KYBERNETIKA — VOLUME 21 (1985), NUMBER 6

ROOT m-TISSUES: SYSTEMS UNDER AN ACTION OF THE m-PARAMETERS

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The root loci of the characteristic polynomial of the degree n with the coefficients parametrized polynomially by the m parameters are treated. All these root loci are embedded into the structure of the group action on the set. The image of the action of the additive group of m real parameters is the m-tissue: through every root there go the m tissues. The group action is used to compute the single- or multivalued differentials to obtain the root loci. The main use of these root m-tissues is to check the robustness of the assigned roots of the characteristic polynomial — the boundary of these are generally $2m$-gons, under the simultaneous change of the m parameters. The inverse problem to the root locus analysis is stated and solved as the shift of $n - \varrho$ roots of the characteristic polynomial of the degree n on some paths to the prescribed $n - \varrho$ positions and the computation of both the $n - \varrho$ gains on these paths and the root loci of the ϱ remaining roots.

0. INTRODUCTION

Systems undergo changes. To design a feedback system an open-loop system is changed to the closed-loop one. During the performance of the feedback system the open-loop system changes again. Even before closing the loop, the open-loop system had been changed to make both the description and the design better treatable. To treat the changes, it has been proved useful to resolve the time changes into the frequency spectrum and to describe the system in the frequency domain or in the complex frequency domain and describe the changes by some parametrization of the former description. The transfer function between any two points of the system is a ratio of two polynomials. These are the polynomials both in the complex frequency and in some real parameters. Then to describe the changes of the system, it suffices to parametrize the real and complex conjugate roots of both polynomials. A simple affine parametrization of a denominator polynomial by a single parameter treats the classical root locus method. This method has been proved useful for decades and may be due to this the method changed a little from its very beginning. Usually there is a list of a dozen rules of the root locus to help a paper and pencil root locus drawing. In the following we present a single rule (a multivalued differential) to be used for the computer drawing of the classical root locus on the screen. Moreover we give a classification of the classical root loci to classify all possible changes of the system under a change of an affinely-acting real parameter. The mentioned rule is also extended to hold even for any polynomial scalar- or vector-parametrization.

The stability of the linear, finite-dimensional, time-invariant systems is determined by the eigenvalues of the state matrix or equivalently, eliminating the eigenvectors, by the roots of the characteristic polynomial which are identical with the eigenvalues. In this paper we shall be concerned with the latter description, nevertheless it had been the former one where the problem had been conceived and solved firstly – in the connection with the celestial mechanics [18]. For the present state of the solution of the former problem see the perturbation theory for linear operators in finite-dimensional space [20]. It has been treated also in the electrical circuits theory [14], [25] and lately even in the connection with the classical root locus [16].

These perturbation or sensitivity approaches are of the local nature only. We have observed no attempt to integrate the local behaviour to obtain the global results. But there is clearly a demarcation line between the local sensitivity and robustness to finite changes. The discrete-time deadbeat control which shifts the closed-loop eigenvalues to the single point farthest from the stability boundary is both maximally sensitive and maximally stable [32] or robust control. In this paper we shall be concerned with the derivation of the local equations and afterwards with their integration. We shall embed both the local and global behaviour into the action of the additive group of the real parameters to the system roots, giving the tissues of these roots.

1. DERIVATION OF THE BASIC EQUATIONS

A system

$$sx = Fx + \alpha$$

with a complex frequency $s \in C$, a state $x \in \mathbb{R}^n$, the system order $n \in \mathbb{N}$, the state matrix $F: \mathbb{R}^n \to \mathbb{R}^n$, and the initial state $\alpha \in \mathbb{R}^n$ can describe a controlled system with a plant, a regulator, a reference, and a disturbance. The system response

$$x = \frac{\operatorname{adj}(sI - F)}{\det(sI - F)}\,\alpha$$

has a characteristic polynomial

$$\det(sI - F) = \sum_{m_1,\ldots,m_n} \delta^{1,\ldots,n}_{m_1,\ldots,m_n}(\delta_{1,m_1}s - F_{1,m_1})\ldots(\delta_{n,m_n}s - F_{n,m_n})$$

where the summation with respect to $m_1, \ldots, m_n$ is over all permutations on $1, \ldots, n$, see e.g. [3]. $\delta^{1,\ldots,n}_{m_1,\ldots,m_n} = 1(-1)$ for the even (odd) permutation $\begin{pmatrix} 1 & \ldots & n \\ m_1 & \ldots & m_n \end{pmatrix}$. $\delta_{ij} = 1(0)$ for $i = (\neq)\, j$. The i, j-th element of the adjoint matrix

$$[\operatorname{adj}(sI - F)]_{i,j} = \det \begin{bmatrix} s - F_{11} & \ldots & 0 & \ldots & -F_{1n} \\ \vdots & & \vdots & & \vdots \\ -F_{i1} & \ldots & 1 & \ldots & -F_{in} \\ \vdots & & \vdots & & \vdots \\ -F_{n1} & \ldots & 0 & \ldots & s - F_{nn} \end{bmatrix}$$

where 1 is at the j-th column. To analyse the system response, consider the causes of increasing complexity.

RL^1_1. $F_{i_1,j_1} = K$ is a simple parameter of F, i.e. if $F_{i_1,j_1} = F_{i_2,j_2}$ then $i_1 = i_2$, $j_1 = j_2$. Then the root locus of $\det[sI - F(K)]$ is

$$(RL^1_1) \qquad a(s) + K\, b(s) = 0$$

where $K \in \mathbb{R}$, so the characteristic polynomial is affine in K (or it is a pencil of the polynomials $a(s)$, $b(s)$).

RL^v_1. $F_{i_1,j_1} = F_{i_2,j_2} = \ldots = F_{i_v,j_v} = K(1 < v \leqq n^2)$ is a multiple parameter of F. Then the root locus of $\det[sI - F(K)]$ is

$$(RL^v_1) \qquad a(s) + \sum_{k=1}^{v} K^k\, b_k(s) = 0$$

where $K \in \mathbb{R}$, so the characteristic polynomial is an algebraic function, [10], of a single parameter K. The multiplicity of K is only an upper bound of the power of K — the $b_v(s)$ may be zero.

RL^v_m. Leaving out any constraints of F_{ij} and denoting $k_1, k_2, \ldots, k_m$ of them as $K_1, K_2, \ldots, K_m$, the root locus of $\det[sI - F(K_1, K_2, \ldots, K_m)]$ is

$$(RL^v_m) \qquad a(s) + \sum_{k_1,k_2,\ldots,k_m} K_1^{k_1} K_2^{k_2} \ldots K_m^{k_m} b_{k_1,k_2,\ldots,k_m}(s) = 0$$

where $K_1, K_2, \ldots, K_m \in \mathbb{R}$; $k_i = 1, 2, \ldots, m_i$; $i = 1, 2, \ldots, m$; $m \leqq n^2$. The characteristic polynomial is an algebraic function of the parameter $K_1, K_2, \ldots, K_m$. The root locus RL^v_m covers part of the s-plane, may be even the whole s-plane, as we know from the poles shift under the state feedback where $m = n$.

In the following it is supposed that the characteristic polynomials are irreducible as polynomials in s, K or $s, K_1, K_2, \ldots, K_m$. I.e. if $c(s, K) = c_1(s, K) \,.\, c_2(s, K)$, then c_1 or c_2 is a polynomial of the 0 th degree both in s and K. Similarly for $c(s, K_1, \ldots, K_m)$: if $c(s, K_1, \ldots, K_m) = c_1(s, K_1, \ldots, K_m) \,.\, c_2(s, K_1, \ldots, K_m)$ then c_1 or c_2 has the 0 th degree in $s, K_1, \ldots, K_m$. The 0 th degree polynomials have no roots. Examples of reducible characteristic polynomials RL^1_1 are the polynomials with common roots of $a(s)$, $b(s)$, that of RL^v_m are the polynomials which are the product of the affinely parametrized characteristic polynomials RL^1_1.

Similarly as the $\det(sI - F)$, we can analyse $[\operatorname{adj}(sI - F)]_{i,j}$ $(i, j = 1, \ldots, n)$. Notice, [23], that the roots of the characteristic polynomial change with the change of the feedback loop parameters and the roots of the numerator polynomials change both with the change of feed-forward paths and with the feedback loops.

The effective computation of the characteristic polynomial as polynomial in nonnumeric or symbolic parameters see [22], [23], [30], [31], [34].

2. APPLICATIONS

With respect to the control design there exists fundamental dichotomy for the class of the characteristic polynomials: the polynomial $\det(sI - F)$ is either asymptotically stable or not. Equivalently all roots of $\det(sI - F) = 0$ either lie at the open left half of the s-plane or not.

RL_1^1. This root locus was introduced first at [7], it was transformed into the method at [13], for a survey see [21], some results relevant for us see [16], [24], [33]. The root locus have been well established tool for the classical trial and error design, see e.g. [5, 21]. Only recently the counterexamples [11, 12] had shown that even the modern analytical (or algebraic) design (i.e. a poles shift and/or a quadratic optimization) guarantees generally (– unless we can measure or control the state directly) no gain margin and have to be tested like the classical trial and error design.

RL_1^1 symmetrical with respect to the stability boundary is, see e.g. [19]

$$a(s)\,a(-s) + K\,b(s)\,b(-s) = 0 \quad (K \geqq 0)$$

The stable n roots are the roots of the optimal control characteristic polynomial, $b(s)/a(s)$ is the open-loop control transfer function, the K is the weight of input (actuator) energy – for 1 being the weight of the output energy. Similar equation see e.g. [19]

$$a(s)\,a(-s) + L\,c(s)\,c(-s) = 0 \quad (L \geqq 0)$$

holds for the optimal reconstruction. The stable n roots are the roots of the optimal reconstruction characteristic polynomial, $c(s)/a(s)$ is the open-loop reconstruction transfer function, the L is the weight of the power of output disturbances – for 1 being the weight of the input disturbances. For feasible (robust) control based on the reconstruction, the K or L have to be adjusted (and so the characteristed polynomials have to be deoptimized) to achieve the sufficient gain and phase margins of the control loop [12]. Moreover, solving these root loci gives a method for a spectral factorization [17].

RL_1^2. In the model building from elementary branches (at the opposite to the model identification from input-output data) a single branch K may occur m-times. As there exists the other multiple branch – the n differentiators s, the characteristic polynomial $\det[sI - F(K)]$ is the polynomial both in K and s. E.g. for a salt solution mixer with two inputs, [9],

$$F = \begin{bmatrix} -Q_1/L_1 & Q_2/L_2 \\ Q_1/L_1 & -Q_1/L_2 \end{bmatrix}$$

and the characteristic polynomial $\det(sI - F)$ has root locus

$$s^2 + \left(\frac{Q_1}{L_1} + \frac{Q_2}{L_2}\right) s + \frac{Q_1^2 - Q_1 Q_2}{L_1 L_2} = 0$$

where $Q_i(L_i)$ are flow rates (volumes). For $1/L_1 = 1/L_2 = K$ we have the root locus RL_1^2. Similar example from aeronautics, [5], motivated first the root locus RL_1^2 at [28].

RL_1^ν. Characterising the properties of multivariable control, a characteristic named multivariable root locus was introduced, [27]. For $F = A - KBC$, $K \in \mathbb{R}$, $BC: \mathbb{R}^\nu \to \to \mathbb{R}^\nu$, the root locus of the characteristic polynomial

$$s^n - \operatorname{tr}(A - KBC)\, s^{n-1} + \ldots + (-1)^n \det(A - KBC) = 0$$

(i.e. $\det(A - KBC) = K^n$ const.) have been considered, esp. with the emphasis on $K \to \infty$. The introduction of this particular feedback has been artificial and the use of this as a design tool is not established yet. Surveys see [26, 28].

RL_1^2. In the construction of the optimal state feedback under the multivariable input we have, see e.g. [19], $F = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^TC & -A^T \end{bmatrix}$ and the optimal characteristic polynomial root locus

$$\text{num} \det \left[R + H^T(-s)\, H(s)\right] = 0$$

where $H(s) = C(sI - A)^{-1} B$. Then (unless the case of input weighting matrix R) any element of F has the multiplicity $\nu = 2$. (We have considered the special case of $R \in \mathbb{R}$, at the root locus RL_1^2 for $K = R^{-1}$, $H(s) = b(s)/a(s)$.) Similarly for the multivariable reconstruction.

Consider now the complex frequency $s \in \mathbb{C}$ not as the Laplace transform variable but as the Z-transform variable. Now the characteristic polynomial $\det(sI - F)$ is asymptotically stable if its roots are inside the circle $|s| - 1 = 0$. Then instead of the mirror symmetry $(s, -s)$ we have the inversion symmetry (s, s^{-1}). In the RL_1^2 we have to change $-s$ for s^{-1}.

RL_m^ν. For $m = 2$, $\nu = 1$ this root locus was treated at [21]. There have been the gap between the naturalness and the potential use of this most general root locus and the difficulties of its understanding and construction.

In all cases the degree of stability have to be measured by the least parametric change to reach the stability boundary i.e. by some gain or better multiparametric margins. The degree of stability has not to be generally measured by the distance of the poles from the stability border: at [11] there is a counterexample showing that just the poles farthest from the stability boundary may become unstable first.

3. ROOT m-TISSUE AS AN IMAGE OF AN ACTION OF THE m PARAMETERS

In this section we shall use one of the central structures of the geometry of manifolds, see e.g. [1, 8], an action of a group on a set. This makes possible to put together the highly structured group with only poorly structured set. Then we use the special structure of the textile geometry, [6], the m-tissue. This makes possible to put together the highly dimensional space of the m parameters with the points from 2-dimensional plane. The examples show that the main effort in the constructions of RL^1_m is at the computation of homologies of the m-cubes. Some combinatorial argument helps to classify RL^1_1.

Definition 1. The action of the group G on the set S is a mapping

$$G \times S \to S : (g, \sigma) \mapsto \sigma^g$$

$(g \in G, \sigma \in S)$ such that

$$
\begin{aligned}
&(g_2, \sigma^{g_1}) \mapsto \sigma^{g_2 \circ g_1} && \text{(compatibility)}\\
&(e, \sigma) \mapsto \sigma && \text{(existence of the neutral element)}\\
&(g^{-1}, \sigma^g) \mapsto \sigma && \text{(existence of the inverse element)}
\end{aligned}
$$

hold. A subset

$$\{\sigma^g \mid g \in G\}$$

is called an orbit containing $\sigma \in S$.

Theorem 1. The roots $s_1, s_2, \dots, s_n$ of the equations RL^1_1, RL^v_1, respectively RL^v_m have the structure of the action of the additive group $\{K\} = \mathbb{R}$, respectively $\{K_1, K_2, \dots, K_m\} = \mathbb{R}^m$ on the set $\mathbb{C}^n_1[s]$ of the n roots.

Proof. (i) Let us start with RL^1_1. Choose $K = K_1 + K_2, K, K_1, K_2 \in \mathbb{R}$. Then the n roots of $a(s) + (K_1 + K_2)\, b(s) = 0$ are the same as the n roots of the equation $(a(s) + K_1\, b(s)) + K_2\, b(s) = 0$. The compatibility of the action means just this. There exists the neutral element $e = 0$ and the inverse element $g^{-1} = -K$. (ii) Let us continue with RL^v_1: again for the compatibility the roots of $a(s) + \sum (K_1 + K_2)^k \,.\, b_k(s) = 0$ remains the same even after the reordering terms of the equation to obtain the equation $A(s, K_1) + \sum B_k(s, K_1)\, K_2^k = 0$ where A, B_k are obtained comparing the coefficients. Again $e = 0$, $g^{-1} = -K$. (iii) Finally the case of RL^v_m. Choose $K, \hat{K}, \tilde{K} \in \mathbb{R}^m$ where $K = (K_1, \dots, K_m), K = (\hat{K}_1, \dots, \hat{K}_m), \tilde{K} = (\tilde{K}_1, \dots, \tilde{K}_m)$. Even here the roots of $a(s) + \sum (\hat{K}_1 + \tilde{K}_2)^{k_1}\, (\hat{K}_2 + \tilde{K}_2)^{k_2} \dots (\hat{K}_m + \tilde{K}_m)^{k_m} \,.\, b_{k_1, k_2, \dots, k_m}(s) = 0$ are invariant with respect to reordering the terms to obtain $A(s, \hat{K}_1, \dots, \hat{K}_m) + \sum B_{k_1, k_2, \dots, k_m}(s, \hat{K}_1, \dots, \hat{K}_m)\, \tilde{K}_1^{k_1} \tilde{K}_2^{k_2} \dots \tilde{K}_m^{k_m} = 0$ where A, $B_{k_1, \dots k_m}$ were obtained by the comparison of the coefficients. Now $e = (0, 0, \dots, 0)$, $g^{-1} = (-K_1, -K_2, \dots, -K_m)$. □

Definition 2. The image $s_i = T(K_1, \dots, K_m)$ of the linear mapping

$$T : \mathbb{R}^m \to \mathbb{C} : (K_1, \dots, K_m) \to s_i \quad (m \in \mathbb{N}_+)$$

is called the m-tissue at the point s_i.

Theorem 2. Let the equations RL_1^1, RL_m^1, RL_m^ν be irreducible and let n_i be the multiplicity of the root s_i $(i = 1, 2, \dots, n - n_1 \dots n_i - N)$ where N is the number of the different roots. Then at every root s_i of the RL_1^1, RL_m^1, resp. RL_m^ν, the n_i-valued differential is

$$(DS_1^1) \qquad \mathrm{d}s_i = \left[- \frac{b(s_i)\,\mathrm{d}K}{\prod\limits_{\substack{j=1\\ j \neq i}}^{n} (s_i - s_j)} \right]^{1/n_i}$$

$$(DS_1^\nu) \qquad \mathrm{d}s_i = \left[- \frac{\sum\limits_{k=1}^{\nu} B_k(s_i, K)\,\mathrm{d}K^k}{\prod\limits_{\substack{j=1\\ j \neq i}}^{n} (s_i - s_j)} \right]^{1/n_i}$$

where $B_k(s, K) = \sum\limits_{l=k}^{\nu} \binom{k}{l} K^{l-k}\, b_l(s)$, resp.

$$(DS_m^\nu) \qquad \mathrm{d}s_i = \left[- \frac{\sum\limits_{k_1,\dots,k_m}^{\nu} B_{k_1,\dots,k_m}(s_i, K_1, \dots, K_m)\,\mathrm{d}K_1^{k_1}\,\mathrm{d}K_2^{k_2} \dots \mathrm{d}K_m^{k_m}}{\prod\limits_{\substack{j=1\\ j \neq i}}^{n} (s_i - s_j)} \right]^{1/n_i}$$

where

$$B_{k_1,\dots,k_m}(s, K_1, \dots, K_m) =$$
$$= \sum_{l_1,\dots,l_m} \binom{k_1}{l_1} \cdots \binom{k_m}{l_m} K_1^{l_1 - k_1} K_2^{l_2 - k_2} \dots K_m^{l_m - k_m}\, b_{l_1,\dots,l_m}(s)$$

At the roots of the multiplicity 1, the $\mathrm{d}s_i/\mathrm{d}K$ forms locally the root 1-tissue, the $\{\mathrm{d}s_i/\mathrm{d}K_1, \mathrm{d}s_i/\mathrm{d}K_2, \dots, \mathrm{d}s_i/\mathrm{d}K_m\}$ forms locally the root m-tissue.

Proof. Let us start again with RL_1^1. We shall apply the additive group properties to the previous parameter value K and its differential $\mathrm{d}K$, i.e. $K_1 + K_2 = K + \mathrm{d}K$. Now we shall use the invariance of the roots of

$$a(s) + (K + \mathrm{d}K)\, b(s) = (a(s) + K\, b(s)) + \mathrm{d}K\, b(s) =$$
$$= (s - s_i)^{n_i} \prod_{j \neq i} (s - s_j) + \mathrm{d}K\, b(s) \to \mathrm{d}s_i^{n_i} \prod_{j \neq i} (s_i - s_j) + \mathrm{d}K\, b(s_i) =$$
$$= 0 \quad \text{as} \quad s \to s_i + \mathrm{d}s_i\,.$$

Solving for $\mathrm{d}s_i$ we have, under the irreducibility assumption, the DS_1^1. Now for

RL_m^1, RL_m^v: the only difference is that we have, to compare the coefficients of $B_k(s, K)$, resp. $B_{k_1,\ldots,k_m})(s, K_1, \ldots, K_m)$. But the determination of these is straightforward. Such is even the verification that, under the assumptions DS_1^1, DS_m^1, resp. DS_m^v forms the root tissues. □

Note 1. For $n_i > 1$ the formula DS_1^1 is used to obtain the phase properties near the multiple root. Approaching this root (this can be easily tested as there are n_i points which approach each other in the n_ith root in a symmetric pattern) the phase properties of the n_ith root are used and the integration either jumps straight over (for n_i odd) or jumps and bends out (for n_i even). Due to the continuity of $K(s)$, the integration error can be controlled. At the root of $n_i > 1$ of RL_1^1, the $|ds_i/dK|$ is improper and only any branch of $|ds_i/dK^{1/n_i}|$ is proper. At the formula DS_m^1 one power of dK majorises the others: nevertheless, during the integration all the powers are evaluated which eliminates the necessity of analysis which power is major. The same holds for DS_m^v.

Example 1. We have to compute the root locus of the type RL_1^1

$$(s(K)) \qquad s^3 - s - K = 0 .$$

For $K = 0$ we have $s_{1,2} = \pm 1$, $s_3 = 0$. We shall start from this neutral element of the group $\mathbb{R}$ and compute the differentials

$$(DS_1^1) \qquad ds_1 = \frac{dK}{(s_1 - s_2)(s_1 - s_3)}, \quad ds_2 = \frac{dK}{(s_2 - s_3)(s_2 - s_1)},$$
$$ds_3 = \frac{dK}{(s_3 - s_1)(s_3 - s_2)}$$

firstly for $K > 0$. Near $K = 0{\cdot}4$ we have the double root. As this is even, we change the phase of the differential about $\pi/2$ and jump over the double point. Then we integrate DS_1^1 till near $K = 12$ where the RL_1^1 goes outside the chart of the s-plane. Then we start again at $s_{1,2} = \pm 1$, $s_3 = 0$ and integrate similarly, now for $K < 0$. For high complex frequencies s and high gain K the $s(K)$ is outside the Fig. 1a. Substituting $s = z^{-1}$, $K = G^{-1}$ at $(s(K))$ we obtain

$$(z(G)) \qquad z^3 + Gz^2 - G = 0$$

To compute the inverse root locus $z(G)$ we slightly unfold the triple root using the symmetry of $1^{1/3}$ and start at $z_1 = 0{\cdot}1$, $z_{2,3} = -0{\cdot}05 \pm j\,0{\cdot}086$ and $G = 0{\cdot}0$. Computing the differentials

$$(DS_1^1) \qquad dz_1 = \frac{-(z_1^2 - 1)\,dG}{(z_1 - z_2)(z_1 - z_3)}, \quad dz_2 = \frac{-(z_2^2 - 1)\,dG}{(z_2 - z_1)(z_2 - z_1)},$$
$$dz_3 = \frac{-(z_3^2 - 1)\,dG}{(z_3 - z_1)(z_3 - z_2)}$$

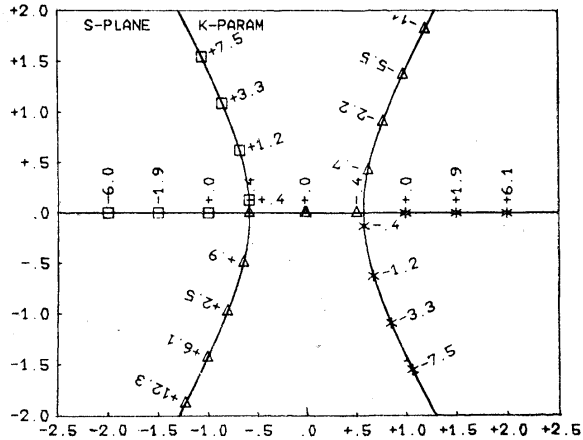
S-PLANE
K-PARAM
+2.0
+1.5
+1.0
+.5
.0
-.5
-1.0
-1.5
-2.0
-2.5 -2.0 -1.5 -1.0 -.5 .0 +.5 +1.0 +1.5 +2.0 +2.5

Fig. 1a.

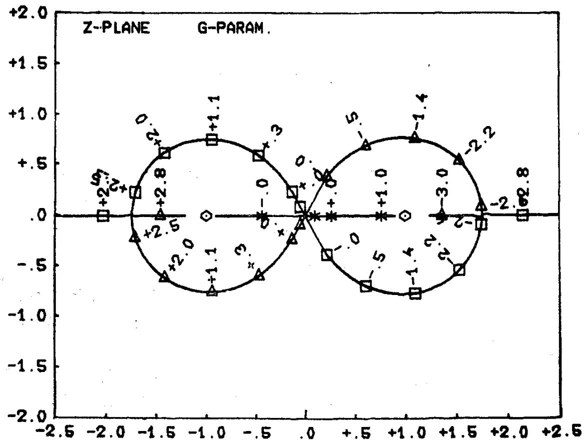
Z-PLANE
G-PARAM.
+2.0
+1.5
+1.0
+.5
.0
-.5
-1.0
-1.5
-2.0
-2.5 -2.0 -1.5 -1.0 -.5 .0 +.5 +1.0 +1.5 +2.0 +2.5

Fig. 1b.

fisrtly for $G > 0$ and then for $G < 0$ we obtain the Fig. 1b. Both charts overlap, so we have the whole root locus at two compact charts. Now consider the $s(K)$ with extra $-K^2$ term, or with quadratic action of the $\mathbb{R}$:

$$s^3 - s - K^2 - K = 0$$

For $K = 0$ we have again $s_{1,2} = \pm 1$, $s_3 = 0$. Starting from these points we compute the differentials

$$(DS_1^1) \qquad \mathrm{d}s_1 = \frac{\mathrm{d}K^2 + \mathrm{d}K}{(s_1 - s_2)(s_1 - s_3)}, \quad \mathrm{d}s_2 = \frac{\mathrm{d}K^2 + \mathrm{d}K}{(s_2 - s_3)(s_2 - s_1)},$$

$$\mathrm{d}s_3 = \frac{\mathrm{d}K^2 + \mathrm{d}K}{(s_3 - s_1)(s_3 - s_2)}$$

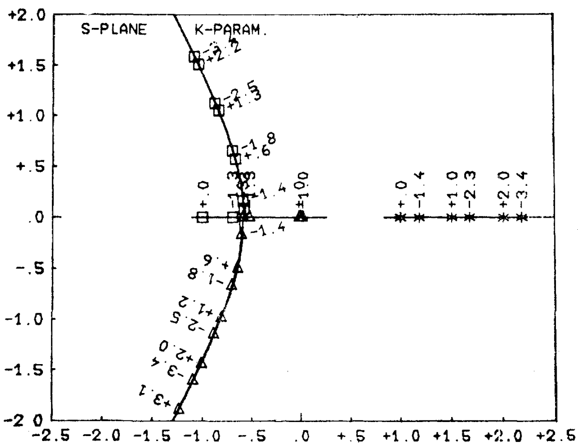

Fig. 1c.

for $K > 0$. Near $K = 0{\cdot}5$ there is a turning point and the root locus parametrizes the part of $\mathbb{R}$ twice, the remaining part leaving uncovered, see Fig. 1c. Similarly for $K < 0$ with the turning point near $K = -0{\cdot}5$.

Example 2. We shall compute how the roots of

$$(s - p_1)(s - p_2)(s - p_3) + K_1(s - z_1) + K_2(s - z_2) + K_3(s - z_3) = 0$$

$$(p_1 = -1,\ p_{2,3} = \pm j,\ z_1 = -2,\ z_2 = 0,\ z_3 = 2)$$

are shifted using the positive, bounded gains K_1, K_2, K_3. The equations of DS_3^1 are $(i = 1, 2, 3)$

$$(DS_3^1) \qquad \mathrm{d}s_i = -\sum_{j=1}^{3} (s_i - z_j)\, \mathrm{d}K_j / \prod_{\substack{j=1 \\ j \neq i}}^{3} (s_i - s_j)$$

We shall integrate these equations starting with the point boundary equations at p_1, p_2, p_3, using the properties of a root 3-tissue for the complex conjugate roots s_2, s_3 and three root 1-tissues for the real root s_1. Consider p_2 and p_3: through p_2 there go root 3-tissue which is the smooth linear map of $\mathrm{d}K_1$, $\mathrm{d}K_2$, $\mathrm{d}K_3$, the same holds for $p_3 = \bar{p}_2$. For p_1 the 3 root 1-tissues lie at $\mathbb{R}$. At first we shall integrate DS_3^1 for the first tissue, i.e. for $\mathrm{d}K_2 = \mathrm{d}K_3 = 0$, starting at p_1, p_2, p_3 and integrating from $K_1 = 0$ to $\Delta K > 0$. Then we shall integrate DS_3^1 for the second tissue, i.e. for $\mathrm{d}K_3 = \mathrm{d}K_1 = 0$, starting again at p_1, p_2, p_3 and integrating from $K_2 = 0$ to $\Delta K > 0$. Finally we shall integrate DS_3^1 for the third tissue, i.e. for $\mathrm{d}K_1 = \mathrm{d}K_2 = 0$, starting again at p_1, p_2, p_3 and integrating from $K_3 = 0$ to $\Delta K > 0$. Similarly we shall integrate from the roots reached at the previous steps – always for $\mathrm{d}K_j = 0$ $(j \neq i)$ at DS_3^1 and obtain the smoothly deformed 3-cubes with the corners at p_2 – see Fig. 2 and $p_3 = \bar{p}_2$. On $\mathbb{R}$ the 3-cube degenerates to an 1-cube, i.e. abscissa with an end point at p_1. Now consider the smoothly deformed 3-cube with the corner at p_2. For the gains $K_i \geqq \Delta K$ $(i = 1, 2, 3)$ there are three 3-cubes with the common sides with the considered basic 3-cube. Now integrating on edges of these 3-cubes for the gains $\Delta K \leqq K_i \leqq 2\Delta K$ we shall obtain the 2nd level of the 3-cubes, see Fig. 2.

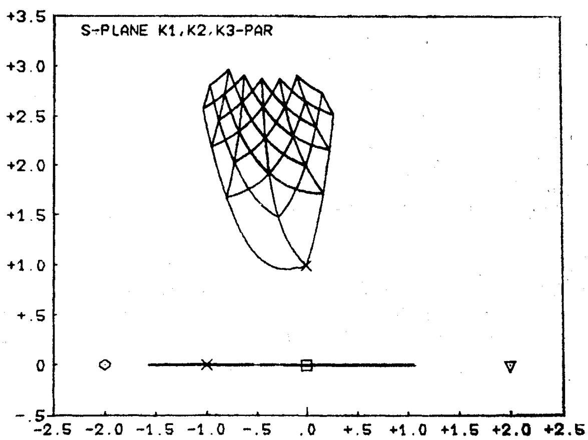

Fig. 2.

Similarly for the cubes which are the neighbours of the basic 3-cube with the corner at p_3. For the continuation of the abscissa on p_1 we shall obtain the double extended abscissas. Then integrating for the gains $2\Delta K \leqq K_i \leqq 3\Delta K$ we shall obtain the 3rd level of the smoothly deformed 3-cubes, see Fig. 2 and the 3rd level of ascissas at $\mathbb{R}$. The body of the Fortran code in the complex arithmetics is

```
NS 1 = S 1 - DK * (S 1 - Z(I))/((S 1 - S 2) * (S 1 - S 3))
NS 2 = S 2 - DK * (S 2 - Z(I))/((S 2 - S 3) * (S 2 - S 1))
NS 3 = S 3 - DK * (S 3 - Z(I))/((S 3 - S 1) * (S 3 - S 2))
```

The main part of the code is concerned with the homology of elementary cubes. The speed of the drawing at EAI Pacer 600 of all the root loci mentioned in the paper had been approx. from 0·5 to 1 cm/sec, drawing running in parallel for all roots.

Example 3. Consider the characteristic polynomial equation

$$(RL_5^1) \qquad \prod_{i=1}^{3} (s - s_i) + \sum_{i=1}^{5} K_i(s - z_i) = 0$$

$$(s_1 = -2\,, \quad s_{2,3} = -1 \pm 1{\cdot}75\,\mathrm{j}\,, \quad z_1 = -1\,, \quad z_2 = -0{\cdot}5\,,$$
$$z_3 = 1\,, \quad z_4 = 1{\cdot}5\,, \quad z_5 = 2)\,.$$

We have to find its roots for $-K \leqq K_i \leqq K$ $(i = 1, \ldots, 5)$, i.e. for the gains from 5-cube with the centre at $(0, 0, 0, 0, 0)$. To construct the root 5-tissue from $\mathbb{C}$ we construct first the 5-cube from $\mathbb{R}^5$ as a Boolean algebra of the subsets of the set of 5 elements, see Fig. 3a. From every vertex of the 5-cube there go 5 edges, every

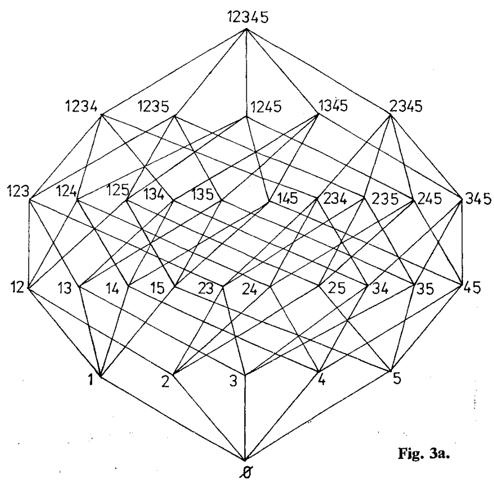

Fig. 3a.

face of the 5-cube is square. The homology of the vertices, edges and faces will be used to integrate the 3 equations $(i = 1, 2, 3)$

$$(DS_5^1) \qquad \mathrm{d}s_i = -\sum_{i=1}^{5} \mathrm{d}K_i(s_i - z_i) / \prod_{\substack{j=1 \\ j \neq i}}^{3} (s_i - s_j)$$

First, integrating for $0 \leqq K_1 \leqq K$ $(\mathrm{d}K_l = 0,\ l \neq 1)$ then for

$$0 \leqq K_2 \leqq K\ (\mathrm{d}K_l = 0,\ l \neq 2), \ldots, 0 \leqq K_5 \leqq K\ (\mathrm{d}K_l = 0,\ l \neq 5)$$

we found the vertex $\emptyset$. Second, repeating the integration for

$$-K \leqq K_i \leqq K\ (i = 1, \ldots, 5;\ \mathrm{d}K_l = 0;\ l = 1, \ldots, 5;\ l \neq i)$$

we found the opposite vertex 12 345. Then starting at the vertex $\emptyset$, we integrate on the edges of the faces incident with the vertex $\emptyset$, i.e. the faces $\emptyset$, 1, 2, 12 till $\emptyset$, 4, 5, 45. Then we integrate over the faces from 1, 12, 13, 123 to 5, 35, 45, 345. Then we integrate over the opposite half: over the faces 12345, 1234, 1235, 123 till 12345, 1345, 2345, 345. Finally over faces from 1234, 123, 124, 12 till 2345, 245, 345, 45. For the resulting image of the all faces, see Fig. 3b. At every image of the vertex near the complex (real) root there go one 5-tissue (five 1-tissues). The boundary of the shift of the complex (real) pole in the Fig. 3b under the action of the 5 parameters is a smoothly deformed 10-gon (abscissa). The boundary edges of the 10-gon we can find more directly. Starting from the vertex $\emptyset$ we choose those two from five edges 1, 2, 3, 4, 5 which start depart from $\emptyset$ at the maximal angle. Then integrating over

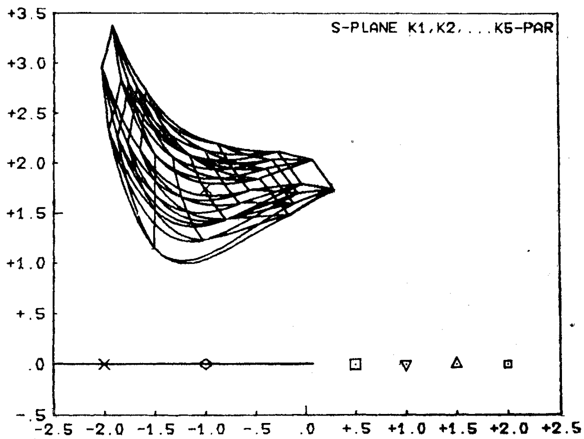

Fig. 3b.

these extremal edges, we obtain two extremal vertices. At these we choose from the remaining 4 edges which start to depart at maximal angle and integrate to the next two vertices. At these we choose from the 3 edges, ..., at the remaining two points we have no choice and integrate to obtain the edges meeting at the remaining vertex. The obtained 10-gon is generally nonconvex, so the boundaries for the different levels of parameters variations may overlap.

Definition 3. The irreducible root loci of the type RL_1^1 and no multiplicity at $\mathbb{C} \cup \{\infty\}$ different from the multiplicities at $\mathbb{C}$ (shortly: no multiplicity at ∞) are equivalent if they have the same unordered set of the multiplicities, where the multiplicity of the point is defined as $\emptyset$.

Note 2. The assumption of no multiplicity of RL_1^1 at ∞ is not restrictive. For the classification based on the multiplicities it suffices to have a compact map of that part of RL_1^1 which contains no multiplicities. Instead of $z = 1/s$ as in the Example 1, consider the Möbius group of the bilinear transformation $z = (as + b)/(s - d)$, $b + ad \neq 0$. This transformation maps conformally the s-plane minus the point d onto the z-plane, minus the point a, see e.g. [4]. So we have to select a, d in such a way that neither at the point d of the s-plane, not at the point a of the z-plane occur the singularities of $s(K)$ or $z(G)$. The other possibility, already used in the Example 1 is to use two overlapping maps at the s- and z-plane and count the multiplicities either at the s- or z-plane.

Theorem 3. Consider the root loci RL_1^1 which are irreducible and have no multiplicity at ∞. Then there exist no root loci with odd number of even multiplicities.

Proof. It suffices to consider only the even multiplicity from $\mathbb{R}$. From it there eave to the top half of $\mathbb{C}$ the odd number of branches. On the other side from any odd multiplicity there leave the even number of branches. But there exists no pairing between the odd and even number of the branches. So there exists no partitioning of the root loci multiplicities between single even multiplicity and an odd multiplicity or multiplicities. □

Fact 1. Consider the root loci from the Theorem 3, moreover with the multiplicities nor exceeding 8. The representants of these under the equivalence from the Definition 3 are given in Fig. 4a — the poles are denoted by a slanting cross, the zeros by a diamond. The representants of all other root loci with the multiplicities not exceeding 8 are given by the union of these loci.

Fact 2. Consider the representants of the root loci in Fig. 4a. As the poles (zeros) are connected on their circles only with zeros (poles), it is not necessary to distinguish between them and it is possible to denote both poles and zeros as vertices. Next instead of a circle or circles connecting the pole and the zero connect the vertices by a single or multiple branch. Then the elementary root loci from the Fig. 4a are represented by the Coxeter diagrams, [8], in Fig. 4b.

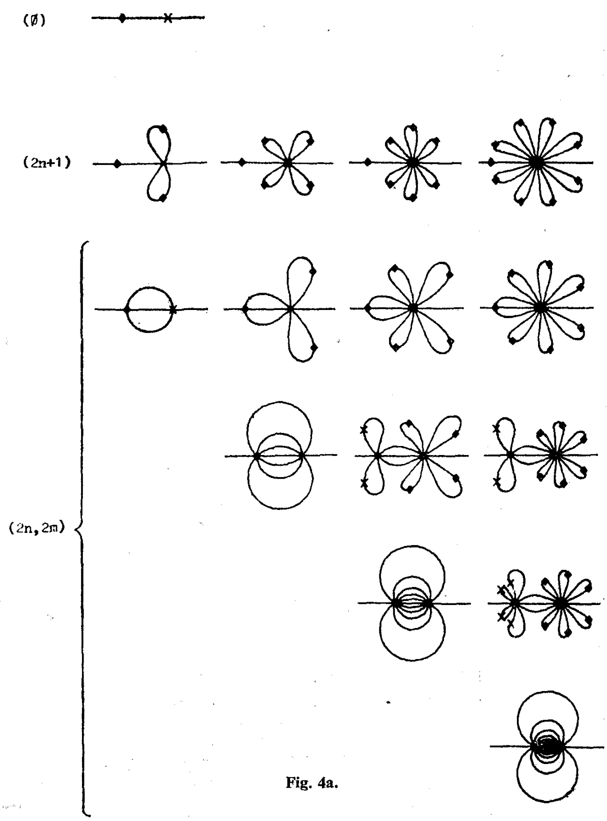

Fig. 4a.

4. INVERSE PROBLEM

Up to now we were concerned mainly with the analysis, mainly with respect to the testing of the robustness. We were mainly testing whether the $2m$-gons, which are in general the boundaries of the root m-tissues generated under the linear action of the m parameters, are in some feasibility regions of the stability half of the s-plane. Now let us consider the inverse, or the synthesis problem: given the poles p_i where

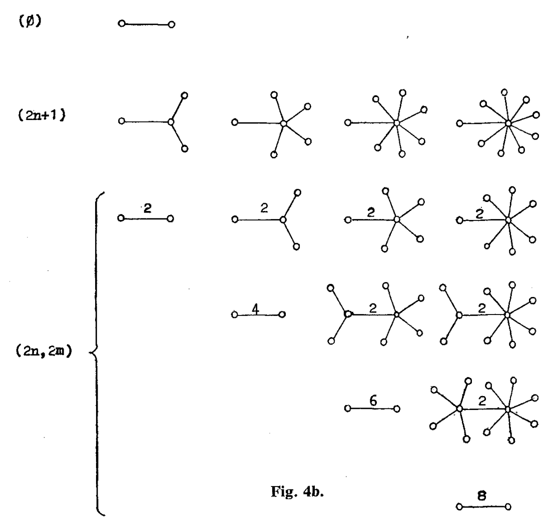

Fig. 4b.

they are we have to find the feedback gains K_i to shift the poles where they should be – at π_i $(i = 1, 2, \ldots, n)$. For the full state measurement feedback and a single input (dually a single output and full access to the state) the polynomial equations are, see e.g. [19]

$$a(s) + KA(s)\,G = c(s)$$

where $K \in \mathbb{R}^n$ (dually the left-hand-side is $a + HAL$, $L \in \mathbb{R}^n$). Let us set the paths s_i on which the poles p_i have to be shifted to the π_i $(i = 1, \ldots, n)$ as disjoint line segments

$$s_i = p_i + \vartheta(\pi_i - p_i) \quad (0 \leqq \vartheta \leqq 1)$$

The assumption on the simple π_i is justified by the properties of RL_n^1 at the multiple root. Then the gains are the integrals

$$K_i = \int_0^1 \mathrm{d}\varkappa_i(s_i(\vartheta), \ldots, s_n(\vartheta))$$

where for the differentials of the gain we have – following the derivation of DS_n^1:

$$\mathrm{d}s_i \prod_{\substack{j=1 \\ j \neq i}} (s_i - s_j) + \mathrm{d}\varkappa_1\, b_1(s_i) + \ldots + \mathrm{d}\varkappa_n\, b_n(s_i) = 0$$

where $b_i(s_i) = \sum A_{ij}(s_i)\, G_j$ $(i, j = 1, \ldots, n)$. In the direct problem we solve the equation for the roots differentials $\mathrm{d}s_i$, in the inverse problem we shall solve it with respect of the gains differentials. The linear algebraic equations for the differentials $d\varkappa_1, \dots, d\varkappa_n$

$$\begin{bmatrix} b_1(s_1) \dots b_1(s_n) \\ \dots\dots\dots\dots \\ b_n(s_1) \dots b_n(s_n) \end{bmatrix} \begin{bmatrix} d\varkappa_1 \\ \dots \\ d\varkappa_n \end{bmatrix} = \begin{bmatrix} -ds_1 \prod_{j=2}^{n} (s_1 - s_j) \\ \dots\dots\dots\dots \\ -ds_n \prod_{j=1}^{n-1} (s_n - s_j) \end{bmatrix}$$

The regularity of the matrix gives us the shiftability condition for the paths $s_1, \dots, s_n$. The poles shift equations are symmetric with respect to such permutation of the roots p_i or π_i which keeps the coefficients of $c(s)$ real. From these possible n-tuples of the shifting paths we can choose any – e.g. with respect of the time of computation the shortest.

Now consider the equation for the polynomial assignment for the output feedback and single input

$$a(s) + KHA(s)\, G = c(s)$$

where $K \in \mathbb{R}^\varrho$, $\varrho < n$. (Dually $a + HAGL$.) Denote $b_i(s) = \sum H_{ij} A_{jk}(s)\, G_k$ $(i = 1, \dots \dots, \varrho; j, k = 1, \dots, n)$. (Dually $b_i(s) = \sum H_j A_{jk}(s)\, G_{ki}$.) Now we have n linear algebraic equations for ϱ gains $d\varkappa_1, \dots, d\varkappa_\varrho$ and directly unspecified $n - \varrho$ roots $ds_{\varrho+1}, \dots, ds_n$, given the prescribed ϱ root paths $ds_1, \dots, ds_\varrho$. The ϱ equations for the gains are

$$\begin{bmatrix} b_1(s_1) \dots b_\varrho(s_1) \\ \dots\dots\dots\dots \\ b_1(s_\varrho) \dots b_\varrho(s_\varrho) \end{bmatrix} \begin{bmatrix} d\varkappa_1 \\ \dots \\ d\varkappa_\varrho \end{bmatrix} = \begin{bmatrix} -ds_1 \prod_{j=2}^{n} (s_1 - s_j) \\ \dots\dots\dots\dots \\ -ds_\varrho \prod_{\substack{j=1 \\ j \neq \varrho}}^{n} (s_\varrho - s_j) \end{bmatrix}$$

and the $n - \varrho$ equations for the roots are

$$ds_i = - \frac{\sum_{l=1}^{\varrho} b_l(s_i)\, d\varkappa_l}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (s_i - s_j)} \quad (i = \varrho + 1, \dots, n)$$

(For some of the roots $s_{\varrho+1}, \dots, s_n$ of the multiplicity n_i – but different from the roots $s_1, \dots, s_{\varrho-1}$, we use the n_ith root of the right-hand-side.)

Example 4. We have to find such feedback gains K_1, K_2 and the root π_3 to obey the polynomial assignment equation

$$(s - p_1)(s - p_2)(s - p_3) + K_1(s - z_1)(s - z_2) + K_2(s - z_3)(s - z_4) =$$
$$= (s - \pi_1)(s - \pi_2)(s - \pi_3)$$

where $p_{1,2} = 2 \pm 3\,j$, $p_3 = 0$, $z_1 = 2{\cdot}25$, $z_2 = 2$, $z_3 = 1{\cdot}75$, $z_4 = 1{\cdot}5$, $\pi_{1,2} = = -0{\cdot}5 \pm 0{\cdot}5\,j$. The prescribed paths for the roots $s_{1,2}$ are $s_1 = p_1 + \vartheta(\pi_1 - p_1)$,

$s_2 = p_2 + \vartheta(\pi_2 - p_2)$, $0 \leqq \vartheta \leqq 1$. The equations for $d\varkappa_1$, $d\varkappa_2$ are

$$\begin{bmatrix} (s_1 - 2{\cdot}25)(s_1 - 2) & (s_1 - 1{\cdot}75)(s_1 - 1{\cdot}5) \\ (s_2 - 2{\cdot}25)(s_2 - 2) & (s_2 - 1{\cdot}75)(s_2 - 1{\cdot}5) \end{bmatrix} \begin{bmatrix} d\varkappa_1 \\ d\varkappa_2 \end{bmatrix} = \begin{bmatrix} -ds_1(s_1 - s_2)(s_1 - s_3) \\ -ds_2(s_2 - s_3)(s_2 - s_1) \end{bmatrix}$$

the formula for ds_3 is

$$ds_3 = -\frac{(s_3 - 2{\cdot}25)(s_3 - 2)\, d\varkappa_1 + (s_3 - 1{\cdot}75)(s_3 - 1{\cdot}5)\, d\varkappa_2}{(s_3 - s_1)(s_3 - s_2)}$$

Integrating the equations for $d\varkappa_1$, $d\varkappa_2$ starting at $\varkappa_1$, $\varkappa_2 = 0$ for $\vartheta = 0$, and $s_{1,2} = 2 \pm 3j$ and the equation for ds_3 for $s_3 = 0$ we obtain $\varkappa_1 = K_1 = -8{\cdot}0$, $\varkappa_2 = K_2 = 14{\cdot}0$ and $s_3 = \pi_3 = -10{\cdot}$ at $\vartheta = 1$, at the end of the prescribed paths

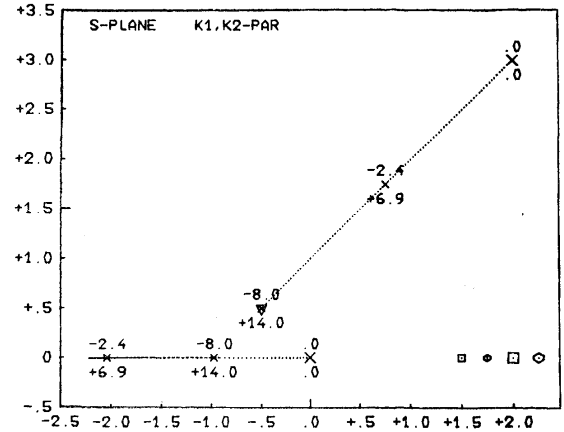

Fig. 5.

$s_{1,2}$ and computed folded paths s_3, see Fig. 5. For $\varrho = n$, another n integration paths can be the part of RL_1^1 from the stability half of the s-plane which is the solution of $a(s)\, a(-s) + \lambda\, b(s)\, b(-s) = 0$, i.e. the optimal poles parametrized by the scalar weight $0 \leqq \lambda \leqq \Lambda$. In this case the n optimal poles s_i (together with their n mirror images $-s_i$) and the n gains K_i $(i = 1, \ldots, n)$ are computed simultaneously.

Even dynamic controllers with single input (dually single output) are described by the characteristic polynomial linear in the gains and can be computed for the prescribed poles shift.

The situation changes for the control synthesis for several inputs and several outputs. E.g. for full state feedback we start with the n equations $(i = 1, \ldots, n)$ for $s = s_i$

$$\text{num det}\left[I + \frac{GA(s)\,K}{a(s)}\right] = 0$$

Then we can obtain the equation of the RL_n^v type. The number of the gains have to be again $\varrho \leqq n$, the number of directly unprescribed roots again $n - \varrho \geqq 0$, and the number of the prescribed root paths again ϱ.

Even in the inverse problem we can generalize the problem of shift to the prescribed point to that of shifting to the prescribed boundary.

Finally, even the inverse problem can be treated as the action of a group. Consider ϱ prescribed paths not as ϱ parametrized segments but as ϱ parametrized loops on which orbits belong to the couples $\{[p_i = s_i(0),\ \pi_i = s_i(1)] \mid i = 1, \ldots, \varrho\}$. This is a Lie group, see e.g. [1, 8], acting on the manifold of the product of the vector space $\mathbb{R}^\varrho$ of the parameters $K_1(\vartheta), \ldots, K_\varrho(\vartheta)$ and of the Lie group of the root loci of $s_{\varrho+1}(K_1, \ldots, K_\varrho), \ldots, s_n(K_1, \ldots, K_\varrho)$. In this paper we prefer only to touch the inverse problem. It is treated, for multiple inputs, in some other problem parametrization elsewhere, [35].

5. CONCLUSION

The multiple real parameters for both analysis and synthesis of the linear systems had been embedded into the complex plane. For the analysis, the root m-tissues are the image of the action of the additive group of the m parameters. For the synthesis, the m parameters are the image of the action of the Lie group of the prescribed m paths of the roots. This concern with the m bounded subsets ($2m$-gons or m abscissas in the complex plane) we take more pertinent to the control theory than the usual concern with some fixed m points. After the solution of the synthesis - action of the roots on the gain parameters, the analysis problem can be easily solved as the action of the system parameters on the roots.

ACKNOWLEDGEMENT

The persisting computational assistance of Mrs. Marie Nováková at the EAI Pacer 600 computer is gratefully acknowledged. Discussions with author's colleagues Jan Ježek, Vladimír Kučera and Milan Schlegel at the author's lectures on the theme of the paper in Prague 1984, are gratefully acknowledged.

(Received June 25, 1984.)

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