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SYMMETRIC MATRIX POLYNOMIAL EQUATIONS

JAN JEŽEK

The linear symmetric matrix polynomial equation is investigated. It occurs in the synthesis of discrete quadratically optimal multivariable controllers in connection with the matrix spectral factorization problem. A new efficient algorithm for numerical solution is also presented.

1. INTRODUCTION

For problems of linear control system synthesis, an apparatus of polynomial equations (for single-variable case) and of matrix polynomial equations (for multivariable case) was successfully developed in recent times, cf. [1]. In connection with quadratic criteria, we are led to equations of special type, containing an operation of conjugation $a \mapsto a^*$ representing $a(s) \mapsto a(-s)$ for continuous-time systems and $a(d) \mapsto a(d^{-1})$ for discrete-time ones, cf. [2], [3]. The key problem is solution of a quadratic polynomial equation $x^*x = b$ ($b = b^*$, $b > 0$ on the boundary of stability, x stable), known also as a spectral factorization problem. The solution can be found by iterating a linear equation $a^*x + x^*a = 2b$ (a stable), see [4]. Such equations were investigated in [5].

In the matrix case, the conjugation operation includes matrix transpose: $A \mapsto A^*$ means $A(s) \mapsto A^{\mathrm{T}}(-s)$ or $A(d) \mapsto A^{\mathrm{T}}(d^{-1})$, the relevant equation is

$$A^*X + X^*A = 2B\,. \tag{1}$$

An algorithm for matrix spectral factoıization was reported in [6].

Understanding the structure of equation (1) and of its solution is fundamental for construction of efficient numerical algorithms. This papeı aims at a rigorous and comprehensive theory of (1) as a natural generalization of two simpler cases: that of constant matrices and that of scalar polynomials. A new algorithm for numerical solution is also presented, superior to that previously published in [6].

2. FIRST PRELUDE: THE MATRIX EQUATION

Before tackling the general matrix polynomial equation, it is worth to note the special case of equation

$$A^{\mathrm{T}}X + X^{\mathrm{T}}A = 2B \tag{2}$$

where A, B, X are matrices of real numbers. A bit of terminology: real matrices with properties $S = S^{\mathrm{T}}$, $Q = -Q^{\mathrm{T}}$ are called symmetric, skew-symmetric; complex matrices with $S = \bar{S}^{\mathrm{T}}$, $Q = -\bar{Q}^{\mathrm{T}}$ hermitian, skew-hermitian. For hermitian matrices, $R \geqq S$ means $R - S$ positive semidefinite, $R > S$ means $R - S$ positive definite. Principal minors of a square matrix are those obtained by selecting some subset of rows and the same subset of columns. Corner principal minors are those situated in the left upper corner.

Theorem M1. The homogeneous matrix equation

$$A^{\mathrm{T}}X + X^{\mathrm{T}}A = 0 \tag{3}$$

where A is nonsingular, has the general solution

$$X = QA \tag{4}$$

where Q is an arbitrary skew-symmetric matrix. Further, if A is upper triangular and if the same is required for X, then the equation has only trivial solution.

Proof. Premultiplying (3) by $A^{-\mathrm{T}}$, postmultiplying by A^{-1} and introducing the substitution

$$\hat{X} = XA^{-1} \tag{5}$$

we obtain an equivalent equation

$$\hat{X} + \hat{X}^{\mathrm{T}} = 0 \tag{6}$$

whose general solution is evident: the symmetric part of $\hat{X}$ is zero, the skew-symmetric one is arbitrary. By the backward substitution, we get (4).

Now, with A upper triangular, the triangularity is conserved by (5): X is triangular iff $\hat{X}$ is. So $\hat{X}$ must be triangular as well as skew-symmetric; the only such $\hat{X}$ is zero, and so is X. □

Theorem M2. The matrix equation

$$A^{\mathrm{T}}X + X^{\mathrm{T}}A = 2B \tag{7}$$

where A is nonsingular, B symmetric, is always solvable and its general solution is

$$X = X_{\mathrm{P}} + QA \tag{8}$$

X_{P} being a particular solution and Q an arbitrary skew-symmetric matrix. Furthermore:

a) If A is upper triangular then there exists the unique upper triangular solution.
b) If $B > 0$ then every X is nonsingular.

c) If $B > 0$ and A upper triangular with positive diagonal entries then the upper triangular X has also positive diagonal entries.

Proof. By multiplying $A^{-T}(\ )A^{-1}$ and by (5) we get an equivalent equation

$$\hat{X} + \hat{X}^{T} = 2\hat{B} \tag{9}$$

where $\hat{B} = A^{-T}BA^{-1}$. Its general solution is

$$\hat{X} = \hat{B} + Q \tag{10}$$

with an arbitrary skew-symmetric Q.

With A upper triangular, $\hat{X}$ is taken (uniquely) as the triangular part of $2\hat{B}$: $\hat{x}_{ij} = 2\hat{b}_{ij}$ for $i < j$, $\hat{x}_{ii} = \hat{b}_{ii}$. The uniqueness is conserved by (5).

With $B > 0$, the same holds for $\hat{B}$ and every $\hat{X}$ is nonsingular according to Theorem A1, see Appendix A. The same is X.

With $\hat{b}_{ii} > 0$, it is $\hat{x}_{ii} > 0$, and with $a_{ii} > 0$ we have $x_{ii} > 0$. □

Theorem M2 abc) can be strengthened:

Theorem M3. If $B > 0$ and A is nonsingular upper triangular then every solution X has nonzero principal minors. In that case, if A has positive diagonal entries then the mentioned minors are also positive.

Proof. The $\hat{X}$ in (10) has positive principal minors, according to Theorem A2. see Appendix A. By the backward substitution, the minors remain nonzero and with $a_{ii} > 0$, they remain positive. □

Theorem M3 has a converse:

Theorem M4. If $B > 0$ and A has nonzero corner principal minors, then there exists the unique upper triangular solution X. If the mentioned minors are positive then the X has positive diagonal entries.

Proof. The matrix A having nonzero principal corner minors can be decomposed

$$A = LU \tag{11}$$

with L lower and U upper triangular. Use (11) in (7), multiply $U^{-T}(\ )U^{-1}$ and introduce the substitution

$$\hat{X} = L^{T}XU^{-1} \tag{12}$$

We get (9) with $\hat{B} = U^{-T}BU^{-1} > 0$. The unique triangular solution was proved to exist.

If A has positive principal corner minors then L and U have positive diagonal entries. The entries $\hat{x}_{ii} = \hat{b}_{ii}$ are positive and so are x_{ii}. □

Theorem M5. (Algorithm of solution.)

The upper triangular solution X of equation (7) where A is nonsingular upper triangular and $B > 0$, can be obtained by the recurrent formulas

$$x_{ii} = \frac{1}{a_{ii}} \left[b_{ii} - \sum_{k=1}^{i-1} a_{ki} x_{ki} \right] \quad i = 1, \ldots, n \tag{13}$$

$$x_{ij} = \frac{1}{a_{ii}} \left[2b_{ij} - \sum_{k=1}^{i-1} a_{ki} x_{kj} - \sum_{k=1}^{i} x_{ki} a_{kj} \right] \quad i = 1, \ldots, n\,; \quad j = i + 1, \ldots, n\,. \tag{14}$$

Proof. Write (7) using subscripts:

$$\sum_{k=1}^{i} (a_{ki} x_{kj} + x_{ki} a_{kj}) = 2b_{ij} \quad i = 1, \ldots, n\,, \quad j = i, \ldots, n\,. \tag{15}$$

For the diagonal entries

$$\sum_{k=1}^{i} a_{ki} x_{ki} = b_{ii} \quad i = 1, \ldots, n\,. \tag{16}$$

In (15), isolate the last term in the first sum, in (16) the last term in the sum. From that, we obtain (13), (14); $a_{ii} \neq 0$ as A is nonsingular. □

3. SECOND PRELUDE: THE POLYNOMIAL EQUATION

As a second preparatory stuff, we shall deal with properties of an equation

$$a^* x + x^* a = 2b \tag{17}$$

where a, b, x are polynomials with real coefficients, $b = b^*$ (in the discrete-time case, b is a 'two-sided polynomial'). A related equation

$$a^* x + y^* b = c + d^* \tag{18}$$

is also dealt with. Proofs are omitted as they can be easily derived from [5] where the equations are investigated under general assumptions. Here we limit our attention to the case most important for control theory, especially for the spectral factorization problem: that of stable polynomials a, b in (18). Stability is meant here in the strict sense: $a(s) \neq 0$ for $\operatorname{Re} s \geqq 0$ in the continuous case and $a(d) \neq 0$ for $|d| \leqq 1$ in the discrete case. With stable polynomials, a^*, b in (18) are coprime and their absolute terms a_0, $b_0 \neq 0$ (the latter property plays a role only in the discrete-time case). By that, the structure gets simpler.

The discrete-time case and the continuous-time one are somewhat different. It is caused by different properties of conjugation operation $a(s) \mapsto a(-s)$ and $a(d) \mapsto$ $\mapsto a(d^{-1})$. The general solution of continuous-time equations contains polynomials of arbitrarily high degrees unlike the discrete-time equations where the degrees of solution are bounded, see [5]. To obtain mutually corresponding results, we must add assumptions and requirements for degrees.

Theorem P1. The homogeneous polynomial equation

$$a^* x + x^* a = 0 \tag{19}$$

where a is stable (in the continuous-time case with requirement $\deg x \leqq \deg a$) has only trivial solution.

Theorem P2. The polynomial equation

$$a^*x + x^*a = 2b \tag{20}$$

where a is stable (in the continuous-time case with assumption $\deg b \leqq 2 \deg a$ and with requirement $\deg x \leqq \deg a$) is always solvable and has unique solution.

Theorem P3. If a in (20) is stable and if $b > 0$ on the boundary of stability then the solution x is also stable. In the discrete-time case, if the absolute coefficient $a_0 > 0$ then $x_0 > 0$. In the continuous-time case, $\deg b = 2 \deg a$ implies $\deg x = \deg a$ and if the leading coefficient $a_H > 0$ then $x_H > 0$.

Proof is based on properties of harmonic functions of the complex variable, see [4].

Theorem P4. The homogeneous polynomial equation

$$a^*x + y^*b = 0 \tag{21}$$

where a, b are stable (in the continuous-time case with requirements $\deg x \leqq \deg b$, $\deg y \leqq \deg a$) has the general solution $x = qb$, $y = -qa$ where q is an arbitrary number.

Theorem P5. The polynomial equation

$$a^*x + y^*b = c + d^* \tag{22}$$

where a, b are stable (in the continuous-time case with assumption $\deg(c + d^*) \leqq \leqq \deg a + \deg b$ and with requirement $\deg x \leqq \deg b$, $\deg y \leqq \deg a$), is always solvable and its general solution is $x = x_P + qb$, $y = y_P - qa$ where (x_P, y_P) is a particular solution and q an arbitrary number.

With an additional requirement $y_0 = 0$ (i.e. the absolute term zero) in the discrete-time case, or $\deg y < \deg a$ (i.e. the leading term zero) in the continuous-time case, the solution is unique.

The above mentioned equations can be solved numerically using the Euclidean algorithm. In [4], [5], such algorithms are presented which conserve the symmetry during the process. It causes savings in computational operations.

4. THE MATRIX POLYNOMIAL EQUATION

Now we are ready to tackle equation (1) where A, B, X are real matrix polynomials. Again, only equations with stable matrix polynomial A are investigated, i.e. those with $\det A$ stable. Note that stability includes nonsingularity ($\det A \neq 0$).

The discrete-time case and the continuous-time one are treated in a unified way; relations between degrees are replaced by properness of polynomial fractions: $\deg x \leqq \deg a$ iff xa^{-1} is proper, $\deg b \leqq 2 \deg a$ iff $a^{-*}ba^{-1}$ is proper. Similarly, $\deg x = \deg a$ iff xa^{-1} is biproper (proper together with an inverse).

Theorem MP1. The homogeneous matrix polynomial equation

$$A^*X + X^*A = 0 \tag{23}$$

where A is stable (in the continuous-time case with requirement XA^{-1} = proper) has the general solution

$$X = QA \tag{24}$$

where Q is an arbitrary constant skew-symmetric matrix.

Proof. Denote $\tilde{A} = \operatorname{adj} A$, $a = \det A$. Multiply (23) by $\tilde{A}^*(\ )\tilde{A}$ getting an equivalent equation

$$a^*X\tilde{A} + \tilde{A}^*X^*a = 0 \tag{25}$$

By substitution $X\tilde{A} = \hat{X}$ we get an equation

$$a^*\hat{X} + \hat{X}^*a = 0 \tag{26}$$

Properness of XA^{-1} is equivalent to that of $\hat{X}a^{-1}$. We have not yet proved equivalence of (23) and (26): for every X satisfying (23) there exists $\hat{X} = X\tilde{A}$ satisfying (26) but from the other side, for every $\hat{X}$ the expression $X = \hat{X}\tilde{A}^{-1}$ is not evident to be a polynomial. That we shall prove later.

The general solution of (26) can be found elementwise. For diagonal entries we have

$$a^*\hat{x}_{ii} + \hat{x}_{ii}^*a = 0\,, \quad i = 1, \ldots, n\,. \tag{27}$$

As a is stable, (27) has only trivial solution according to Theorem P1. For non-diagonal entries:

$$a^*\hat{x}_{ij} + \hat{x}_{ji}^*a = 0\,, \quad i = 1, \ldots, n\,; \quad j = i + 1, \ldots, n\,. \tag{28}$$

According to Theorem P4, the general solution is

$$\hat{x}_{ij} = q_{ij}a\,, \quad \hat{x}_{ji} = -q_{ij}a\,,$$

where q_{ij} are arbitrary numbers. It can be written $\hat{X} = aQ$ with a skew-symmetric Q.

Now, for every $\hat{X}$ we see that $X = \hat{X}\tilde{A}^{-1} = QA$ is a polynomial. It proves the equivalence of (23), (26) and the theorem. □

Theorem MP2. The matrix polynomial equation

$$A^*X + X^*A = 2B \tag{29}$$

where A is stable, $B = B^*$ (in the continuous-time case with assumption $A^{-*}BA^{-1} =$ = proper and with requirement XA^{-1} = proper) is always solvable and its general solution is

(30) $$X = X_{\mathrm{P}} + QA$$

where X_{P} is a particular solution and Q an arbitrary constant skew-symmetric matrix.

Proof. The form of the general solution is evident from Theorem MP1; only the existence of X_{P} remains to be proved.

Like in Theorem MP1, we convert (29) to

(31) $$a^*\hat{X} + \hat{X}^*a = 2\hat{B}$$

where $\hat{B} = \tilde{A}^*B\tilde{A}$, the properness of $A^{-*}BA^{-1}$ being equivalent to that of $a^{-*}\hat{B}a^{-1}$. We shall prove equivalence of (29), (31), i.e. we prove that for every $\hat{X}$ satisfying (31), $X = \hat{X}\tilde{A}^{-1} = \hat{X}A/a$ is a polynomial. Multiply (31) by A:

$$a^*\hat{X}A + \hat{X}^*aA = \tilde{A}^*Ba$$

The right-hand side is divisible by a, so is the term $\hat{X}^*aA$ and so must be the first term. But a is stable, a, a^* are coprime, $\hat{X}A$ must be divisible by a. So $\hat{X}A/a$ is polynomial and the equivalence is proved.

Solve (31) elementwise. For diagonal entries:

(32) $$a^*\hat{x}_{ii} + \hat{x}_{ii}^*a = 2\hat{b}_{ii}, \quad i = 1, \ldots, n.$$

The fraction $a^{-*}\hat{b}_{ii}a^{-1}$ is proper, (32) is solvable according to Theorem P2. For non-diagonal entries:

(33) $$a^*\hat{x}_{ij} + \hat{x}_{ji}^*a = 2\hat{b}_{ij}, \quad i = 1, \ldots, n; \quad j = i + 1, \ldots, n.$$

The fraction $a^{-*}\hat{b}_{ij}a^{-1}$ is proper, (33) is solvable according to Theorem P5. The solvability of (29) is proved. □

To establish a matrix polynomial analogy of Theorems M2abc) and P3, we need more on properness of matrix polynomial fractions. We define column degrees of A as $p_j = \operatorname{col\,deg} a_{ij} = \max_i \deg a_{ij}$. Then we define a columnwise-leading matrix A_{H} as a matrix whose jth column contains p_jth coefficients of $a_{ij}(s)$. It can be expressed as $A_{\mathrm{H}} = \lim_{s\to\infty} A(s) \operatorname{diag}_i (s^{-p_j})$. Assuming A_{H} nonsingular (i.e. $A(s)$ column-reduced), XA^{-1} is proper iff $\operatorname{col\,deg}_j X \leqq \operatorname{col\,deg}_j A$ for all j. Assuming both A_{H} and X_{H} nonsingular, $\operatorname{col\,deg}_j X = \operatorname{col\,deg}_j A$ means XA^{-1} biproper.

For symmetric matrix polynomials, we have diagonal degrees $q_j = \deg b_{jj}$ and a diagonalwise-leading matrix B_{H} whose (i, j)th entry is the $\frac{1}{2}(q_i + q_j)$th coefficient of $b_{ij}(s)$ multiplied by $(-1)^{q_i/2}$. It can be expressed as

$$B_{\mathrm{H}} = \lim_{s\to\infty} \operatorname{diag}_i ((-s)^{-q_i}) \,.\, B(s) \,.\, \operatorname{diag}_j (s^{-q_j}).$$

Theorem MP2. Under the assumptions of Theorem MP2, let the absolute matrix A_0 (in the discrete-time case) or the columnwise-leading matrix A_{H} (in the continuous-time case) be upper triangular. In the continuous-time case, we also assume A_H nonsingular, i.e. $A(s)$ column-reduced; in the discrete-time case, A_0 is always nonsingular when A is stable. Then there exists the unique solution with X_0 or X_H upper triangular (in the latter case, X_H is defined via the column degrees of A).

Proof. Let X_P be a particular solution, from (30) we have

$$X_0 = X_{P0} + QA_0 \quad \text{or} \quad X_H = X_{PH} + QA_H \,. \tag{34}$$

Given A_0, X_{P0}, we look for such Q which makes X_0 upper triangular. Write (34) using subscripts:

$$0 = \xi_{ij} - \sum_{k=1}^{j} q_{ki}\alpha_{kj} \,, \quad i = 2, \ldots, n \,; \quad j = 1, \ldots, i - 1 \,. \tag{35}$$

It can be easily seen that the unique solution of system (35) is given by the recurrent formula

$$q_{ji} = \frac{1}{\alpha_{jj}} \left(\xi_{ij} - \sum_{k=1}^{j-1} q_{ki}\alpha_{kj}\right) . \tag{36}$$

□

Theorem MP4. If the matrix B in (29) satisfies $B > 0$ on the boundary of stability then every X is stable. In that case, if A_0 or A_H is upper triangular with positive diagonal entries and if $A^{-*}BA^{-1}$ is biproper (in the continuous-time case only) then the solution with X_0 or X_H upper triangular has also positive diagonal entries of these matrices. In the continuous-time case, this XA^{-1} is biproper.

Proof. For the discrete-time case, multiply (29) by $A^{-*}(\;)\,A^{-1}$:

$$XA^{-1} + A^{-*}X^* = 2A^{-*}BA^{-1} \,. \tag{37}$$

As A is stable, $F = XA^{-1}$ is an analytic function of complex variable d for $|d| \leqq 1$. The function

$$G(d) = \tfrac{1}{2}[F(d) + \overline{F^{T}(\bar{d})}] = \tfrac{1}{2}[X(d)\,A^{-1}(d) + \overline{A^{-T}(\bar{d})}\,\overline{X^{T}(\bar{d})}] \,, \tag{38}$$

as its hermitian part, is harmonic in that region, see Appendix B. For $|d| = 1$, $\overline{F^{T}(\bar{d})} = F^{T}(d^{-1})$ holds, hence it is here

$$G = \tfrac{1}{2}(F + F^*) = A^{-*}BA^{-1} > 0 \,.$$

For $|d| < 1$ it must also hold $G > 0$, otherwise the function G would attain a minimum inside the region, an impossible thing for a harmonic function. So it is $F + \overline{F^{T}} > 0$, and according to Theorem A1, F is nonsingular for $|d| \leqq 1$. Hence $\det X \neq 0$ for $|d| \leqq 1$, X is stable.

Consider the absolute terms. For $d = 0$, (38) leads to

$$A_0^{T}X_0 + X_0^{T}A_0 = 2A_0^{T}\,G(0)\,A_0 > 0 \,.$$

This is an equation of type (7); according to Theorem M2, the diagonal entries of X_0 are positive.

For the continuous-time case, the region of analyticity of F is bounded by the imaginary axis and the right half circle with radius R. As it can be arbitrarily large, X is stable. To consider the leading terms, note that XA^{-1} is proper, $F(\infty)$ finite, $\lim_{s\to\infty} X(s)\, A^{-1}(s) = X_H A_H^{-1}$. As $A^{-*}BA^{-1}$ is biproper, $G(\infty) = A_H^{T-1} B_H A_H^{-1}$ is finite and > 0. For $s \to \infty$, we obtain

$$A_H^T X_H + X_H^T A_H = 2A_H^T\, G(\infty)\, A_H > 0$$

and the diagonal entries of X_H positive. So $\operatorname{col}\deg_j X = \operatorname{col}\deg_j A$ and, as X_H is nonsingular, XA^{-1} is biproper. □

5. THE ALGORITHM

The idea of the proof of Theorem MP2 — conversion of the original matrix equation (29) into a scalar form (31) can be used as a computational algorithm [6]:

a) $a = \det A$, $\tilde{A} = \operatorname{adj} A$ are computed
b) $\hat{B} = \tilde{A}^* B \tilde{A}$ is transformed
c) equations (31) are solved for each ij. For a nondiagonal entry, a choice is made to select a particular solution, possibly the simplest one
d) the backward substitution is performed
e) when the solution with triangular X_0 is needed, the fitting of Q is performed, see the proof of Theorem MP3.

The scheme works well and the computation complexity is reasonably low. But the need to compute a matrix polynomial adjoint is something not to be too happy with. The another disadvantage is the necessity of the fitting e); it would be better to obtain the triangular solution directly.

In the discrete-time case, we can transform matrix polynomial A to an upper triangular matrix polynomial $\hat{A}$ by a unimodular U from the right:

$$\hat{A} = AU\,. \tag{39}$$

Applying it to (29) we obtain

$$\hat{A}^* \hat{X} + \hat{X}^* \hat{A} = 2\hat{B} \tag{40}$$

where

$$\hat{B} = U^* B U \tag{41}$$

and

$$X = U^{-1} \hat{X}\,. \tag{42}$$

As a unimodular substitution has a polynomial inverse, (29) and (40) are equivalent. Moreover, from $\hat{A}_0 = A_0 U_0$ we see that U_0 is triangular and X_0 is triangular together with $\hat{X}_0$. Equation (40) can be solved in a way resembling Theorem M5:

Theorem MP5. (Algorithm of solution.) The solution of equation (40) with $\hat{X}_0$

upper triangular can be obtained by recurrent solving of polynomial equations (omitting the $\hat{}$):

$$a_{ii}^* x_{ii} + x_{ii}^* a_{ii} = 2b_{ii} - \sum_{k=1}^{i-1} (a_{ki}^* x_{ki} + x_{ki}^* a_{ki}), \quad i = 1, \ldots, n \tag{43}$$

$$a_{ii}^* x_{ij} + x_{ji}^* a_{jj} = 2b_{ij} - \sum_{k=1}^{i-1} a_{ki}^* x_{kj} - \sum_{k=1}^{j-1} x_{ki}^* a_{kj}, \tag{44}$$

$$i = 1, \ldots, n; \quad j = i + 1, \ldots, n$$

where in (44), the solution with the absolute term of x_{ji} equal to zero is selected.

Proof. Write (40) using subscripts:

$$\sum_{k=1}^{i} a_{ki}^* x_{kj} + \sum_{k=1}^{j} x_{ki}^* a_{kj} = 2b_{ij}.$$

Isolating the last terms in sums we get (43) and (44). □

The computational scheme is evident. The transformation (39) is realized as a sequence of elementary transformations $U = \prod_k U_k$. For every U_k, it is easy to construct U_k^*, so (41) can be performed by elementary steps as well. Moreover, $U^{-1} = \prod_k U_k^{-1}$ (in reverse order); every U_k^{-1} is easily constructed and U^{-1} can be stored as a matrix polynomial or as a coded sequence of elementary operations. It waits for the backward run (42).

The algorithm was implemented with Fortran on IBM 370/135 computer using double precision format. In comparison with the previous algorithm, number of operations was significantly reduced. Another advantage more was noted: when operating with polynomials, it may happen due to round-off errors that degrees of actually computed polynomials are greater than the theoretical ones, the 'leading' terms being e.g. 10^{-13} times less than the true ones. It is difficult for an algorithm not to generate these parasitic terms; this case occurs in the computation of an adjoint. In the new algorithm when only elementary unimodular transformations are used, this unwanted effect is greatly reduced.

Unfortunately, no continuous-time version of the algorithm is known to the author. The cause is in (39) – $\hat{A}$ cannot be made upper triangular and in the same time column-reduced (needed for selecting the proper solution).

6. CONCLUSIONS

The symmetric matrix polynomial equation was investigated to the extent needed for the matrix spectral factorization problem. No attempt was made to cover the general case of A singular or not stable. Although not urgently demanded from control problems, this case would deserve a further study from the mathematician's point of view. Another theme is generalizing symmetric equations for other algebras than those of matrices and polynomials.

APPENDIX A: SOME PROPERTIES OF MATRICES

In this section, two theorems concerning properties of matrices are proved which cannot be found in standard matrix algebra books.

Theorem A1. Let A be a complex matrix, let its hermitian part S be positive definite. Then A is nonsingular.

Proof. By contradiction: let A be singular, then a vector $u \neq 0$ exists satisfying $Au = 0$, $\bar{u}^{\mathrm{T}}Au = 0$, $\operatorname{Re} \bar{u}^{\mathrm{T}}Au = \frac{1}{2}\bar{u}^{\mathrm{T}}(A + \bar{A}^{\mathrm{T}})\, u = \bar{u}^{\mathrm{T}}Su = 0$. But for positive definite S, it must be $\bar{u}^{\mathrm{T}}Su > 0$. $\square$

Theorem A2. Let A be a real matrix, let its symmetric part S be positive definite. Then

a) all eigenvalues of A have positive real parts

b) all principal minors of A are positive.

Proof. Let α be an eigenvalue of A (complex in general), u a corresponding eigenvector (also complex). Write $A = S + Q$ with symmetric S and skew-symmetric Q and consider

$$f = \bar{u}^{\mathrm{T}}Au = \bar{u}^{\mathrm{T}}Su + \bar{u}^{\mathrm{T}}Qu\,.$$

The expression $\bar{u}^{\mathrm{T}}Su$ is real positive as it can be thought as a hermitian form. The expression $u^{\mathrm{T}}Qu$ is imaginary as it can be thought as a skew-hermitian form. So we have $\operatorname{Re} f > 0$, $f = \bar{u}^{\mathrm{T}}Au = \alpha\|u\|^2$, $\operatorname{Re} \alpha > 0$.

As a real matrix, A has all eigenvalues real or pairwise complex conjugated. The $\det A$, as a product of eigenvalues, satisfies $\det A > 0$. All principal corner submatrices S_{IJ} being positive definite, $\det S_{IJ} > 0$ must hold for all principal corner minors. Finally, every principal minor can be permuted into the corner position without loss of positive definiteness. $\square$

Note. For a complex matrix A, only the part a) of Theorem A2 holds.

APPENDIX B: MATRICIAL HARMONIC FUNCTIONS

In this section, properties of harmonic function of complex variable are generalized to a matrix case, especially to hermitian positive semidefinite matrices. We begin with recalling the scalar case:

A real function $u(x, y)$ of complex variable $z = x + \mathrm{i}y$ is called *harmonic* in a region if it satisfies Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

there. For every analytic function $f(z)$, its real part $u(z)$ and imaginary part $v(z)$

are harmonic, they are related by Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

In a singly connected region, each of the functions u, v is determined by the other uniquely up to an additive constant.

The maximum theorem: if $u(z) \neq$ const is harmonic in a closed region then $u(z)$ cannot attain its maximum (nor minimum) at an internal point.

Similarly to the decomposition of a complex number into the real and imaginary parts, a complex square matrix F can be decomposed into the hermitian part G and the skew-hermitian part H:

$$G = \tfrac{1}{2}(F + \bar{F}^{\mathrm{T}}), \quad H = \tfrac{1}{2}(F - \bar{F}^{\mathrm{T}}).$$

Each of the matrices G, H contains n^2 real entries.

If $F(z)$ is a matricial analytic function then $G(z)$ and $H(z)$ are harmonic, i.e. all their entries are harmonic. From the other side, given n^2 harmonic functions $g_{ii}, g_{ij} = u_{ij} + iv_{ij}$ $(i = 1, \ldots, n;\ j = i + 1, \ldots, n)$ in a singly connected region, all functions f_{ij} are determined uniquely up to an additive skew-hermitian constant. They are given by an integrable partial differential equation system consisting of the Cauchy-Riemann equations as well as the decomposition equations.

Exactly as in the scalar case, the maximum theorem can be proved, the maximum being in the sense of comparison of hermitian positive semidefinite matrices.

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Ing. Jan Ježek, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.