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LONG MEMORY TIME SERIES MODELS

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The paper deals with the fractionally differenced white noise and with other long memory processes of this type. It is a review of methods published recently, complemented with many new proofs. Some new procedures for estimating parameters are proposed and the seasonal persistent process is analyzed in detail.

1. INTRODUCTION

For a long time the most frequently used models in time series analysis were the AR, MA and ARMA processes. Their spectral densities are continuous and therefore bounded functions on $[-\pi, \pi]$. If the periodogram of real data reached significantly high values, it was considered as an indication of the trend or of a periodic component. The bias arising after trend removal in the spectral density estimators was corrected using special factors (see [7] and [19]). However, the statistical analysis of many hydrological time series has led in the last time to the conclusion that the peak of the periodogram near to the origin should be rather explained by a model with a spectral density, which is not bounded in the neighbourhood of the zero frequency. From this reason models with long memory have been investigated, because they appear to be suitable for applications of such kind. Their definition reads as follows. Let $\{X_t\}$ be a stationary (discrete) process with a covariance function R_k. Then $\{X_t\}$ is called a process with long memory, if $\sum|R_k| = \infty$. In the case that $\sum|R_k| < \infty$ we say that the process $\{X_t\}$ has short memory. From practical point of view we restrict ourselves to the processes with $R_0 \neq 0$. Then the above definitions can be formulated in the same way using the correlation function. The definition itself was proposed in [15].

2. FRACTIONALLY DIFFERENCED WHITE NOISE

2.1. Fundamental properties

Let $\{\varepsilon_t\}$ be a white noise with $\mathsf{E}\varepsilon_t = 0$, $\operatorname{var} \varepsilon_t = \sigma^2 > 0$. Let B be the back-shift operator; i.e., $\mathsf{B}X_t = X_{t-1}$, $\mathsf{B}\varepsilon_t = \varepsilon_{t-1}$ etc. If $\{X_t\}$ is a linear process satisfying

(2.1) $$(1 - \mathsf{B})^\delta X_t = \varepsilon_t\,, \quad \delta \in (-\tfrac{1}{2}, \tfrac{1}{2})\,,$$

then $\{X_t\}$ is called (simple) fractionally differenced white noise (FDWN). Instead of (2.1) we can use equivalently

(2.2) $$X_t = (1 - \mathsf{B})^{-\delta}\,\varepsilon_t\,.$$

The process $\{X_t\}$ possesses the spectral density

(2.3) $$f(\lambda) = \frac{\sigma^2}{2\pi}\,|1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{-2\delta} = \frac{\sigma^2}{2\pi}\left(4 \sin^2 \frac{\lambda}{2}\right)^{-\delta}$$

(see Figs. 1 and 2). A detailed derivation of (2.3) can be done using the methods explained in [1], Chap. 9.1. Obviously, $f(\lambda) \to \infty$ for $\lambda \to 0$ iff $\delta > 0$. Because we are interested especially in models of this kind, we shall assume everywhere in this paper that $\delta \in (0, \frac{1}{2})$.

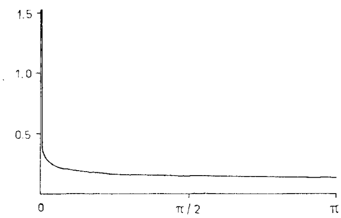

Fig. 1. Spectral density of FDWN for $\sigma^2 = 1$ and $\delta = 0{\cdot}1$.

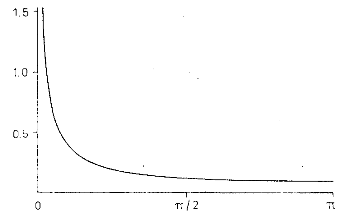

Fig. 2. Spectral density of FDWN for $\sigma^2 = 1$ and $\delta = 0{\cdot}4$.

Theorem 2.1. The covariance function R_k and the correlation function ϱ_k of the process $\{X_t\}$ with the spectral density (2.3) are

(2.4) $$R_k = \frac{(-1)^k\,\sigma^2\,\Gamma(1 - 2\delta)}{\Gamma(k + 1 - \delta)\,\Gamma(-k + 1 - \delta)} \quad (k = 0, \pm 1, \ldots)\,,$$

and

(2.5) $$\varrho_k = \frac{\delta(1 + \delta)\ldots(k - 1 + \delta)}{(1 - \delta)(2 - \delta)\ldots(k - \delta)} \quad (k = 1, 2, \ldots)\,,$$

respectively, where Γ is the gamma function.

Proof (see [8]). We start with

$$R_k = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\lambda} f(\lambda)\,\mathrm{d}\lambda = 2\int_0^{\pi} \cos k\lambda\, f(\lambda)\,\mathrm{d}\lambda = 4\int_0^{\pi/2} \cos 2kx\, f(2x)\,\mathrm{d}x\,.$$

It can be checked that $q(x) = \cos 2kx\, f(2x)$ is such that $q(x) = q(\pi - x)$, $x \in (0, \pi/2)$. Thus

$$R_k = 2 \int_0^{\pi} \cos 2kx\, f(2x)\, \mathrm{d}x .$$

We insert for f from (2.3). Using (6.2) and the well known formulas

$$\cos k\pi = (-1)^k , \quad \mathrm{B}(a, b) = \Gamma(a)\, \Gamma(b)/\Gamma(a + b) ,$$
$$\Gamma(a) = (a - 1)\, \Gamma(a - 1) ,$$

we obtain (2.4). Formula (2.5) follows from $\varrho_k = R_k/R_0$. □

Our results can be also written in the form

$$R_0 = \sigma^2\, \Gamma(1 - 2\delta)/\Gamma^2(1 - \delta) , \tag{2.6}$$
$$R_{k+1} = (k + 1 - \delta)^{-1} (k + \delta)\, R_k \quad (k \geqq 0) ,$$

$$\varrho_0 = 1 , \quad \varrho_1 = \delta/(1 - \delta) , \quad \varrho_{k+1} = (k + 1 - \delta)^{-1} (k + \delta)\, \varrho_k \quad (k \geqq 0) . \tag{2.7}$$

These formulas will be used for estimating δ and for simulations.

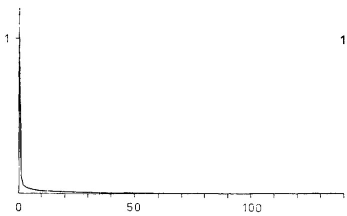

Fig. 3. Correlation function of FDWN for $\delta = 0{\cdot}1$.

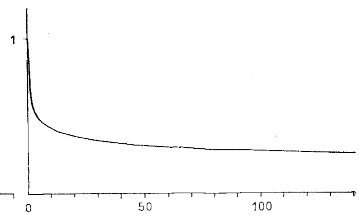

Fig. 4. Correlation function of FDWN for $\delta = 0{\cdot}4$.

Two examples of ϱ_k are given in Figs. 3 and 4.

Formula (2.5) is equivalent to

$$\varrho_k = [\Gamma(1 - \delta)/\Gamma(\delta)]\, [\Gamma(k + \delta)/\Gamma(k + 1 - \delta)] .$$

Expanding $\Gamma(k + \delta)$ and $\Gamma(k + 1 - \delta)$ by the Stirling formula, we get after some computations

$$\varrho_k \sim [\Gamma(1 - \delta)/\Gamma(\delta)]\, k^{2\delta - 1} \quad (k \to \infty) . \tag{2.8}$$

From here it is clear that $\sum |\varrho_k| = \infty$. It means that FDWN belongs to the set of models with long memory.

Theorem 2.2. The AR(∞) and the MA(∞) representations of FDWN are

$$\sum_{j=0}^{\infty} a_j X_{t-j} = \varepsilon_t \tag{2.9}$$

and

$$X_t = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}, \tag{2.10}$$

respectively, where the coefficients a_j and b_j are given in (6.6) and (6.7).

Proof (see [8]). The assertions follow from (2.1), (2.2) and (6.5), since

$$\sum_{j=0}^{\infty} |a_j| < \infty \quad \text{and} \quad \sum_{j=0}^{\infty} |b_j|^2 < \infty \quad \text{for} \quad \delta \in (0, \tfrac{1}{2}). \qquad \square$$

From the historical point of view, Mandelbrot and van Ness in [13] introduced originally so called "fractional Gaussian noise" (FGN) in the following way. Let $B(s)$ be a process of Brownian motion. Define

$$\eta_t = \int_{-\infty}^{t} (t - s)^{\delta} \, \mathrm{d}B(s).$$

Then $X_t = \eta_t - \eta_{t-1}$ has been called FGN. In this connection the symbol $H = \delta + \frac{1}{2}$ is used. The relation between FGN and FDWN is discussed in [5], [8] and [11]. A generalization is described in [2].

2.2. Methods for simulations

The processes FGN and FDWN are tightly connected. Therefore, we introduce also the methods, which were proposed originally for FGN [see (a), (b), (c)].

(a) A method based on inverting the correlation matrix (McLeod and Hipel [15]) is numerically cumbersome and not suitable for sample size greater than 100.

(b) An approximation by ARMA (1, 1) process (O'Connel [16] and [17]).

(c) An approximation based on a sum of independent AR(1) processes ("fast fractional Gaussian noise" – see [12]).

(d) An approximation based on an MA process of high order. Instead of (2.9) a similar but finite series is considered (Mandelbrot and Wallis [14]). Since the convergence $b_j \to 0$ is very slow [see (6.8)], this method needs an extraordinary high order of the considered MA model.

(e) An approximation based on an AR process of high order. The infinite series (2.8) is substituted by a finite sum. Since the convergence $a_j \to 0$ is faster, Hosking [9] prefers this method to (d).

(f) A method based on partial correlation coefficients. The procedure is proposed by Hosking in [9] and its steps use some results from [8]. Since it is considered as an effective method, we introduce here some details. The procedure is used for simulating FDWN with normal distribution. We describe the case with $\sigma^2 = 1$.

(i) Generate a random variable $X_0 \sim N(0, v_0)$, where $v_0 = R_0 = \Gamma(1 - 2\delta) : \Gamma^2(1 - \delta)$.

(ii) Calculate Φ_{jt} $(j = 1, 2, \ldots, t)$ recurrently from

$$\Phi_{tt} = \delta/(t - \delta), \quad \Phi_{tj} = \Phi_{t-1,j} - \Phi_{tt}\Phi_{t-1,t-j} \quad (j = 1, \ldots, t - 1).$$

(iii) Calculate

$$m_t = \sum_{j=1}^{t} \Phi_{tj} X_{t-j}\,, \quad v_t = (1 - \Phi_{tt}^2)\, v_{t-1}\,.$$

(iv) Generate $X_t \sim N(m_t, v_t)$.

(v) Repeat (ii)–(iv) for $t = 1, 2, \ldots, N$.

If it is necessary to simulate several realizations with the same δ, it is recommended to store the coefficients Φ_{tj}.

The main idea of this procedure is that

$$\Phi_{kk} = \delta/(k - \delta) \quad (k = 1, 2, \ldots) \tag{2.11}$$

is the partial correlation function of the process $\{X_t\}$ (see [8]).

2.3. Estimation of parameters

The model (2.1) has two parameters, δ and σ^2. The main problem is to estimate δ, because then an estimator for σ^2 can be based on (2.6).

The simplest procedure is the moment method. Let $\hat{R}_k$ and $\hat{\sigma}_k$ be estimators for R_k and ϱ_k, respectively. Using (2.7) one can see that $\delta = \varrho_1/(1 + \varrho_1)$. Thus an estimator for δ is

$$\hat{\delta} = \hat{\varrho}_1/(1 + \hat{\varrho}_1) \tag{2.12}$$

and an estimator for σ^2 is

$$\hat{\sigma}^2 = \hat{R}_0\, \Gamma^2(1 - \hat{\delta})/\Gamma(1 - 2\hat{\delta})\,. \tag{2.13}$$

If $\hat{R}_k$ and $\hat{\varrho}_k$ are consistent estimators, the same is true for $\hat{\delta}$ and $\hat{\sigma}^2$.

Another possibility is to use an estimator $\hat{\Phi}_{kk}$ of the partial correlation function ($k = 1, 2, \ldots, M \ll N$). From (2.11) we have estimators

$$\hat{\delta}_k = k\hat{\Phi}_{kk}/(1 + \hat{\Phi}_{kk})$$

which can serve for a final estimator

$$\delta^* = (\hat{\delta}_1 + \ldots + \hat{\delta}_M)/M\,.$$

This procedure is able to check the adequacy of the model. However, further theoretical investigations in this direction are needed.

Geweke and Porter-Hudak [2] proposed the following method. From (2.3) we get

$$\ln f(\lambda) = \ln \frac{\sigma^2}{2\pi} - \delta \ln \left(4 \sin^2 \frac{\lambda}{2}\right). \tag{2.14}$$

Let a realization $X_1, \ldots, X_N$ be given. Consider the values of its periodogram $I(\lambda)$ in the points $\lambda = \lambda_j$, where

$$\lambda_j = 2\pi j/N \quad (j = 0, 1, \ldots, N - 1)\,.$$

With respect to (2.14) we have

$$\ln I(\lambda_j) = \ln \frac{\sigma^2}{2\pi} - \delta \ln \left(4 \sin^2 \frac{\lambda_j}{2}\right) + \ln \frac{I(\lambda_j)}{f(\lambda_j)}\,. \tag{2.15}$$

The variables $\ln [I(\lambda_j)/f(\lambda_j)]$ are asymptotically independent identically distributed with the asymptotic expectation $-C$ and with the asymptotic variance $\pi^2/6$, where $C = 0.57721 \ldots$ is the Euler constant. Thus (2.15) can be considered as a regression model

$$Y_j = a + \delta x_j + e_j \quad (j = 0, 1, \ldots, N - 1),$$

where $Y_j = \ln I(\lambda_j)$ and $x_j = -\ln(4 \sin^2(\lambda_j/2))$. If $\tilde{a}$ and $\tilde{\delta}$ are the mean squares estimates of the regression parameters, then $\tilde{\delta}$ can serve as an estimator of our original parameter δ. Since $\tilde{a}$ is an estimator of $\ln(\sigma^2/(2\pi)) + C$, an estimator for σ^2 is

$$\tilde{\sigma}^2 = 2\pi \, e^{\tilde{a} - C}.$$

If the couples (Y_j, x_j) do not correspond to the linear dependence, we can doubt if FDWN is a correct model for our data.

Still another procedure can be based on the following idea. If we approximate in (2.3) $\sin x \doteq x$ for small x, then

$$f(\lambda) \doteq (2\pi)^{-1} \sigma^2 \lambda^{-2\delta}.$$

Denote $F(\lambda)$ the spectral distribution function corresponding to the density $f(\lambda)$. Then

$$F(x) - F(0) = \int_0^x f(\lambda) \, d\lambda \doteq (2\pi)^{-1} \sigma^2 (1 - 2\delta)^{-1} x^{1-2\delta}.$$

But $F(x) - F(0)$ can be estimated by $\int_0^x I(\lambda) \, d\lambda$. Since

$$I(\lambda) = (2\pi)^{-1} (C_0 + 2 \sum_{k=1}^{N-1} C_k \cos k\lambda),$$

where

$$C_k = N^{-1} \sum_{t=1}^{N-k} X_t X_{t+k} \quad \text{for} \quad k = 0, 1, \ldots, N - 1,$$

we have

$$\int_0^x I(\lambda) \, d\lambda = (2\pi)^{-1} (x C_0 + 2 \sum_{k=1}^{N-1} k^{-1} C_k \sin kx).$$

Because

$$\ln [F(x) - F(0)] = \ln \{\sigma^2/[2\pi(1 - 2\delta)]\} + (1 - 2\delta) \ln x,$$

we can use again estimators for parameters of the linear regression.

2.4. Estimation of the mean value

If a process $\{X_t\}$ has the spectral density (2.3) and a constant mean value $\mu = \mathsf{E} X_t$, we can estimate μ by $\bar{X}_n = (X_1 + \ldots + X_n)/n$. Clearly $\mathsf{E} \bar{X}_n = \mu$. Because $\operatorname{var} \bar{X}_n$ does not depend on μ, we shall assume in the next considerations that $\mu = 0$. We also keep the assumption $\delta \in (0, \frac{1}{2})$.

Theorem 2.3. Let $R_0 = \operatorname{var} X_t$. Then

$$\operatorname{var} \bar{X}_n = n^{-2} R_0 \frac{\delta}{1 + 2\delta} \left[1 + \frac{(1 + \delta)(2 + \delta)\dots(n + \delta)}{\delta(1 - \delta)(2 - \delta)\dots(n - 1 - \delta)} \right].$$

Proof. Let $Z(\cdot)$ be the random measure corresponding to the process $\{X_t\}$. Then

$$\bar{X}_n = n^{-1} \sum_{t=1}^{n} X_t = n^{-1} \sum_{t=1}^{n} \int_{-\pi}^{\pi} e^{it\lambda} \, dZ(\lambda) = \int_{-\pi}^{\pi} \sum_{t=1}^{n} n^{-1} e^{it\lambda} \, dZ(\lambda).$$

Using Theorem 3.14 c) in [1] we get

$$E\bar{X}_n^2 = \int_{-\pi}^{\pi} |n^{-1} \sum_{t=1}^{n} e^{it\lambda}|^2 f(\lambda) \, d\lambda.$$

After some computations we obtain

$$E\bar{X}_n^2 = n^{-2} \, 2^{-2\delta} \pi^{-1} \sigma^2 \int_0^{\pi} \sin^{-2-2\delta} x \sin^2 nx \, dx.$$

The result follows by using (6.10) and (2.6). □

Theorem 2.4. If $n \to \infty$, then

$$\operatorname{var} \bar{X}_n \sim R_0 (1 + 2\delta)^{-1} \Gamma(1 - \delta) \Gamma^{-1}(1 + \delta) \, n^{2\delta - 1}.$$

Proof. The assertion follows from Theorem 2.3 by help of the Stirling formula. □

Theorems 2.3 and 2.4 are introduced without proof in [9].

3. GENERAL FRACTIONALLY DIFFERENCED WHITE NOISE

Let $f(\lambda)$ be the spectral density given in (2.3) and let $f_u(\lambda)$ be a positive continuous even function on $[-\pi, \pi]$. Then any process having the spectral density

$$g(\lambda) = f(\lambda) f_u(\lambda) \tag{3.1}$$

fs called the general fractionally differenced white noise (GFDWN). As a rule, i$_u(\lambda)$ is the spectral density of a suitable ARMA process. Without any loss of generality

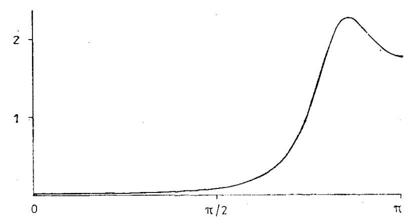

Fig. 5. Spectral density $f_u(\lambda)$ of the AR(2) model $X_t + 1{\cdot}2 X_{t-1} + 0{\cdot}5 X_{t-2} = \varepsilon_t$.

we can assume $\sigma^2 = 1$, because any multiplicative factor can be included into $f_u(\lambda)$.

Fig. 5 shows the spectral density $f_u(\lambda) = (2\pi)^{-1} |1 + 1{\cdot}2\, e^{-i\lambda} + 0{\cdot}5\, e^{-2i\lambda}|^{-2}$, which corresponds to the AR(2) model $X_t + 1{\cdot}2X_{t-1} + 0{\cdot}5X_{t-2} = \varepsilon_t$. The spectral density $g(\lambda)$ corresponding to $f(\lambda)$ with $\delta = 0{\cdot}45$ is plotted in Fig. 6.

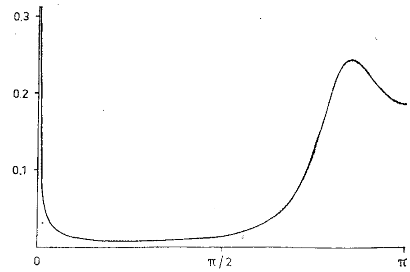

Fig. 6. Spectral density $g(\lambda) = f(\lambda) f_u(\lambda)$ of GFDWN for $\delta = 0{\cdot}45$.

The parameters of (3.1) are estimated in two steps. First, an estimator for δ is constructed. Thus we have in our disposal an estimator $\hat{f}(\lambda)$ for $f(\lambda)$. Then we use an estimator $\hat{g}(\lambda)$ for $g(\lambda)$ and calculate $\hat{f}_u(\lambda) = \hat{g}(\lambda)/\hat{f}(\lambda)$ as an estimator for $f_u(\lambda)$.

Gewerke and Porter-Hudak [2] propose the following method for estimating δ. Let $I(\lambda)$ be the periodogram calculated from a realization $X_1, \ldots, X_N$ of GFDWN. From the formula

$$\ln g(\lambda) = \ln [(2\pi)^{-1} f_u(0)] - \delta \ln (4 \sin^2 (\lambda/2)) + \ln [f_u(\lambda)/f_u(0)]$$

one gets

$$\ln I(\lambda) = \ln [(2\pi)^{-1} f_u(0)] - \delta \ln (4 \sin^2 (\lambda/2)) + \ln [f_u(\lambda)/f_u(0)] + \\ + \ln [I(\lambda)/g(\lambda)] \,.$$

If we consider only some frequencies $\lambda = \lambda_j$ near to 0, the value of $\ln [f_u(\lambda_j)/f_u(0)]$ is negligible. An estimator for δ we get again from the linear regression with dependent variables $Y_j = \ln I(\lambda_j)$ and with independent variables $x_j = -\ln (4 \sin^2 (\lambda_j/2))$.

Another procedure follows from the approximations $\sin \lambda \doteq \lambda$ and $f_u(\lambda) \doteq f_u(0)$ near to 0. Then $g(\lambda) \doteq (2\pi)^{-1} \lambda^{-2\delta} f_u(0)$ and

$$G(x) - G(0) = \int_0^x g(\lambda)\, d\lambda \doteq (2\pi)^{-1} f_u(0) (1 - 2\delta)^{-1} x^{1-2\delta} \,.$$

Further we can proceed in the same way as in the Section 2.3.

4. SEASONAL FRACTIONALLY DIFFERENCED WHITE NOISE

If it is known in advance that a model should be constructed for a seasonal time series with the seasons of the length s, we can use the linear process $\{X_t\}$ defined by

$$(1 - \mathrm{B}^s)^\delta X_t = \varepsilon_t .$$

Such a process $\{X_t\}$ is called the "seasonal fractionally differenced white noise" – SFDWN. Again, let $\delta \in (0, \frac{1}{2})$. SFDWN has the spectral density

$$f_s(\lambda) = (2\pi)^{-1} \sigma^2 |1 - e^{-is\lambda}|^{-2} = (2\pi)^{-1} \sigma^2 (4 \sin^2 (s\lambda/2))^{-\delta} .$$

Obviously, $f_s(\lambda) \to \infty$ for $\lambda \to 2k\pi/s$ ($k = 0, \pm 1, \ldots, \pm[s/2]$), where [ ] denotes the integer part. In Fig. 7 we can see $f_s(\lambda)$ for $\delta = 0{\cdot}3$, $s = 5$ and $\sigma^2 = 1$.

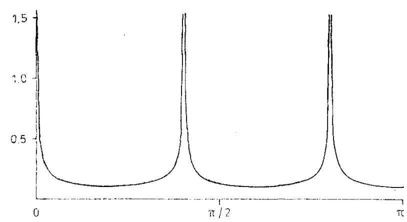

Fig. 7. Spectral density $f_s(\lambda)$ of SFDWN for $\delta = 0{\cdot}3$, $s = 5$ and $\sigma^2 = 1$.

SFDWN has the AR(∞) and MA(∞) representations

$$\sum_{j=0}^{\infty} a_j X_{t-sj} = \varepsilon_t \quad \text{and} \quad X_t = \sum_{j=0}^{\infty} b_j \varepsilon_{t-sj} ,$$

respectively, where a_j and b_j are the same coefficients as in Theorem 2.2. It follows from the MA(∞) representation that the covariance function $R_t^{(s)}$ of the SFDWN is given by

$$R_t^{(s)} = \begin{cases} R_m & \text{if} \quad t = sm \quad \text{for an integer} \quad m , \\ 0 & \text{otherwise} , \end{cases}$$

where R_m is the covariance function of FDWN introduced in Theorem 2.1. Because

$$R_t^{(s)} = 2 \int_0^\pi \cos t\lambda \, f_s(\lambda) \, d\lambda = \pi^{-1} 2^{-2\delta} \sigma^2 \int_0^\pi \cos t\lambda \, (\sin^2 (s\lambda/2))^{-\delta} \, d\lambda =$$

$$= 2^{1-2\delta} \pi^{-1} \sigma^2 \int_0^{\pi/2} \cos 2tx \, (\sin^2 sx)^{-\delta} \, dx ,$$

we have proved an interesting formula

$$\int_0^{\pi/2} \cos 2tx\,(\sin^2 sx)^{-\delta}\,\mathrm{d}x = \begin{cases} \dfrac{(-1)^m\, 2^{2\delta-1}\pi\, \Gamma(1-2\delta)}{\Gamma(m+1-\delta)\,\Gamma(-m+1-\delta)} & \text{for } t = ms, \\ 0 \quad \text{otherwise}, \end{cases}$$

where m, s are integers and $\delta \in (0, \frac{1}{2})$.

Quite analogously it can be defined the general seasonal fractionally differenced white noise as the process with a spectral density of the type $g_s(\lambda) = f_s(\lambda)\, f_u(\lambda)$.

5. SEASONAL PERSISTENT PROCESS

Let $\delta \in (0, \frac{1}{2})$ and $\omega \in (0, \pi)$. A seasonal persistent process (SPP) is such a linear process $\{X_t\}$ which satisfies the relation

$$[(1 - e^{i\omega}B)(1 - e^{-i\omega}B)]^{\delta} X_t = \varepsilon_t .$$

This can be written in a more convenient form

$$(1 - 2\cos\omega\,.\,B + B^2)^{\delta} X_t = \varepsilon_t .$$

The spectral density of the SPP is

$$f(\lambda) = (2\pi)^{-1}\sigma^2 |(1 - e^{i\omega - i\lambda})(1 - e^{-i\omega - i\lambda})|^{-2\delta} =$$

(5.1) $$= (2\pi)^{-1}\, 2^{-4\delta}\, \sigma^2 \left(\sin^2 \frac{\omega - \lambda}{2} \sin^2 \frac{\omega + \lambda}{2}\right)^{-\delta} .$$

We can also write

$$f(\lambda) = (2\pi)^{-1}\, 2^{-2\delta}\, \sigma^2 |\cos\lambda - \cos\omega|^{-2\delta} ,$$

but (5.1) is more suitable for further investigations. Sometimes the authors use the notation $\Phi = \cos\omega$. Some typical shapes of $f(\lambda)$ for $\sigma^2 = 1$ are given in Fig. 8

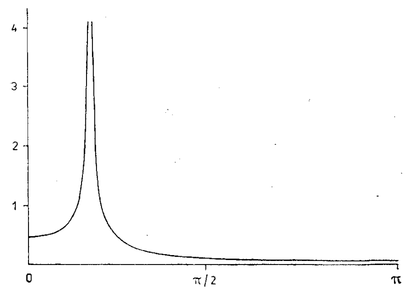

Fig. 8. Spectral density of SPP ($\sigma^2 = 1$, $\delta = 0{\cdot}4$, $\omega = \pi/6$, $\Phi = 0{\cdot}866$).

($\delta = 0{\cdot}4$, $\omega = \pi/6$, $\Phi = 0{\cdot}866$), Fig. 9 ($\delta = 0{\cdot}2$, $\omega = \pi/6$, $\Phi = 0{\cdot}866$), Fig. 10 ($\delta = 0{\cdot}3$, $\Phi = -0{\cdot}6$, $\omega = 2{\cdot}214$) and Fig. 11 ($\delta = 0{\cdot}3$, $\Phi = 0{\cdot}995$, $\omega = 0{\cdot}1$).

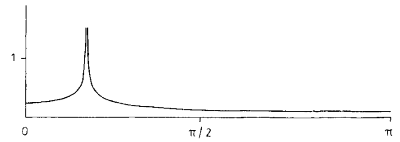

Fig. 9. Spectral density of SPP ($\sigma^2 = 1$, $\delta = 0{\cdot}2$, $\omega = \pi/6$, $\Phi = 0{\cdot}866$).

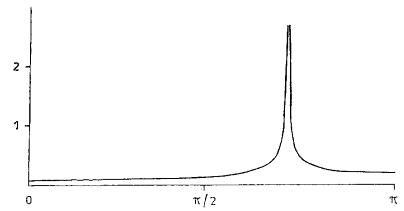

Fig. 10. Spectral density of SPP ($\sigma^2 = 1$, $\delta = 0{\cdot}3$, $\omega = 2{\cdot}214$, $\Phi = -0{\cdot}6$).

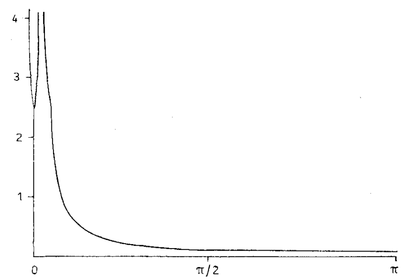

Fig. 11. Spectral density of SPP ($\sigma^2 = 1$, $\delta = 0{\cdot}3$, $\omega = 0{\cdot}1$, $\Phi = 0{\cdot}955$).

The covariance function $R(k)$ of the SPP can be expressed by the formula

$$(5.2) \qquad R(k) = \pi^{-1}\, 2^{-2\delta} \sigma^2 \int_0^\pi \cos k\lambda \left|\cos \lambda - \cos \omega\right|^{-2\delta} \mathrm{d}\lambda\,.$$

An explicit solution is known only in special cases. Let $\omega = \pi/2$. From the relation $\cos k(x - \pi) = (-1)^k \cos kx$ we get

$$R(2k + 1) = 0 \quad (k = 0, 1, \ldots)$$

and

$$R(2k) = \pi^{-1}\, 2^{1-2\delta}\sigma^2 \int_0^{\pi/2} \cos^{-2\delta} x \cos 2kx\, \mathrm{d}x \qquad (k = 0, 1, \ldots)\,.$$

Using (6.4) we come to

$$R(2k) = (-1)^k (1 - 2\delta)^{-1}\, \sigma^2/\mathrm{B}(1 - \delta + k, 1 - \delta - k) =$$
$$= (-1)^k \sigma^2\, \Gamma(1 - 2\delta)/[\Gamma(1 - \delta + k)\, \Gamma(1 - \delta - k)] \quad (k = 0, 1, 2, \ldots)\,.$$

The asymptotic behaviour of the $R(k)$ in the general case is described in the following theorem.

Theorem 5.1. If $k \to \infty$, then

$$R(k) \sim \sigma^2\, 2^{1-2\delta} \sin^{-2\delta} \omega \frac{\Gamma(1 - 2\delta)}{\Gamma(1 - \delta)\, \Gamma(\delta)}\, k^{2\delta-1} \cos k\omega\,. \tag{5.3}$$

Proof. We start with

$$R(k) = 2 \int_0^{\pi} \cos k\lambda\, f(\lambda)\, \mathrm{d}\lambda\,,$$

where $f(\lambda)$ is given in (5.1). Make substitution $(\omega - \lambda)/2 = x$. After elementary computations we obtain

$$R(k) = \pi^{-1}\, 2^{1-4\delta}\sigma^2 \sin^{-2\delta} \omega\,.\, (J_1 \cos k\omega + J_2 \cos k\omega +$$
$$+ J_3 \sin k\omega + J_4 \sin k\omega)\,,$$

where

$$J_1 = \int_{-(\pi-\omega)/2}^{\omega/2} \cos 2kx (\sin^2 x)^{-\delta}\, \mathrm{d}x\,,$$

$$J_2 = \int_{-(\pi-\omega)/2}^{\omega/2} \cos 2kx (\sin^2 x)^{-\delta} \{\sin^{2\delta} \omega - [\sin^2(\omega - x)]^{\delta}\}\, [\sin^2(\omega - x)]^{-\delta}\, \mathrm{d}x\,,$$

$$J_3 = \int_{-(\pi-\omega)/2}^{\omega/2} \sin 2kx (\sin^2 x)^{-\delta}\, \mathrm{d}x\,,$$

$$J_4 = \int_{-(\pi-\omega)/2}^{\omega/2} \sin 2kx (\sin^2 x)^{-\delta} \{\sin^{2\delta} \omega - [\sin^2(\omega - x)]^{\delta}\}\, [\sin^2(\omega - x)]^{-\delta}\, \mathrm{d}x\,.$$

First, consider J_1. From the periodicity we can see that the integral over $[-(\pi - \omega)/2, 0]$ is the same as the integral over $[(\pi + \omega)/2, \pi]$. Thus

$$J_1 = J_1^* - J_1^{**}\,,$$

where

$$J_1^* = \int_0^{\pi} \cos 2kx \sin^{-2\delta} x\, \mathrm{d}x\,, \quad J_1^{**} = \int_{\omega/2}^{(\pi+\omega)/2} \cos 2kx \sin^{-2\delta} x\, \mathrm{d}x\,.$$

The value of J_1^* is given by (6.2) and its asymptotics was investigated in the derivation of the formula (2.8). Thus

$$J_1^* \sim \pi\, 2^{2\delta} k^{2\delta-1}\, \Gamma(1-2\delta)/[\Gamma(1-\delta)\,\Gamma(\delta)]\,.$$

Theorem 6.4 ensures that $J_1^{**} = O(k^{-1})$.

The similar procedure is valid also for J_3. In this case we have $J_3 = J_3^* - J_3^{**}$, where

$$J_3^* = \int_0^\pi \sin 2kx \sin^{-2\delta} x \,\mathrm{d}x\,, \quad J_3^{**} = \int_{\omega/2}^{(\omega+\pi)/2} \sin 2kx \sin^{-2\delta} x \,\mathrm{d}x\,.$$

But $J_3^* = 0$ according to (6.3), whereas $J_3^{**} = O(k^{-1})$.

Now consider J_2. Since

$$\lim_{x\to 0} (\sin^2 x)^{-\delta} \{\sin^{2\delta} \omega - [\sin^2(\omega - x)]^\delta\} = 0\,,$$

Theorem 6.4 yields $J_2 = O(k^{-1})$. Quite analogously, $J_4 = O(k^{-1})$. Summarizing these results we come to (5.3). □

Hosking [9] introduces without proof that $R_k \sim ak^{2\delta-1} \cos k\omega$. The constant a is not specified there.

Theorem 5.2. SPP is a long memory process.

Proof. We prove that $\sum|R(k)| = \infty$. Let $\sum|R(k)| < \infty$. Then the function

$$h(x) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-ik\lambda} R(k)$$

is continuous (uniformly convergent series of continuous functions has a continuous sum). Since $h(x)$ is the uniformly convergent Fourier series of the function $f(x)$ on $[-\pi, \pi]$, it holds $f(x) = h(x)$ and this equality may be violated maximally in a finite number of points (e.g. [6], p. 388). This is a controversy, because $h(x)$ being continuous on $[-\pi, \pi]$ must be bounded, whereas the Lebesgue measure of the set $\{x \in [-\pi, \pi]: f(x) \geqq K\}$ is positive for arbitrary large K. □

Note that the proof of Theorem 5.2 suits also for other long memory models.

Theorem 5.3. Let $U = \{z: |z| < 1\}$, $\bar{U} = \{z: |z| \leqq 1\}$. Put $\psi(z) = (1 - \mathrm{e}^{\mathrm{i}\omega}z)^{-\delta} \times (1 - \mathrm{e}^{-\mathrm{i}\omega}z)^{-\delta}$, $z \in \bar{U} - \{e^{\mathrm{i}\omega}, \mathrm{e}^{-\mathrm{i}\omega}\}$, where we define $w^\alpha = \exp\{\alpha \operatorname{Log} w\}$ for every complex α and $w \notin (-\infty, 0)$. Then ψ is analytic in U. If

$$\psi(z) = \sum_{n=0}^{\infty} c_n z^n \quad (z \in U)\,,$$

then

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty\,.$$

Proof (Netuka). For $\Theta \in (0, \pi) - \{\omega, -\omega\}$ we have

$$|\psi(\mathrm{e}^{\mathrm{i}\Theta})|^2 = 2^{-4\delta} \left(\sin^2 \frac{\Theta + \omega}{2}\right)^{-\delta} \left(\sin^2 \frac{\Theta - \omega}{2}\right)^{-\delta}.$$

Because $\delta \in (0, \frac{1}{2})$, we can see that

$$\int_{-\pi}^{\pi} |\psi(e^{i\Theta})|^2 \, d\Theta < \infty .$$

The Fourier coefficients with negative indices of the function $\psi(e^{i\Theta})$ are zeros and thus Theorem 17·10 in Rudin [18] implies that $\sum |c_n|^2 < \infty$. □

Now, we derive the MA(∞) representation of SPP. We have

$$X_t = (1 - e^{i\omega}B)^{-\delta} (1 - e^{-i\omega}B)^{-\delta} \varepsilon_t =$$

$$= (\sum_{j=0}^{\infty} b_j e^{ij\omega} B^j \sum_{k=0}^{\infty} b_k e^{-ik\omega} B^k) \varepsilon_t = \sum_{n=0}^{\infty} c_n \varepsilon_{t-n} ,$$

where the coefficients b_j are given in (6.7) and

$$c_n = \sum_{k=0}^{n} b_k b_{n-k} \cos \omega(n - 2k) .$$

Theorem 5.3 ensure that $\sum |c_n|^2 < \infty$, and thus the series $\sum c_n \varepsilon_{t-n}$ converges in the quadratic mean. The coefficients c_n can be calculated as follows. Let $\beta_k^{(n)} = b_k b_{n-k}$. Then $\beta_0^{(0)} = 1$ and

$$\beta_0^{(n+1)} = \frac{\delta + n}{n + 1} \beta_0^{(n)} , \quad \beta_{k+1}^{(n)} = \frac{k + \delta}{n + \delta - 1 - k} \frac{n - k}{k + 1} \beta_k^{(n)} \quad (k = 0, \ldots, n - 1) .$$

Because $\beta_k^{(n)} = \beta_{n-k}^{(n)}$, it is sufficient to calculate $\beta_k^{(n)}$ only for $k \leqq [n/2]$. Some coefficients $\beta_k^{(n)}$ are introduced in the following table.

Coefficients $\beta_k^{(n)}$

n	k		
	0	1	2
0	1		
1	δ		
2	$\delta(\delta + 1)/2$	δ^2	
3	$\delta(\delta + 1)(\delta + 2)/6$	$\delta^2(\delta + 1)/2$	
4	$\delta(\delta + 1)(\delta + 2)(\delta + 3)/24$	$\delta^2(\delta + 1)(\delta + 2)/6$	$\delta^2(\delta + 1)^2/4$

The AR(∞) representation of the SPP can be derived similarly. We have

$$\varepsilon_t = (1 - e^{i\omega}B)^{\delta} (1 - e^{-i\omega}B)^{\delta} = \sum_{n=0}^{\infty} h_n X_{t-n} ,$$

where

$$h_n = \sum_{k=0}^{n} a_k a_{n-k} \cos \omega(n - 2k) ,$$

and the coefficients a_k are given in (6.6). The series $\sum h_n X_{t-n}$ converges in the quadratic

mean because
$$\sum_{n=0}^{\infty}|h_n| \leqq \sum_{n=0}^{\infty}\sum_{k=0}^{n}|a_k|\,.\,|a_{n-k}| = (\sum_{j=0}^{\infty}|a_j|)^2 < \infty\,.$$
Denote $\alpha_k^{(n)} = a_k a_{n-k}$. Then $\alpha_0^{(0)} = 1$ and
$$\alpha_0^{(n+1)} = \frac{n-\delta}{n+1}\alpha_0^{(n)}\,,\quad \alpha_{k+1}^{(n)} = \frac{k-\delta}{n-\delta-1-k}\,\frac{n-k}{k+1}\alpha_k^{(n)}\quad (k = 0, \ldots, n-1).$$
Also here $\alpha_k^{(n)} = \alpha_{n-k}^{(n)}$ and thus it suffices to calculate $\alpha_k^{(n)}$ only for $k = 0, \ldots, [n/2]$. Some $\alpha_k^{(n)}$ are given here.

Coefficients $\alpha_k^{(n)}$

n	k		
	0	1	2
0	1		
1	$-\delta$		
2	$(1-\delta)(-\delta)/2$	δ^2	
3	$(2-\delta)(1-\delta)(-\delta)/6$	$(1-\delta)\delta^2/2$	
4	$(3-\delta)(2-\delta)(1-\delta)(-\delta)/24$	$(2-\delta)(1-\delta)\delta^2/6$	$(1-\delta)^2\delta^2/4$

It is possible also to define the generalized seasonal persistent process (GSPP) as the process having a spectral density of the type $f(\lambda) f_u(\lambda)$, where $f(\lambda)$ is given in (5.1) and $f_u(\lambda)$ is a positive continuous even function on $[-\pi, \pi]$.

If SPP has a constant mean value μ, we can estimate it also by $\bar{X}_n = (X_1 + \ldots + X_n)/n$.

Theorem 5.4. If $\{X_t\}$ is a SPP, then $\operatorname{var}\bar{X}_n = O(n^{-1})$.

Proof. In the same way as in the proof of Theorem 2.3 we get
$$\operatorname{var}\bar{X}_n = n^{-2}\int_{-\pi}^{\pi}\sin^2\frac{n\lambda}{2}\sin^{-2}\frac{\lambda}{2}f(\lambda)\,\mathrm{d}\lambda\,,$$
where $f(\lambda)$ is given in (5.1). Denote
$$q(x) = \left[\sin^2\left(\frac{\omega}{2} - x\right)\sin^2\left(\frac{\omega}{2} + x\right)\right]^{-\delta}.$$
After some computations we obtain
$$\operatorname{var}\bar{X}_n = \sigma^2\pi^{-1}\,2^{1-4\delta}(J_1 + J_2)\,,$$
where
$$J_1 = n^{-2}\int_0^{\omega/4}\sin^{-2}x\sin^2 nx\; q(x)\,\mathrm{d}x\,,$$
$$J_2 = n^{-2}\int_{\omega/4}^{\pi/2}\sin^{-2}x\sin^2 nx\; q(x)\,\mathrm{d}x\,.$$

Let

$$m = \min_{0 \leqq x \leqq \omega/4} q(x), \quad M = \max_{0 \leqq x \leqq \omega/4} q(x).$$

Taking into account the inequalities

$$x \cos x \leqq \sin x \leqq x \quad (0 \leqq x \leqq \pi)$$

and the relation

$$\int_0^\pi \sin^{-2} x \sin^2 nx \, dx = \int_0^\pi |\sum_{k=1}^n e^{2ikx}|^2 \, dx = n/2,$$

we obtain for every $n \geqq 2\pi/\omega$

$$J_1 \leqq m^{-1}n^{-2} \int_0^\pi \sin^{-2} x \sin^2 nx \, dx = (2mn)^{-1},$$

$$J_1 \geqq M^{-1}n^{-2} \int_0^{\pi/(2n)} x^{-2}n^2x^2 \cos^2 nx \, dx = \pi(4Mn)^{-1}.$$

Denote

$$m' = \min_{\omega/4 \leqq x \leqq \pi/2} \left[\sin^2\left(\frac{\omega}{2} + x\right)\right]^\delta.$$

Then

$$J_2 \leqq m'^{-1}n^{-2} \sin^{-2} \frac{\omega}{4} \int_{\omega/4}^{\pi/2} \left[\sin^2\left(\frac{\omega}{2} - x\right)\right]^{-\delta} dx = O(n^{-2}). \qquad \square$$

The proof shows that the order of var $\bar{X}_n$ is really n^{-1} and not, for example, $o(n^{-1})$.

6. APPENDIX

Theorem 6.1. If μ and ν are complex numbers such that Re $\mu > 0$, Re $\nu > 0$, then

$$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x \, dx = \frac{1}{2} B\left(\frac{\mu}{2}, \frac{\nu}{2}\right), \tag{6.1}$$

$$\int_0^\pi \sin^{\nu-1} x \cos ax \, dx = \frac{\pi \cos \frac{a\pi}{2}}{2^{\nu-1} \nu B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}, \tag{6.2}$$

$$\int_0^\pi \sin^{\nu-1} x \sin ax \, dx = \frac{\pi \sin \frac{a\pi}{2}}{2^{\nu-1} \nu B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)}, \tag{6.3}$$

(6.4) $$\int_0^{\pi/2} \cos^{\nu-1} x \cos ax \, dx = \frac{\pi}{2^\nu \nu \, B\left(\frac{\nu + a + 1}{2}, \frac{\nu - a + 1}{2}\right)}.$$

Proof. See Gradštejn and Ryžik [3], formulas 3.621, 5; 3.631,8; 3.631,1; 3.631,9. □

Theorem 6.2. Let $\delta \in (-\frac{1}{2}, \frac{1}{2})$ and $|z| < 1$. Then

(6.5) $$(1 - z)^\delta = \sum_{j=0}^{\infty} a_j z^j, \quad (1 - z)^{-\delta} = \sum_{j=0}^{\infty} b_j z^j,$$

where

(6.6) $$a_j = \frac{\Gamma(j - \delta)}{\Gamma(-\delta)\,\Gamma(j + 1)} = \frac{(j - 1 - \delta)(j - 2 - \delta)\ldots(1 - \delta)(-\delta)}{j!},$$

(6.7) $$b_j = \frac{\Gamma(j + \delta)}{\Gamma(\delta)\,\Gamma(j + 1)} = \frac{(j - 1 + \delta)(j - 2 + \delta)\ldots(1 + \delta)\,\delta}{j!}.$$

If $j \to \infty$, then

(6.8) $$j^{1+\delta} a_j \to 1/\Gamma(-\delta), \quad j^{1-\delta} b_j \to 1/\Gamma(\delta).$$

Proof. The assertions (6.5), (6.6) and (6.7) follow from the Maclaurin formula. Using the Stirling formula, we get (6.8). □

Theorem 6.3. Let f be a real integrable function on a finite non-degenerated interval $[a, b]$. Let g be monotonous and finite on $[a, b]$. Then there exists a number $\xi \in [a, b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = g(a) \int_a^\xi f(x)\, dx + g(b) \int_\xi^b f(x)\, dx.$$

Proof. See [10], p. 198. □

Theorem 6.4. Let f be a real integrable function on a finite non-degenerated interval $[a, b]$. Let

$$a_k = \int_a^b f(x) \cos kx \, dx, \quad b_k = \int_a^b f(x) \sin kx \, dx \quad (k = 1, 2, \ldots).$$

If the variation of f is finite on $[a, b]$, then the sequences $\{ka_k\}$ and $\{kb_k\}$ are bounded.

Proof (cf. [10], p. 484, Ex. 2). Let f be nondecreasing. Then Theorem 6.3 yields

$$a_k = f(a) \int_a^\xi \cos kx \, dx + f(b) \int_\xi^b \cos kx \, dx =$$
$$= k^{-1} f(a)(\sin k\xi - \sin ka) + k^{-1} f(b)(\sin kb - \sin k\xi),$$

where ξ depends on k. From here we have $|ka_k| \leqq C$, where $C = 2|f(a)| + 2|f(b)|$ is a constant. The proof for b_k is similar. Finally, every function with finite variation can be written as the difference of two nondecreasing functions. □

Theorem 6.5. Let

$$K_n(a) = \int \cos x \sin^{-1-a} x \sin 2nx \,dx \quad (a \neq 0).$$

Then

$$K_n(a) = 2a^{-1}n \int \sin^{-a} x \cos 2nx \,dx - a^{-1} \sin^{-a} x \sin 2nx .$$

Proof. We can write $K_n(a) = \int u'(x)\, v(x)\,dx$, where $u'(x) = \sin^{-2} x \cos x$, $v(x) = \sin^{1-a} x \sin 2nx$. Then $u(x) = -\sin^{-1} x$ and the integration by parts leads to an equation for $K_n(a)$. □

Theorem 6.6. Let

$$J_n(a) = \int \sin^{-2-a} x \sin^2 nx \,dx \quad (a \neq -1).$$

Then

$$J_n(a) = \tfrac{1}{2}(1 + a)^{-1}\, a \int \sin^{-a} x \,dx - \tfrac{1}{2}(1 + a)^{-1}\, a \int \sin^{-a} x \cos 2nx \,dx -$$
$$- (1 + a)^{-1} \cos x \sin^{-1-a} x \sin^2 nx + (1 + a)^{-1}\, n\, K_n(a),$$

where $K_n(a)$ is defined in Theorem 6.5.

Proof. In this case we have $J_n(a) = \int u'(x)\, v(x)\,dx$, where $u'(x) = \sin^{-2} x$, $v(x) = \sin^{-a} x \sin^2 nx$. Then $u(x) = -\cos x \sin^{-1} x$ and the integration by parts gives an equation, which is equivalent to our assertion. □

Theorem 6.7. Let $\delta \in (0, \frac{1}{2})$ and $n = 1, 2, \ldots$. Then

$$\int_0^\pi \cos x \sin^{-1-2\delta} x \sin 2nx \,dx = \frac{(-1)^n \pi n\, 2^{2\delta}}{\delta(1 - 2\delta)\, \mathrm{B}(1 - \delta + n, 1 - \delta - n)}, \tag{6.9}$$

$$\int_0^\pi \sin^{-2-2\delta} x \sin^2 nx \,dx = \tag{6.10}$$
$$= \frac{\pi\delta\, 2^{2\delta}}{1 + 2\delta} \frac{\Gamma(1 - 2\delta)}{\Gamma^2(1 - \delta)} \left[1 + \frac{(1 + \delta)(2 + \delta)\ldots(n + \delta)}{\delta(1 - \delta)(2 - \delta)\ldots(n - 1 - \delta)}\right].$$

Proof. Formula (6.9) follows from Theorem 6.5 by using (6.2). It is easy to see that $\sin^{-2} x \sin 2nx \to 0$ for $x \to 0+$ as well as for $x \to \pi-$.

From (6.2) we get that

$$\int_0^\pi \sin^{-2\delta} x \,dx = \pi\, 2^{2\delta}\, \Gamma(1 - 2\delta)/\Gamma^2(1 - \delta).$$

The integral $\int_0^\pi \sin^{-2\delta} x \cos 2nx \,dx$ is given by (6.2). Since $\cos x \sin^{-1-2\delta} x \sin^2 nx \to 0$ for $x \to 0+$, $x \to \pi-$, (6.10) follows from Theorem 6.6 and from (6.9). □

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