2(K \times \Gamma)$ in the corresponsing way using $|||\cdot|||$-norms instead of $\|\cdot\|$-norms.

Theorem 2. Let K, Γ be compact balls in R^m or $\mathbb{R}^n$, respectively. Let $v \in \overline{W}^2(K \times \Gamma)$,

$$y(t) = y(0) + \int_0^t \delta(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}V(s)$$

where δ, σ are bounded progressively measurable functions with respect to the filtration of σ-fields generated by $\{V(s);\ 0 \leqq s \leqq t\}$ for $t \in [0, T]$ is an $(m + n)$-dimensional Wiener process, the values of δ are n-dimensional vectors, and the values of σ are the matrices of dimension $n \times (m + n)$. Let τ be the time of the first exit of $K \times \Gamma$. Then

$$\text{(5)} \qquad v(y(h \wedge \tau)) - v(y(0)) = \int_0^{h\wedge\tau} \{(\delta(t), \nabla v(y(t))) +$$

$$+ \tfrac{1}{2} \operatorname{Tr}[\sigma(t)\, \sigma(t)^* \, \nabla\nabla v(y(t))]\} \, \mathrm{d}t + \int_0^{h \wedge \tau} \nabla v(y(t))\, \sigma(t)\, \mathrm{d}V(t)$$

where the generalized derivatives are employed.

Theorem 2 has the following corollary:

Corollary. Let v: $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ fulfil the following assumptions:

(a) v is bounded and has continuous and bounded first derivatives everywhere,

(b) v has bounded measurable generalized second derivatives which are continuous in the first m variables for fixed last n variables.

Let $y(t)$ be a continuous solution of (4) and let

$$\tfrac{1}{2} \operatorname{Tr}[\sigma(y)\, \sigma(y)^* \, \nabla\nabla v(y)] + (\delta(y), \nabla v(y))$$

be continuous (and bounded). Then

$$\lim_{h \to 0+} (\mathsf{E}v(y(h)) - v(y(0)))/h = \mathsf{D}v(y(0)) \equiv$$

$$\equiv \tfrac{1}{2} \operatorname{Tr}[\sigma(y(0))\, \sigma(y(0))^* \, \nabla\nabla v(y(0))] + (\delta(y(0)), \nabla v(y(0))) .$$

Proof. We can show that $v/(K \times \Gamma)$ fulfils the assumptions of Theorem 2 for compact balls K, Γ, in $\mathbb{R}^m$, or $\mathbb{R}^n$ respectively, using a standard procedure of approximation of $v/(K \times \Gamma)$ by convolution (in last n variables).

We can use the formula (5) from Theorem 2 for $\tau = \infty$ due to the boundedness of coefficients of (4) and derivatives of v. More precisely we can use Theorem 2 for $K_r \times \Gamma_r$ converging to $\mathbb{R}^{m+n}$. Thus we can write

$$\lim_{h \to 0+} (1/h)\, \mathsf{E}(v(y(h)) - v(y(0))) =$$

$$= \lim_{h \to 0+} \left\{ \mathsf{E}(1/h) \int_0^h (\tfrac{1}{2}) \operatorname{Tr}[\sigma(y(t))\, \sigma(y(t))^* \, \nabla\nabla v(y(t))] + \right.$$

$$\left. + (\delta(y(t)), \nabla v(y)))\, \mathrm{d}t + \int_0^h \nabla v(y(t))\, \sigma(t)\, \mathrm{d}V(t) \right\} .$$

The mean value of the stochastic integral is zero. The limit and the 'E' can be interchanged because $(1/h) \int_0^h (\ldots)\, \mathrm{d}t$ are uniformly bounded variables. Since the integrand is continuous, the assertion of the corollary is proved.

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