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INTERNAL PROPERNESS AND STABILITY IN LINEAR SYSTEMS*

VLADIMÍR KUČERA

The concept of internal properness is introduced for systems $E\dot{x} = Fx + Gu$, $y = Hx$ so as to parallel the concept of internal stability. The system is internally proper if its free response is devoid of impulsive modes. The use of internal properness and stability in the design of control systems is demonstrated. The examples include disturbance rejection, model matching, output regulation and block decoupling problems.

1. SINGULAR SYSTEMS

Consider the linear system

$$E\,\dot{x}(t) = F\,x(t) + G\,u(t)\,, \quad t \geqq 0 \qquad (1.1)$$
$$y(x) = H\,x(t)$$

where E and F are $n \times n$ real matrices, G is $n \times q$ and H is $p \times n$. Taking the Laplace transform,

$$x(s) = (sE - F)^{-1}\,[E\,x(0-) + G\,u(s)] \qquad (1.2)$$
$$y(s) = H\,x(s)\,.$$

It is assumed that $sE - F$ is nonsingular so that unique solutions of (1.1) are obtained for all $E\,x(0-)$ and $u(s)$.

When E is nonsingular, (1.1) is a standard (or regular) linear system. In the case of arbitrary E we speak of *generalized* (or singular) systems. The free response $x(t)$, $t \geqq 0$ of such a system may exhibit not only *exponential* modes associated with finite zeros of $sE - F$ but also *impulsive* modes (i.e. Dirac distributions and their

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derivatives) associated with infinite zeros of $sE - F$. The transfer matrix of (1.1),

$$T(s) = H(sE - F)^{-1} G \tag{1.3}$$

may be any, possibly improper and unstable, rational matrix.

From the modelling point of view, the systems described by (1.1) coincide with the set of linear systems generated by various interconnections of *integrators*, *differentiators* and *scalors*. The state of the system (1.1) at time t is $E\,x(t-)$; this vector embodies all information on the integrators and differentiators that is necessary to solve (1.1). The order of (1.1) thus equals rank $E(\leqq n)$.

As an example, the pure integrator $\boldsymbol{I}$ can be described by (1.1) with

$$E = 1\,,\quad F = 0\,,\quad G = 1\,,\quad H = 1$$

the pure differentiator $\boldsymbol{D}$ by

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\quad F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and the scalor $\boldsymbol{k}$ by

$$E = 0\,,\quad F = -1\,,\quad G = 1\,,\quad H = \boldsymbol{k}\,.$$

2. CONTROLLABILITY AND OBSERVABILITY

The fundamental theory of singular or generalized systems was pioneered by Rosenbrock [30] and developed by Verghese, Lévy and Kailath [32]. Further insight is given in Callier and Desoer [5].

By definition, a free-response mode of (1.1) that can be alternatively excited from zero initial conditions by means of an input that contains no component at the modal (finite or infinite) frequency is termed a *controllable* mode. The system is completely controllable if all its modes are controllable. The controllability of exponential modes is characterized by a matrix

$$\begin{bmatrix} sE - F & G \end{bmatrix} \tag{2.1}$$

that has no finite zeros, i.e. has full row rank for all finite s. The controllability of impulsive modes is equivalent to (2.1) having no zero at $s = \infty$, i.e. to (2.1) having a proper right inverse.

Dually, a free-response mode of the system (1.1) that gives rise to a zero output is said to be an *unobservable* mode. The system is completely observable if it has no unobservable mode. The observability of exponential modes is characterized by a matrix

$$\begin{bmatrix} sE - F \\ H \end{bmatrix} \tag{2.2}$$

that has no finite zeros, i.e. has full column rank for all finite s. The observability of impulsive modes is equivalent to (2.2) having no zeros at $s = \infty$, i.e. to (2.2) having a proper left inverse.

One can bring the generalized system (1.1) by allowed transformations (Verghese, Lévy and Kailath [32]) into a form that displays 1) the controllable and observable part, 2) the controllable but unobservable part, 3) the observable but uncontrollable part, and 4) the uncontrollable and unobservable part. Clearly, the transfer matrix of (1.1) depends only on part 1). Systems that are completely controllable and completely observable will be termed *irreducible*.

3. PROPERNESS AND STABILITY

The behaviour of the system (1.1) at $t = 0$ and $t \to \infty$ is of considerable importance, see Callier and Desoer [5] or Kučera [20]. The limiting behaviour as $t \to \infty$ is studied by means of stability. The aim of this paper is to make explicit the notion of properness reflecting the behaviour at $t = 0$ in a manner which would let appear the conceptual similarity of the two notions.

Definition 1. The system (1.1) is internally proper if the matrix $sE - F$ has no infinite zeros, i.e. $(sE - F)^{-1}$ is proper.

Definition 2. The system (1.1) is internally stable if the matrix $sE - F$ is devoid of finite zeros in the closed right half-plane $\operatorname{Re} s \geqq 0$, i.e. if $(sE - F)^{-1}$ is stable.

Thus, in the light of (1.2), the internal properness of (1.1) means that the free motion $x(t)$, $t \geqq 0$ of the system comprises no impulsive modes at $t = 0$ for every initial condition $E\,x(0-)$. On the other hand, the internal stability of (1.1) means that $x(t)$ tends to the origin as $t \to \infty$ irrespective of $E\,x(0-)$, i.e. the system is asymptotically stable in the sense of Lyapunov.

It is to be noted that regular systems, made up of integrators and scalors, are internally proper. On the other hand, an internally proper system may well consist of differentiators and scalors only, e.g.

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The system is shown in Fig. 1.

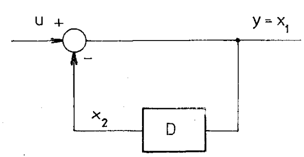

Fig. 1. An internally proper system.

The above notions of properness and stability reflect the *internal* properties of the system and are to be strictly distinguished from the properness and stability as viewed by an *external* observer. The latter simply amounts to the properness and stability of the system transfer matrix (1.3). Evidently, the internal properness (or stability) implies the external one and the two are equivalent if and only if the system is irreducible. The example of a system which is externally but not internally proper is given by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

see Fig. 2.

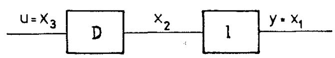

Fig. 2. An externally but not internally proper system.

4. FEEDBACK SYSTEMS

Consider two *irreducible* systems

$$E_1 \dot{x}_1(t) = F_1 x_1(t) + G_1 u_1(t), \quad t \geqq 0 \qquad (4.1)$$
$$y_1(t) = H_1 x_1(t)$$

where E_1 and F_1 are $n_1 \times n_1$, G_1 is $n_1 \times q$, H_1 is $p \times n_1$ and

$$E_2 \dot{x}_2(t) = F_2 x_2(t) + G_2 u_2(t), \quad t \geqq 0 \qquad (4.2)$$
$$y_2(t) = H_2 x_2(t)$$

where E_2 and F_2 are $n_2 \times n_2$, G_2 is $n_2 \times p$ and H_2 is $q \times n_2$. Connect them according to

$$u_1(t) = v_2(t) - y_2(t) \qquad (4.3)$$
$$u_2(t) = v_1(t) + y_1(t)$$

where v_1 and v_2 are external inputs.

The special structure of the composite system implied by (4.3) together with irreducibility of (4.1) and (4.2) implies that the internal properness and stability of the feedback system is equivalent to the properness and stability of the transfer matrix relating inputs v_1, v_2 and outputs u_1, u_2.

The internal properties of the feedback system can thus be studied by means of transfer matrices of the individual components. In order to achieve this goal, it is covenient to factorize the component transfer matrices in terms of *proper, stable, rational* matrices (i.e. those having neither finite nor infinite poles in Re $s \geqq 0$).

To be specific, let $\mathbb{R}_{ps}(s)$ denote the set of proper stable rational functions and $\mathbb{R}_{ps}^{m \times n}$ the set of $m \times n$ matrices with entries from $\mathbb{R}_{ps}(s)$. Write the transfer matrix of (4.1) in the form

$$T_1(s) = A^{-1}(s)\, B(s) = B_1(s)\, A_1^{-1}(s) \tag{4.4}$$

and the transfer matrix of (4.2) as

$$T_2(s) = P^{-1}(s)\, Q(s) = Q_1(s)\, P_1^{-1}(s)\,. \tag{4.5}$$

Here $A \in \mathbb{R}_{ps}^{p \times p}(s)$, $B \in \mathbb{R}_{ps}^{p \times q}(s)$ as well as $P \in \mathbb{R}_{ps}^{q \times q}(s)$, $Q \in \mathbb{R}_{ps}^{q \times p}(s)$ are relatively left prime and $B_1 \in \mathbb{R}_{ps}^{p \times q}(s)$, $A_1 \in \mathbb{R}_{ps}^{q \times q}(s)$ as well as $Q_1 \in \mathbb{R}_{ps}^{q \times p}(s)$, $P_1 \in \mathbb{R}_{ps}^{p \times p}(s)$ are relatively right prime.

The issue of internal properness and stability of the composite system is then settled by the following result, adapted from Desoer and co-workers [10].

Lemma 1. Let both (4.1) and (4.2) be irreducible systems. Then, the composite system (4.1)–(4.3) is internally proper and stable if and only if $AP_1 + BQ_1 = U_1$ is a unit of $\mathbb{R}_{ps}^{p \times p}(s)$ or equivalently, $PA_1 + QB_1 = U$ is a unit of $\mathbb{R}_{ps}^{q \times q}(s)$. □

To recall, the units of $\mathbb{R}_{ps}^{n \times n}$ are the rational matrices having neither pole nor zero, be it finite or infinite, in $\operatorname{Re} s \geqq 0$.

To illustrate Lemma 1, check the internal properness and stability of the feedback system shown in Fig. 1. The first system is a scalor given by

$$E_1 = 0\,, \quad F_1 = -1\,, \quad G_1 = 1\,, \quad H_1 = 1$$

and the second one is a pure differentiator given by

$$E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Hence

$$A = 1\,, \quad B = 1\,, \quad P_1 = \frac{1}{s + \alpha}\,, \quad Q_1 = \frac{s}{s + \alpha}$$

for an arbitrary positive real constant α, and

$$A\,P_1 + B\,Q_1 = \frac{s + 1}{s + \alpha}$$

is indeed a unit of $\mathbb{R}_{ps}(s)$.

5. SYNTHESIS OF FEEDBACK SYSTEMS

The synthesis of feedback systems plays a crucial role in the theory of control. The first subsystem (4.1) can be interpreted as a plant while the second one (4.2) as a compensator or a controller. The free response of a practical control system should contain just desired modes. One usually requires that the control system be internally proper and stable, thus avoiding impulsive and unstable exponential modes.

Given the plant, any compensator that renders the resulting control system internally proper and stable will be called *admissible.* Lemma 1 provides a criterion for a compensator (4.2) to be admissible. However, for the construction of admissible compensators it is useful to note that if P_1, Q_1 and U_1 are as in Lemma 1 then $(Q_1 U_1^{-1})(P_1 U_1^{-1})^{-1}$ is another factorization of T_2 (similarly for P, Q and U). Hence, following Kučera [17] and Desoer and coworkers [10], we now characterize the class of admissible compensators.

Theorem 1. Let (4.1) be an irreducible plant giving rise to the transfer matrix (4.4). Then, (4.2) is an admissible compensator if and only if its transfer matrix can be expressed by (4.5), where P, Q is any (proper, stable, rational) solution of the equation

$$PA_1 + QB_1 = I_q \tag{5.1}$$

such that P is nonsingular and P_1, Q_1 is any (proper, stable, rational) solution of the equation

$$AP_1 + BQ_1 = I_p \tag{5.2}$$

such that P_1 is nonsingular. Here I_p and I_q are the $p \times p$ and $q \times q$ identity matrices, respectively. □

If $\bar{P}$, $\bar{Q}$ is a particular solution of (5.1) and $\bar{P}_1$, $\bar{Q}_1$ is a particular solution of (5.2), then the general solutions of these equations (Kučera [17]) read

$$\begin{aligned} P(s) &= \bar{P}(s) + V(s)\,B(s) \\ Q(s) &= \bar{Q}(s) - V(s)\,A(s) \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} P_1(s) &= \bar{P}_1(s) + B_1(s)\,V_1(s) \\ Q_1(s) &= \bar{Q}_1(s) - A_1(s)\,V_1(s) \end{aligned} \tag{5.4}$$

where V ranges over $\mathbb{R}_{ps}^{q\times p}(s)$ and V_1 ranges over $\mathbb{R}_{ps}^{p\times q}(s)$.

The significance of this result consists in identifying all (irreducible) admissible compensators for a given plant in the *parametric* form (5.3) or (5.4), in terms of one free (proper, stable, rational) parameter V or V_1. More specific compensators can then be singled out simply by specifying the parameter appropriately, according to the additional requirements the control system is to meet.

6. VARIOUS CONTROL PROBLEMS

The implications of Theorem 1 for the solution of control problems will now be given. In particular, the requirement of internal properness and stability will be combined with various control strategies to illustrate the use of proper stable factorizations.

6.1. Disturbance Rejection

Consider an irreducible plant

$$(6.1)\qquad \begin{aligned} E_1 \dot{x}_1(t) &= F_1 x_1(t) + G_1 u(t) + G_1' v(t), \quad t \geqq 0 \\ y(t) &= H_1 x_1(t) \\ z(t) &= H_1' x_1(t) \end{aligned}$$

where E_1 and F_1 are $n_1 \times n_1$, G_1 and G_1' are $n_1 \times 1$, and H_1 and H_1' are $1 \times n_1$. Here u is the control, v is an arbitrary unmeasurable disturbance, y is the measurement and z is the output.

The disturbance rejection problem with internal properness and stability consists in finding an admissible compensator of the form

$$(6.2)\qquad \begin{aligned} E_2 \dot{x}_2(t) &= F_2 x_2(t) + G_2 y(t), \quad t \geqq 0 \\ u(t) &= H_2 x_2(t) \end{aligned}$$

where E_2 and F_2 are $n_2 \times n_2$, G_2 is $n_2 \times 1$, and H_2 is $1 \times n_2$ such that the transfer function from v to z in the composite system (6.1)–(6.2) shown in Fig. 3 be zero.

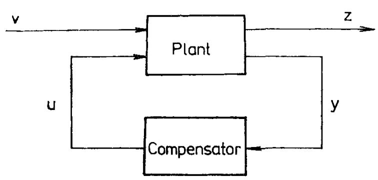

Fig. 3. Disturbance rejection.

To solve the problem, define the factorizations

$$(6.3)\qquad \begin{aligned} H_1(sE_1 - F_1)^{-1} G_1 &= \frac{b(s)}{a(s)} \quad H_1(sE_1 - F_1)^{-1} G_1' = \frac{c(s)}{a(s)} \\ H_1'(sE_1 - F_1)^{-1} G_1 &= \frac{d(s)}{a(s)} \quad H_1'(sE_1 - F_1)^{-1} G_1' = \frac{e(s)}{a(s)} \end{aligned}$$

where a, b, c, d and e is a quintuple of relatively prime elements of $\mathbb{R}_{ps}(s)$.

Similarly we define

$$(6.4)\qquad -H_2(sE_2 - F_2)^{-1} G_2 = \frac{q(s)}{p(s)}$$

for some $p, q \in \mathbb{R}_{ps}(s)$. We shall also need the relatively prime elements g, f of $\mathbb{R}_{ps}(s)$ defined by

$$(6.5)\qquad \frac{be - cd}{a} = \frac{g}{f}.$$

Theorem 2. The disturbance rejection problem with internal properness and stability is solvable if and only if the following four conditions all hold:

(a_1) a and b are relatively prime,

(a_2) a is a divisor of $be - cd$,

(b_1) cd is a divisor of both e and g,

(b_2) $g \neq 0$.

Proof. The conditions (a_1), (a_2) guarantee the internal properness and stability. Indeed, let h be the greatest common divisor of a and $be - cd$. Write $h = h_1, h_2$, where h_1 is the greatest common divisor of a, b and c. Thus

$$a = h_1 a_1\,, \quad a_1 = h_2 f$$
$$b = h_1 b_1$$
$$c = h_1 c_1$$

for some a_1, b_1 and c_1 of $\mathbb{R}_{ps}(s)$. Further let $\bar{x}$, $\bar{y}$ and $\bar{z}$ be any elements of $\mathbb{R}_{ps}(s)$ that satisfy the equations

$$b_1\bar{x} + h_2\bar{y} = d$$
$$c_1\bar{x} + h_2\bar{z} = e\,.$$

Then

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} h_1 & 0 \\ \bar{x} & h_2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ -f\bar{x} & fh_1 \end{bmatrix}$$
$$\begin{bmatrix} b & c \\ d & e \end{bmatrix} = \begin{bmatrix} h_1 & 0 \\ \bar{x} & h_2 \end{bmatrix} \begin{bmatrix} b_1 & c_1 \\ \bar{y} & \bar{z} \end{bmatrix}$$

and hence

$$A = \begin{bmatrix} a_1 & 0 \\ -f\bar{x} & fh_1 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & c_1 \\ \bar{y} & \bar{z} \end{bmatrix}$$

is a relatively left prime factorization of the plant transfer matrix

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{b}{a} & \frac{c}{a} \\ \frac{d}{a} & \frac{e}{a} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

As

$$Q_1 = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$$

is the corresponding factorization for the compensator, it is seen that the matrix

$$AP_1 + BQ_1 = \begin{bmatrix} a_1 p + b_1 q & 0 \\ \bar{y}q & -f\bar{x}p \;\; h_1 f \end{bmatrix}$$

is a unit of $\mathbb{R}_{ps}^{2\times 2}(s)$ if and only if

$$(a_1 p + b_1 q)\, h_1 f = (ap + bq) f$$

is a unit of $\mathbb{R}_{ps}(s)$. Therefore, applying Lemma 1, the overall system is internally proper and stable if and only if both $ap + bq$ and f are units of $\mathbb{R}_{ps}(s)$. Now, $ap + bq$ is a unit for some $p, q \in \mathbb{R}_{ps}(s)$ if and only if (a_1) is verified; and f is a unit if and only if (a_2) is verified.

The conditions (b_1), (b_2) guarantee the disturbance rejection in an internally proper and stable system. For that, the transfer function relating v to z is to be zero,

$$\frac{e}{a} - \frac{cq}{ap + bq} \frac{d}{a} = 0$$

where (a_1) allows one to set

$$ap + bq = 1 . \tag{6.6}$$

Hence q must satisfy the equation $e = cqd$ and this is the case if and only if cd divides e. The associated p is related with q by (6.6); on substituting one has $g = -cpdf$. Since f divides g by (a_2) the equation is solvable if and only if cd divides g. Hence (b_1) follows. The condition $p \neq 0$ then necessitates (b_2). □

The compensator in question has the transfer function

$$-\frac{q}{p} = e\frac{f}{g} .$$

It can be realized arbitrarily, except that its uncontrollable and unobservable modes must be internally proper and stable.

The conditions of Theorem 2 admit the following interpretation:

(a_1) means that the uncontrollable and unobservable modes of the control-to-measurement subsystem are internally proper and stable;

(a_2) means that the non-cyclic modes of the plant are internally proper and stable;

(b_1) means that the zeros in $\operatorname{Re} s \geqq 0$ of the disturbance-to-measurement and control-to-output subsystems are contained in the zeros of the disturbance-to-output subsystem as well as in the zeros of the entire plant:

(b_2) means that the plant transfer matrix has full rank.

In particular, the disturbance cannot be rejected in the absence of cross-coupling (i.e. when $cd = 0$), if the measurement coincides with the output (i.e. when $b = d$, $c = e$) or if the disturbance contaminates only the control (i.e. when $b = c$, $d = e$).

Theorem 2 generalizes the results reported by Kulebakin [21], Basile and Marro [1], Wonham [38], Willems and Commault [34], Bhattacharyya [4], Kučera [18] and Commault, Dion and Perez [7] for single-input single-output singular systems.

6.2. Exact Model Matching

Consider an irreducible plant

$$\begin{aligned} E_1 \dot{x}_1(t) &= F_1 x_1(t) + G_1 u(t), \quad t \geqq 0 \\ y(t) &= H_1 x_1(t) \\ z(t) &= H_1' x_1(t) \end{aligned} \tag{6.7}$$

where E_1 and F_1 are $n_1 \times n_1$, G_1 is $n_1 \times q$, H_1 is $p \times n_1$ and H_2 is $m \times n_1$. Here u is the control, y is the measurement and z is the output.

The exact model matching problem with internal properness and stability consists in finding an admissible compensator of the form

$$\begin{aligned} E_2 \dot{x}_2(t) &= F_2 x_2(t) + G_2 y(t) + G_2' v(t), \quad t \geqq 0 \\ u(t) &= H_2 x_2(t) \end{aligned} \tag{6.8}$$

where E_2 and F_2 are $n_2 \times n_2$, G_2 is $n_2 \times p$, G_2' is $n_2 \times r$ and H_2 is $q \times n_2$, such that the transfer matrix from v to z in the composite system (6.7)–(6.8) shown in Fig. 4 coincide with a given (proper, stable, rational) model matrix $T(s)$.

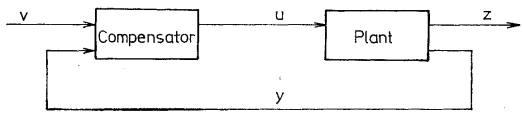

Fig. 4. Exact model matching.

To solve the problem, define the factorizations

$$\begin{aligned} H_1(sE_1 - F_1)^{-1} G_1 &= B_1(s) A_1^{-1}(s) \\ H_1'(sE_1 - F_1)^{-1} G_1 &= C_1(s) A_1^{-1}(s) \end{aligned} \tag{6.9}$$

where A_1, B_1 and C_1 are proper stable rational matrices of compatible sizes. We assume that

$$A_1, \quad \begin{bmatrix} B_1 \\ C_1 \end{bmatrix}$$

are relatively right prime. Similarly we define

$$\begin{aligned} -H_2(sE_2 - F_2)^{-1} G_2 &= P^{-1}(s) Q(s) \\ H_2(sE_2 - F_2)^{-1} G_2' &= P^{-1}(s) R(s) \end{aligned} \tag{6.10}$$

for some proper stable rational matrices P, Q and R.

Theorem 3. The exact model matching problem with properness and stability is solvable if and only if the following two conditions both hold:
(a) A_1 and B_1 are relatively right prime,
(b) C_1 is a left divisor of T.

Proof. Let the composite (6.1)–(6.2) be proper and stable. By Lemma 1, the matrix $PA_1 + QB_1$ is a unit of $\mathbb{R}_{ps}^{q \times q}(s)$. Hence (a) follows. Moreover, let the transfer matrix relating v and z equal $T(s)$. Then

$$T = C_1(PA_1 + QB_1)^{-1} R$$

and $(PA_1 + QB_1)^{-1} R$ is proper and stable. This implies (b).

Conversely let (a) hold. Then there exist proper stable rational matrices $P(s)$, $Q(s)$ with $P(s)$ nonsingular such that

$$PA_1 + QB_1 = I_q\,. \tag{6.11}$$

Condition (b) then entails the existence of a proper stable rational matrix $R(s)$ such that

$$C_1 R = T\,. \tag{6.12}$$

Any triple P, Q, and R satisfying (6.11) and (6.12) defines a compensator, via (6.10), that effects the exact model matching. If it is appropriately realized (e.g. is irreducible) it ensures the properness and stability of the overall system. □

The conditions of Theorem 3 admit a simple intuitive interpretation. Condition (a) means certain disjointness of poles and zeros (both finite and infinite) in $\operatorname{Re} s \geqq 0$ of the control-to-measurement subsystem. Condition (b) then requires that the (finite and infinite) zeros in $\operatorname{Re} s \geqq 0$ of the control-to-output subsystem be in a sense contained in the zeros of the model.

Theorem 3 generalizes for singular systems the plethora of results available in the literature on exact model matching of regular or special singular systems (such as regular systems with a direct feedthrough). Among others, see Morse [25], [26], Wolovich [35], Verghese [31], Pernebo [29], Malabre [22] and Malabre and Kučera [23].

6.3. Output Regulation

Consider an irreducible plant

$$\begin{aligned} E_1 \dot{x}_1(t) &= F_1 x_1(t) + G_1 u(t)\,, \quad t \geqq 0 \\ y(t) &= H_1 x_1(t) \\ z(t) &= H_1' x_1(t) \end{aligned} \tag{6.13}$$

where E_1 and F_1 are $n_1 \times n_1$, G_1 is $n_1 \times q$, H_1 is $p \times n_1$ and H_1' is $m \times n_1$. Here u is the control, y is the measurement and z is the output. Consider also a reference generator

$$\begin{aligned} E_3 \dot{x}_3(t) &= F_3 x_3(t) + G_3 v(t)\,, \quad t \geqq 0 \\ w(t) &= H_3 x_3(t) \end{aligned} \tag{6.14}$$

where E_3 and F_3 are $n_3 \times n_3$, G_3 is $n_3 \times r$ and H_3 is $m \times n_3$. Here v is an exciting input and w is the reference output.

The output regulation problem with internal properness and stability consists in finding an admissible compensator of the form

$$\begin{aligned} E_2 \dot{x}_2(t) &= F_2 x_2(t) + G_2 y(t) + G_2' w(t)\,, \quad t \geqq 0 \\ u(t) &= H_2 x_2(t) \end{aligned} \tag{6.15}$$

where E_2 and F_2 are $n_2 \times n_2$, G_2 is $n_2 \times p$, G'_2 is $n_2 \times m$ and H_2 is $q \times n_2$, such that the transfer function v to $w - z$ in the composite system (6.13)–(6.15) shown in Fig. 5 be proper and stable.

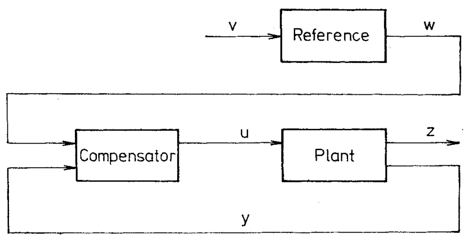

Fig. 5. Output regulation.

Denoting $e(t) = w(t) - z(t)$ the regulation error, the above formulation means that $e(t)$, $t \geqq 0$ is to be free of impulsive and unstable exponential modes for any initial conditions $x_1(0-)$, $x_2(0-)$ and $x_3(0-)$. That is to say, the plant output z is to asymptotically follow the reference output w.

To solve the problem, define the factorizations

$$\begin{aligned} H_1(sE_1 - F_1)^{-1} G_1 &= B_1(s) A_1^{-1}(s) \\ H'_1(sE_1 - F_1)^{-1} G_1 &= C_1(s) A_1^{-1}(s) \\ H_3(sE_3 - F_3)^{-1} G_3 &= F^{-1}(s) G(s) \end{aligned} \tag{6.16}$$

where A_1, B_1, C_1 and F, G are proper stable rational matrices of compatible sizes. We assume that F, G are relatively left prime and

$$A_1, \quad \begin{bmatrix} B_1 \\ C_1 \end{bmatrix}$$

are relatively right prime. Similarly we define

$$\begin{aligned} -H_2(sE_2 - F_2)^{-1} G_2 &= P^{-1}(s) Q(s) \\ H_2(sE_2 - F_2)^{-1} G'_2 &= P^{-1}(s) R(s) \end{aligned} \tag{6.17}$$

for some proper stable rational matrices P, Q and R.

Theorem 4. The output regulation problem with internal properness and stability is solvable if and only if the following two conditions both holds:

(a) A_1 and B_1 are relatively right prime,

(b) F and C_1 are internally skew prime.

Proof. Let the composite (6.13)–(6.15) be internally proper and stable. By Lemma 1, $PA_1 + QB_1$ is a unit of $\mathbb{R}_{ps}^{q \times q}(s)$ whence (a) follows. Moreover, let the transfer function from v to e be proper and stable. It is given by

$$e(s) = [I_p - C_1(PA_1 + QB_1)^{-1} R] F^{-1} G v(s)$$

where $X = (PA_1 + QB_1)^{-1} R$ is proper and stable. Since F and G are left coprime, we conclude that F is a right divisor of $I_p - C_1 X$, i.e. there exists a proper stable rational matrix Y such that

$$C_1 X + YF = I_p .$$

This proves (b) using the terminology of Wolovich [36].

Conversely let (a) hold. Then there exist proper stable rational matrices $P(s)$ and $Q(s)$ with $P(s)$ nonsingular such that

$$PA_1 + QB_1 = I_q . \tag{6.18}$$

Condition (b) implies (Wolovich, [36]) the existence of proper stable rational matrices $R(s)$ and $S(s)$ such that

$$C_1 R + SF = I_p . \tag{6.19}$$

Then

$$e(s) = S\, v(s)$$

in view of (6.18)–(6.19) and it is proper and stable. Hence any P, Q and R satisfying (6.18)–(6.19) define, via (6.17), a compensator that effects the output regulation. If the compensator is appropriately realized, it ensures the internal properness and stability. □

It is again possible to give intuitive interpretation for conditions (a) and (b) of Theorem 4. They both call for certain disjointness; condition (a) for that of poles and zeros (both finite and infinite) in $\operatorname{Re} s \geqq 0$ of the control-to-measurement subsystem and condition (b) for that of reference poles and control-to-output zeros (both finite and infinite) in $\operatorname{Re} s \geqq 0$. Each notion of disjointness is different, however, owing to noncommutativity of matrix multiplication.

Theorem 4 generalizes to singular systems the results of many researchers, including Wonham [38], Wonham and Pearson [39], Francis [12], Bengtsson [3], Wolovich and Ferreira [37], Pernebo [29], Cheng and Pearson [6], Khargonekar and Özgüler [16] and Francis and Vidyasagar [13].

6.4. Block decoupling

Consider an irreducible plant

$$\begin{aligned} E_1 \dot{x}_1(t) &= F_1 x_1(t) + G_1 u(t) \\ y(t) &= H_1 x_1(t) \\ z(t) &= H_1' x_1(t) \end{aligned} \tag{6.20}$$

where E_1 and F_1 are $n_1 \times n_1$, G_1 is $n_1 \times q$, H_1 is $p \times n_1$ and H_1' is $m \times n_1$. Here u

is the control, y is the measurement and z is the output. Let $m_1, m_2, \ldots, m_k$ be a set of positive integers satisfying

$$\sum_{i=1}^{k} m_i = m$$

which defines the partition of z into k blocks z_i of dimension m_i, $i = 1, 2, \ldots, k$.

The block decoupling problem with internal properness and stability consists in finding an admissible compensator of the form

$$\begin{aligned} E_2 \dot{x}_2(t) &= F_2 x_2(t) + G_2 y(t) + G_2' v(t), \quad t \geqq 0 \\ u(t) &= H_2 x_2(t) \end{aligned} \tag{6.21}$$

where E_2 and F_2 are $n_2 \times n_2$, G_2 is $n_2 \times p$, G_2' is $n_2 \times r$ and H_2 is $q \times n_2$, along with positive integers $r_1, r_2, \ldots, r_k$ satisfying

$$\sum_{i=1}^{k} r_k = r$$

such that the transfer matrix from v to z in the composite system (6.20)–(6.21) have the same rank as the plant transfer matrix from u to z has and be block diagonal with diagonal blocks of dimension $m_i + r_i$, $i = 1, 2, \ldots, k$.

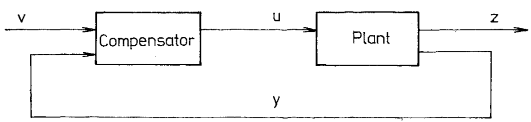

Fig. 6. Block decoupling.

The condition of diagonality is equivalent to saying that the composite system shown in Fig. 6 is decomposed, from the input-output point of view, into a set of k noninteracting subsystems. The rank condition is imposed in order to avoid trivial solutions, such as those leading to a zero overall transfer matrix, and it is equivalent to the preservation of the class of controlled output trajectories. No essential loss of control thus occurs through the decoupling process.

To solve the problem, define the factorizations

$$\begin{aligned} H_1(sE_1 - F_1)^{-1} G_1 &= B_1(s) A_1^{-1}(s) \\ H_1'(sE_1 - F_1)^{-1} G_1 &= C_1(s) A_1^{-1}(s) \end{aligned} \tag{6.22}$$

where A_1, B_1 and C_1 are proper stable rational matrices of compatible sizes. We assume that

$$A_1, \quad \begin{bmatrix} B_1 \\ C_1 \end{bmatrix}$$

are relatively right prime. Similarly we define

$$(6.23) \qquad \begin{aligned} -H_2(sE_2 - F_2)^{-1} G_2 &= P^{-1}(s)\, Q(s) \\ H_2(sE_2 - F_2)^{-1} G_2' &= P^{-1}(s)\, R(s) \end{aligned}$$

for some proper stable rational matrices P, Q and R.

Conforming to the partition of outputs, write

$$C_1 = \begin{bmatrix} C_{11} \\ C_{12} \\ \vdots \\ C_{1k} \end{bmatrix}$$

where C_{1i} is an $m_i \times q$ submatrix. Then we have the following.

Theorem 5. The block decoupling problem with internal properness and stability is solvable if and only if the following two conditions both hold:
(a) A_1 and B_1 are relatively right prime,
(b) $\sum_{i=1}^{k} \operatorname{rank} C_{1i} = \operatorname{rank} C_1$.

Proof. The proof of (a) goes along the lines of the proof given in Theorems 3 and 4, so we prove here only (b).

The transfer matrix relating v to z in the composite system reads

$$T = C_1(PA_1 + QB_1)^{-1} R .$$

Denote

$$X = (PA + QB)^{-1} R .$$

If T is block diagonal, then

$$\operatorname{rank} C_1 X = \sum_{i=1}^{k} \operatorname{rank} C_{1i} X$$

and the rank condition imposed on T gives

$$\operatorname{rank} C_{1i} X = \operatorname{rank} C_{1i}, \quad i = 1, 2, \ldots, k .$$

Therefore (b) holds.

The sufficiency will again be proved by construction. Denote

$$r_i = \operatorname{rank} C_{1i}, \quad i = 1, 2, \ldots, k .$$

Then there exists a nonsingular matrix $U_i \in \mathbb{R}_{ps}^{m_i \times m_i}$ such that

$$C_{1i} = U_i \begin{bmatrix} C'_{1i} \\ 0 \end{bmatrix}$$

where C'_{1i} is of full row rank r_i. If (b) holds, then

$$C'_1 = \begin{bmatrix} C'_{11} \\ C'_{12} \\ \vdots \\ C'_{1k} \end{bmatrix}$$

has full row rank and hence has a right inverse G.

If (a) holds, then there exist proper stable rational matrices $P(s)$, $Q(s)$ with $P(s)$ nonsingular that satisfy

$$PA_1 + QB_1 = I_q .$$

Set $R = Gt$, where the scalar function $t \in \mathbb{R}_{ps}(s)$ it chosen so as to make R a proper stable rational matrix. The compensator defined by P, Q and R according to (6.22) then gives the transfer matrix

$$T = C_1(PA_1 + QB_1)^{-1} R = C_1 Gt$$

which satisfies the rank condition

$$\operatorname{rank} T = \operatorname{rank} C_1 = \operatorname{rank} C_1 A_1^{-1}$$

and is block diagonal

$$T = \begin{bmatrix} U_1 & & \\ & \ddots & \\ & & U_k \end{bmatrix} \begin{bmatrix} I_{r_1} & & \\ 0 & \ddots & \\ & & I_{r_k} \\ & & 0 \end{bmatrix} t .$$

The resulting system is therefore decoupled into $m_i \times r_i$ non-interacting blocks so that the external input vector has the dimension

$$r = \sum_{i=1}^{k} r_i = \operatorname{rank} C_1 . \qquad \square$$

The above result is adapted from Kučera [19] and generalizes the multitude of various results on decoupling of regular systems, to mention just Voznesenskij [33], Kavanagh [15], Morgan [24], Falb and Wolovich [11], Basile and Marro [2], Morse and Wonham [27], [28], Wonham [38], Descusse and Malabre [9], Hautus and Heymann [14] and Descusse, Lafay and Kučera [8].

7. CONCLUSION

The concept of internal properness has been introduced for generalized linear systems as an analogy to the concept of internal stability: a system is internally proper if its free response contains no impulsive motions and it is internally stable if its free response contains no non-decreasing exponential motions.

The design of feedback control systems endowed with these essential properties then has been discussed. The main result is in identifying all compensators that make the given plant internally proper and stable.

The key role of this result has been demonstrated on several examples of control system design. Using the technique of proper stable factorizations, the requirement of internal properness can be easily accommodated in the standard transfer function methods. The essential point is that the requirement of internal properness (like that of stability) is to be imposed on the *overall* system, not on the plant and compensator separately. Of course, it amounts to the same for regular systems, but it is justified in the more general context of singular systems.

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Ing. Vladimír Kučera, DrSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha. Czechoslovakia.