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LOGARITHMIC INFORMATION OF DEGREE q LINKED WITH AN EXTENSION OF FISHER'S INFORMATION

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We introduce the Fisher's information of degree q, $q \in 2\mathbb{N}^*$, of a given random variable X, with a density function depending on a parameter θ, $\theta \in \Theta$, the average Fisher's information of this variable when the parameter is itself random, and the logarithmic information of degree q which characterizes this one. We exhibit relations concerning these 3 quantities and we more particularly obtain extensions of the inequality of Cramér-Rao and properties of non-additivity.

1. INTRODUCTION

We consider the probability space $(\Omega, \mathscr{B}, P)$, where Ω is an Euclidean space, $\mathscr{B}$ the Borel σ-algebra on Ω and P the probability law of a random variable on $\mathscr{B}$; we suppose that P is absolutely continuous with regard to the Lebesgue measure; its density $f(x|\theta)$ with respect to this measure depends on a parameter $\theta \in \Theta \subseteq \mathbb{R}$.

We suppose that the parameter θ is unknown and we estimate a function $g(\theta)$ of this parameter with the help of a statistic T. A well known means of measuring the quality of T is to use the inequality of Cramér-Rao. It gives a good lower bound of the variance $\operatorname{Var}(T)$ of the estimator in the case where T is unbiased, as a function of Fisher's information:

$$I_X(\theta) = \int_\Omega \left(\frac{\log f(x|\theta)}{\partial \theta} \right)^2 f(x|\theta)\, \mathrm{d}x \tag{1}$$

The following inequality

$$\operatorname{Var}(T) \geqq \frac{g'(\theta)}{I_X(\theta)}, \tag{2}$$

is generally obtained, provided that some regularity conditions concerning the density $f(x|\theta)$ are satisfied; more particularly, it requires the possibility of differentiating under the integral sign [7].

The quantity $I_X(\theta)$ measures the information about $g(\theta)$ which is contained in the observation of X and which is deduced from the utilization of T. It can be connected with the comparison of two probability laws defined by near values θ and $\theta + \Delta\theta$ of the parameter, this comparison being established with the help of a divergence or an information gain [1].

Several generalizations have been obtained dealing on the one hand with Fisher's information and on the other hand with the inequality of Cramér-Rao in the non regular case in some cases [9], or regarding θ itself as a random variable [4], introducing new parameters in the definition of Fisher's information [2], establishing a connection with some measures of divergence [8].

In this paper, we propose to define Fisher's information of degree q, $q \in 2\mathbb{N}^*$, leading to a lower bound of the moment of order $r = q/(q-1)$ of the estimator T. In the case where the parameter θ is also random, we define the average Fisher's information of degree q, and we give a new extension of the inequality of Cramér-Rao.

We introduce the logarithmic information of degree q of the random variable θ, in order to exhibit non-additivity properties binding Fisher's information, average Fisher's information and logarithmic information of degree q. We further interprete the logarithmic information of degree q as a measure of the intrinsic information of the random variable θ; we look at the influence of an affine transformation of θ, or of its conditioning by another random variable, on the value of this information. Then we define the joint logarithmic information of two random variables and we exhibit a lower bound of this quantity involving two measures of the logarithmic information of degree q.

By using an intermediate density between two given densities with parameters θ and θ', we establish a lower bound of the rth moment of the unbiased estimator T of $g(\theta)$, not requiring regularity conditions.

Then we study the approximation of Fisher's information of degree q with the help of a φ-divergence for two densities of near parameters θ and θ'.

2. FISHER'S INFORMATION OF DEGREE q

The probability measure of the random variable X is given by means of the density $f(x|\theta)$ depending on the parameter θ, $\theta \in \Theta \subseteq \mathbb{R}$.

Definition 1. For $q \in 2\mathbb{N}^*$, the Fisher's information of degree q of the random variable X is defined by

$$I_X^q(\theta) = \int_\Omega \left(\frac{\partial \log f(x|\theta)}{\partial\theta}\right)^q f(x|\theta)\,\mathrm{d}x \tag{3}$$

It is obvious that $I_X^2(\theta) = I_X(\theta)$, defined in (1).

Theorem 1. Under some regularity conditions, the estimator T of the function $g(\theta)$

satisfies:

(4) $$\left(\mathsf{E}_{X|\theta}\left|T - g(\theta)\right|^r\right)^{1/r} \geqq \frac{\dfrac{\partial \mathsf{E}_{X|\theta}(T)}{\partial\theta}}{I_X^q(\theta)^{1/q}}$$

with $r > 0$ such that $1/q + 1/r = 1$.

Proof. By applying Hölder's inequality, we get

$$\left(\mathsf{E}_{X|\theta}\left|T - g(\theta)\right|^r\right)^{1/r} I^q(X)^{1/q} =$$

$$= \left(\int_\Omega \left|T(x) - g(\theta)\right|^r f(x \mid \theta)\,\mathrm{d}x\right)^{1/r} \left(\int_\Omega \left(\frac{\partial \log f(x \mid \theta)}{\partial\theta}\right)^q f(x \mid \theta)\,\mathrm{d}x\right)^{1/q} \geqq$$

$$\geqq \int_\Omega (T(x) - g(\theta))\,\frac{\partial \log f(x \mid \theta)}{\partial\theta}\,f(x \mid \theta)\,\mathrm{d}x\,.$$

This last quantity is equal to:

$$\int_\Omega (T(x) - g(\theta))\,\frac{\partial f(x \mid \theta)}{\partial\theta}\,\mathrm{d}x = \int_\Omega T(x)\,\frac{\partial f(x \mid \theta)}{\partial\theta}\,\mathrm{d}x - \int_\Omega g(\theta)\,\frac{\partial f(x \mid \theta)}{\partial\theta}\,\mathrm{d}x =$$

$$= \frac{\partial}{\partial\theta}\int_\Omega T(x) f(x \mid \theta)\,\mathrm{d}x - g(\theta)\,\frac{\partial}{\partial\theta}\int_\Omega f(x \mid \theta)\,\mathrm{d}x\,,$$

if we can differentiate under the integral sign, which gives result (4). □

In the case where T is an unbiased estimator of $g(\theta)$, if $\sigma^r(\theta) = \mathsf{E}_{X|\theta}\left|T - g(\theta)\right|^r$, we obtain

(5) $$\sigma^r(\theta) \geqq \frac{g'(\theta)}{I_X^q(\theta)^{r/q}}$$

when $q = 2$ $(r = 2)$, we find the inequality of Cramér-Rao again.

Boekee [2] indicated that the equality holds in (4) if and only if:

$$M(\theta)\,(T - g(\theta)) = K(\theta)\left|\frac{\partial \log f(x \mid \theta)}{\partial\theta}\right|^{1/(r-1)} \operatorname{sign}\left(\frac{\partial \log f(x \mid \theta)}{\partial\theta}\right) p\,.\,p$$

where $M(\theta)$ and $K(\theta)$ are two positive functions which do not equal zero simultaneously.

From the results of Boekee, we also deduce that the utilization of a statistic T with density $f_T(x \mid \theta)$ to estimate $g(\theta)$ entails a loss of the information contained in the random variable.

Theorem 2. With condition of differentiability under the integral sign, every statistic T satisfies;

(6) $$I^q_{T(X)}(\theta) \leqq I^q_X(\theta)$$

the equality holding if T is a sufficient statistic.

3. AVERAGE FISHER'S INFORMATION OF DEGREE q

3.1. Generalized Inequality of Cramér-Rao

We now suppose that the parameter θ is itself a random variable, with density $\xi(\theta)$, for $\theta \in \Theta = [a, b] \subseteq \mathbb{R}$ and that we can differentiate g. Let $f(x, \theta)$ be the joint probability density.

Definition 2. For $q \in 2\mathbb{N}^*$, the average Fisher's information of degree q of the random variable X is defined by

$$J^q(X) = \int_\Theta \int_\Omega \left(\frac{\partial \log f(x, \theta)}{\partial \theta} \right)^q f(x, \theta) \, dx \, d\theta \tag{7}$$

In the case where $q = 2$, Ferreira [4] generalized the inequality of Cramér-Rao in order to exhibit a lower bound of $\mathsf{E}(T - g(\theta))^2$, where E denotes the expectation with regard to $f(x, \theta)$. We show an analogous result, for any value of q, needing no regularity condition.

Theorem 3. Let X be a random variable of density $f(x \mid \theta)$, where θ is itself a random variable of density $\xi(\theta)$. Every statistic T satisfies the following inequality:

$$(\mathsf{E}|T - g(\theta)|^r)^{1/r} \geqq \frac{B + \mathsf{E}_\theta(g'(\theta))}{J^q(X)^{1/q}}, \tag{8}$$

where $B = \lim_{\theta \to b} \xi(\theta) \, \mathsf{E}_{X|\theta}(T - g(\theta)) - \lim_{\theta \to a} \xi(\theta) \, \mathsf{E}_{X|\theta}(T - g(\theta))$, and $r > 0$ is such that $1/q + 1/r = 1$.

Proof. Let $K = \left(\int_\Theta \int_\Omega |T - g(\theta)|^r f(x, \theta) \, dx \, d\theta \right)^{1/r} \left(\int_\Theta \int_\Omega \left(\frac{\partial \log f(x, \theta)}{\partial \theta} \right)^q f(x, \theta) \, dx \, d\theta \right)^{1/q}$, with $1/q + 1/r = 1$. The inequality of Hölder gives:

$$K \geqq \int_\Theta \int_\Omega (T - g(\theta)) f(x, \theta)^{1/r} \frac{\partial \log f(x, \theta)}{\partial \theta} f(x, \theta)^{1/q} \, dx \, d\theta =$$

$$= \int_\Theta \int_\Omega (T - g(\theta)) \frac{\partial f(x, \theta)}{\partial \theta} \, dx \, d\theta .$$

If we integrate by parts the last quantity, we get:

$$\int_\Omega (T - g(\theta)) f(x, \theta) \, dx \Big|_{\theta = a}^{b} - \int_\Theta \int_\Omega \frac{\partial (T - g(\theta))}{\partial \theta} f(x, \theta) \, dx \, d\theta = B + \mathsf{E}_\theta(g'(\theta))$$

with the notation indicated in Theorem 3 and provided that the limit of $\xi(\theta) . \mathsf{E}_X (T - g(\theta))$ exists for $\theta \to a^+$ and $\theta \to b^-$. Inequality (8) follows. □

This results can also be written as,

$$(9)\qquad (\mathsf{E}_\theta(\mathsf{E}_{X|\theta}|T - g(\theta)|^r)^{1/r} \geqq \frac{B + \mathsf{E}_\theta(g'(\theta))}{\left(\mathsf{E}_X\left(\mathsf{E}_{\theta|X}\left(\frac{\partial \log \xi(\theta \mid x)}{\partial \theta}\right)^q\right)\right)^{1/q}}$$

where $\mathsf{E}_{X|\theta}$ represents the expectation with regard to the probability density function $f(x \mid \theta)$ and $\mathsf{E}_{\theta|X}$ the expectation with regard to the probability density function $\xi(\theta \mid x)$.

In the case of an unbiased estimator T of $g(\theta)$, we obtain

$$(10)\qquad (\mathsf{E}_\theta(\sigma^r(\theta)))^{1/r} \geqq \frac{\mathsf{E}_\theta(g'(\theta))}{J^q(X)^{1/q}}.$$

with notation $\sigma^r(\theta)$ introduced in Section 2.

3.2. Non-additivity

We now show that Fisher's information of degree q can be connected with the corresponding average information, by means of a non-additivity property.

Definition 3. For $q \in 2\mathbb{N}^*$, the logarithmic information of degree q of the random variable θ, with density $\xi(\theta)$, is defined by

$$(11)\qquad H^q(\theta) = \int_\Theta \left(\frac{\partial \log \xi(\theta)}{\partial \theta}\right)^q \xi(\theta)\, \mathrm{d}\theta$$

It is easy to prove the following result.

Theorem 4. For $q \in 2\mathbb{N}^*$, Fisher's information of degree q and the corresponding average and logarithmic informations satisfy the following inequality, if $\xi(\theta)$ and $f(x \mid \theta)$ are non-decreasing functions of θ:

$$(12)\qquad J^q(X) \geqq \mathsf{E}_\theta(I^q_X(\theta)) + H^q(\theta)$$

Proof. As $f(x, \theta) = f(x \mid \theta)\, \xi(\theta)$, it comes:

$$J^q(X) = \int_\Omega \int_\Theta \left(\frac{\partial \log f(x, \theta)}{\partial \theta}\right)^q f(x, \theta)\, \mathrm{d}x\, \mathrm{d}\theta =$$

$$= \int_\Omega \int_\Theta \left(\frac{\partial \log f(x \mid \theta)}{\partial \theta} + \frac{\partial \log \xi(\theta)}{\partial \theta}\right)^q f(x \mid \theta)\, \xi(\theta)\, \mathrm{d}x\, \mathrm{d}\theta =$$

$$= \int_\Omega \int_\Theta \left(\frac{\partial \log f(x \mid \theta)}{\partial \theta}\right)^q f(x \mid \theta)\, \xi(\theta)\, \mathrm{d}x\, \mathrm{d}\theta +$$

$$+ \int_\Omega \int_\Theta \left(\frac{\partial \log \xi(\theta)}{\partial \theta}\right)^q f(x \mid \theta)\, \xi(\theta)\, \mathrm{d}x\, \mathrm{d}\theta + F,$$

where F is the sum of the other terms of the binomial formula

$$F = \sum_{i=1}^{q-1} C_q^i \int_\Omega \int_\Theta \left(\frac{\partial \log f(x \mid \theta)}{\partial \theta}\right)^i \left(\frac{\partial \log \xi(\theta)}{\partial \theta}\right)^{q-i} f(x \mid \theta)\, \xi(\theta)\, \mathrm{d}x\, \mathrm{d}\theta\,.$$

Hence:

$$J^q(X) = \int_\Theta I_X^q(\theta)\,.\,\xi(\theta)\,\mathrm{d}\theta + H^q(\theta) + F\,.$$

When $\xi(\theta)$ and $f(x \mid \theta)$ are non-decreasing functions of θ, F is positive and (12) is satisfied. □

For $q = 2$, we get:

$$\begin{aligned} F &= 2 \int_\Omega \int_\Theta \frac{\partial \log f(x \mid \theta)}{\partial \theta} \frac{\partial \log \xi(\theta)}{\partial \theta} f(x \mid \theta)\, \xi(\theta)\, \mathrm{d}x\, \mathrm{d}\theta = \\ &= 2 \int_\Theta \frac{\partial \log \xi(\theta)}{\partial \theta} \xi(\theta)\, \mathrm{d}\theta \int_\Omega \frac{\partial f(x \mid \theta)}{\partial \theta}\, \mathrm{d}x = \\ &= 2 \int_\Theta \frac{\partial \log \xi(\theta)}{\partial \theta} \xi(\theta)\, \mathrm{d}\theta \frac{\partial}{\partial \theta} \int_\Omega f(x \mid \theta)\, \mathrm{d}x\,, \end{aligned}$$

provided that we can differentiate under the integral sign. The last quantity equals zero and this entails the equality in (12).

Conversely, for $q \neq 2$, we can exhibit densities $f(x, \theta)$ and $\xi(\theta)$ such that this equality does not hold. Let us consider, for example:

$$f(x,\theta) \begin{cases} = x\theta^2 & \text{if} \quad x \in [0, \sqrt{(2/\theta)}] \quad \text{et} \quad \theta \in [1, \sqrt{3}] \\ = 0 & \text{otherwise} \end{cases}$$

$$\xi(\theta) \begin{cases} = \theta & \text{if} \quad \theta \in [1, \sqrt{3}] \\ = 0 & \text{otherwise} \end{cases}$$

$$f(x|\theta) \begin{cases} = x\theta & \text{if} \quad x \in [0, \sqrt{(2/\theta)}] \\ = 0 & \text{otherwise.} \end{cases}$$

It comes:

$$J^q(X) = 2^q \frac{\sqrt{(3)}^{2-q} - 1}{2 - q}$$

$$I_X^q(\theta) = \frac{1}{\theta^q}$$

$$\mathsf{E}_\theta(I_X^q(\theta)) = \frac{\sqrt{(3)}^{2-q} - 1}{2 - q} = H^q(\theta)\,.$$

It is easy to see that: $J^q(X) > \mathsf{E}_\theta(I_X^q(\theta)) + H^q(\theta)$.

Corollary. For $q \in \mathbb{N}^*$, Fisher's information of degree q and the corresponding average and logarithmic informations satisfy the following additivity property, whatever may be the densities $f(x \mid \theta)$ and $\xi(\theta)$, provided that we can differentiate under the integral sign:

$$J^q(X) = \mathsf{E}_\theta(I^q_X(\theta)) + H^q(\theta) \tag{13}$$

if and only if $q = 2$.

However, we remark that, in the case where $q \neq 2$, equality (13) may hold for some densities for example:

$$f(x, \theta) \begin{cases} = 1/\theta & \text{if} \quad x \in [0, \theta] \quad \text{and} \quad \theta \in [1, 2] \\ = 0 & \text{otherwise} \end{cases}$$

$$\xi(\theta) \quad \begin{cases} = 1 & \text{if} \quad \theta \in [1, 2] \\ = 0 & \text{otherwise.} \end{cases}$$

Then, we get:

$$J^q(X) = \frac{2^{-q+1} - 1}{1 - q} = \mathsf{E}_\theta(I^q_X(\theta))$$

$$H^q(\theta) = 0 \,.$$

Minkowski's inequality allows us to obtain another inequality concerning the various informations of degree q.

Theorem 5. For $q \in 2\mathbb{N}^*$, Fisher's information of degree q and the corresponding average and logarithmic informations satisfy the following inequality:

$$J^q(X)^{1/q} \leqq \mathsf{E}_\theta(I^q_X(\theta))^{1/q} + H^q(\theta)^{1/q} \,, \tag{14}$$

the equality holding if and only if θ is uniformly distributed.

Proof. We have

$$J^q(X)^{1/q} = \left(\int_\Omega \int_\Theta \left(\frac{\partial \log f(x \mid \theta)}{\partial \theta} + \frac{\partial \log \xi(\theta)}{\partial \theta}\right)^q f(x \mid \theta)\, \xi(\theta) \,\mathrm{d}x \,\mathrm{d}\theta\right)^{1/q}$$

By using Minkowski's inequality, we get:

$$J^q(X)^{1/q} \leqq \left(\int_\Theta \left(\int_\Omega \left(\frac{\partial \log f(x \mid \theta)}{\partial \theta}\right)^q f(x \mid \theta) \,\mathrm{d}x\right) \xi(\theta) \,\mathrm{d}\theta\right)^{1/q} +$$

$$+ \left(\int_\Omega \left(\int_\Theta \left(\frac{\partial \log \xi(\theta)}{\partial \theta}\right)^q \xi(\theta) \,\mathrm{d}\theta\right) f(x \mid \theta) \,\mathrm{d}x\right)^{1/q} \leqq$$

$$\leqq \left(\int_\Theta I^q_X(\theta)\, \xi(\theta) \,\mathrm{d}\theta\right)^{1/q} + H^q(\theta)^{1/q} \,,$$

which yields (14).

The equality holds if and only if

$$\frac{\partial \log f(x \mid \theta)}{\partial \theta} = K \frac{\partial \log \xi(\theta)}{\partial \theta}, \quad \text{for} \quad K > 0 ,$$

therefore $f(x \mid \theta) = \xi(\theta)^K c(x)$.

By integrating both sides with respect to x, we necessarily get $\xi(\theta)$ as a constant. □

4. PROPERTIES OF THE LOGARITHMIC INFORMATION OF DEGREE q

The logarithmic information of degree $q \in 2\mathbb{N}^*$ defined in (11) cannot be regarded as a kind of Fisher's information, since the variable and the parameter are the same. However, it turns out to be important when the parameter θ of the density of X is also a random variable, as it is the case in Theorem 4 and its corollary.

It is easy to check that:

$$H^q(\theta) \geqq 0 ,$$

the equality holding if and only if θ is uniformly distributed.

This property, as well as the additivity property in the case of independent random variables, points out the fact that this quantity is an interesting measure of the intrinsic information of a random variable of continuous type.

4.1. Affine transformation of the random variable

We study the influence on the logarithmic information of the affine transformation $h(y) = cy + d$, for $c \in \mathbb{R}^*$ and $d \in \mathbb{R}$. Let $v = h(\theta)$ be the transformed random variable, with density $\eta(v) = (1/c)\, \xi(\theta)$. It is easy to see that:

Theorem 6. For $q \in 2\mathbb{N}^*$, the logarithmic information of degree q of the random variable v satisfies the equality:

$$H^q(v) = \frac{1}{c^q} H^q(\theta) . \tag{15}$$

4.2. Conditioning of the random variable

We now suppose that θ is conditioned by another random variable α, taking its values in $A \subseteq \mathbb{R}$ of density $\omega(\alpha)$. Let $\zeta(\theta|\alpha)$ be the conditional density function; we prove a sub-additivity property of the logarithmic information of degree q.

Definition 4. For $q \in 2\mathbb{N}^*$, we call logarithmic information of degree q of θ conditioned by α the following quantity;

$$H^q(\theta \mid \alpha) = \int_A \int_\Theta \left(\frac{\partial \log \zeta(\theta \mid \alpha)}{\partial \theta} \right)^q \zeta(\theta|\alpha)\, \omega(\alpha)\, \mathrm{d}\alpha\, \mathrm{d}\theta \tag{16}$$

Then, we get:

Theorem 7. The logarithmic information of degree q of the random variable θ conditioned by α satisfies the following inequality, provided that we can differentiate under the integral sign:

$$H^q(\theta) \leqq H^q(\theta \mid \alpha)\,, \tag{17}$$

the equality holding if θ and α are independent.

Proof. We have

$$\xi(\theta) = \int_A \zeta(\theta \mid \alpha)\,\omega(\alpha)\,\mathrm{d}\alpha \quad \forall \theta \in \Theta\,.$$

It comes:

$$H^q(\theta) = \int_\Theta \left(\frac{\partial \log \xi(\theta)}{\partial \theta}\right)^q \xi(\theta)\,\mathrm{d}\theta = \int_\Theta \left[\frac{\dfrac{\partial \int_A \zeta(\theta \mid \alpha)\,\omega(\alpha)\,\mathrm{d}\alpha}{\partial \theta}}{\int_A \zeta(\theta \mid \alpha)\,\omega(\alpha)\,\mathrm{d}\alpha}\right]^q \xi(\theta)\,\mathrm{d}\theta =$$

$$= \int_\Theta \left(\frac{\int_A \dfrac{\partial \zeta(\theta \mid \alpha)}{\partial \theta}\,\omega(\alpha)\,\mathrm{d}\alpha}{\int_A \zeta(\theta \mid \alpha)\,\omega(\alpha)\,\mathrm{d}\alpha}\right)^q \xi(\theta)\,\mathrm{d}\theta\,.$$

By considering

$$\mu(\alpha \mid \theta) = \frac{\zeta(\theta \mid \alpha)\,\omega(\alpha)}{\int_A \zeta(\theta \mid \alpha)\,\omega(\alpha)\,\mathrm{d}\alpha}\,,$$

we get

$$H^q(\theta) = \int_\Theta \left(\int_A \frac{\dfrac{\partial \zeta(\theta \mid \alpha)}{\partial \theta}}{\zeta(\theta \mid \alpha)}\,\mu(\alpha \mid \theta)\,\mathrm{d}\alpha\right)^q \xi(\theta)\,\mathrm{d}\theta\,.$$

We apply Jensen's inequality

$$H^q(\theta) \leqq \int_\Theta \left(\int_A \left(\frac{\partial \log \zeta(\theta \mid \alpha)}{\partial \theta}\right)^q \mu(\alpha \mid \theta)\,\mathrm{d}\alpha\right) \xi(\theta)\,\mathrm{d}\theta =$$

$$= \int_\Theta \int_A \left(\frac{\partial \log \zeta(\theta \mid \alpha)}{\partial \theta}\right)^q \zeta(\theta \mid \alpha)\,\omega(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\theta = H^q(\theta \mid \alpha)\,. \qquad \square$$

4.3. Joint logarithmic information of degree q

Let θ and α be two random variables of respective densities $\xi(\theta)$ and $\omega(\alpha)$, $\theta \in \Theta$, $\alpha \in A$. We denote by $\varrho(\theta, \alpha)$ their joint density function.

Definition 5. For $q \in 2N$, the joint logarithmic information of degree q of two random variables θ and α is defined by

$$(18)\qquad L^q(\theta, \alpha) = \int_A \int_\Theta \left[\left(\frac{\partial \log \varrho(\theta, \alpha)}{\partial \theta}\right)^q + \left(\frac{\partial \log \varrho(\theta, \alpha)}{\partial \alpha}\right)^q\right] \varrho(\theta, \alpha)\, \mathrm{d}\theta\, \mathrm{d}\alpha$$

This quantity satisfies a super-additivity property with regard to the logarithmic information of degree q, which can be expressed as follows:

Theorem 8. The logarithmic informations of two random variables θ and α and their joint logarithmic information of degree q satisfies the following inequality, provided that we can differentiate under the integral sign:

$$(19)\qquad L^q(\theta, \alpha) \geqq H^q(\alpha) + H^q(\theta \mid \alpha),$$

the equality holding if θ and α are independent,

Proof. The logarithmic information of θ conditioned by α equals

$$(20)\qquad H^q(\theta \mid \alpha) = \int_A \left(\int_\Theta \left(\frac{\partial \log \zeta(\theta \mid \alpha)}{\partial \theta}\right)^q \zeta(\theta \mid \alpha)\, \mathrm{d}\theta\right) \omega(\alpha)\, \mathrm{d}\alpha =$$

$$= \int_A \int_\Theta \left(\frac{\partial \log \varrho(\theta, \alpha)}{\partial \theta}\right)^q \varrho(\theta, \alpha)\, \mathrm{d}\theta\, \mathrm{d}\alpha\,.$$

On the other hand,

$$H^q(\alpha) = \int_A \left(\frac{\partial \log \omega(\alpha)}{\partial \alpha}\right)^q \omega(\alpha)\, \mathrm{d}\alpha = \int_A \left(\frac{\displaystyle\int_\Theta \frac{\partial \varrho(\theta, \alpha)}{\partial \alpha}\, \mathrm{d}\theta}{\displaystyle\int_\Theta \varrho(\theta, \alpha)\, \mathrm{d}\theta}\right)^q \omega(\alpha)\, \mathrm{d}\alpha =$$

$$= \int_A \left(\int_\Theta \frac{\dfrac{\partial \varrho(\theta, \alpha)}{\partial \theta}}{\varrho(\theta, \alpha)} \frac{\varrho(\theta, \alpha)}{\displaystyle\int_\Theta \varrho(\theta, \alpha)\, \mathrm{d}\theta}\, \mathrm{d}\theta\right)^q \omega(\alpha)\, \mathrm{d}\alpha\,.$$

We apply Jensen's inequality:

$$(21)\qquad H^q(\alpha) \leqq \int_A \int_\Theta \left(\frac{\dfrac{\partial \varrho(\theta, \alpha)}{\partial \alpha}}{\varrho(\theta, \alpha)}\right)^q \frac{\varrho(\theta, \alpha)}{\displaystyle\int_\Theta \varrho(\theta, \alpha)\, \mathrm{d}\theta}\, \omega(\alpha)\, \mathrm{d}\theta\, \mathrm{d}\alpha =$$

$$= \int_A \int_\Theta \left(\frac{\partial \log \varrho(\theta, \alpha)}{\partial \alpha}\right)^q \varrho(\theta, \alpha)\, \mathrm{d}\theta\, \mathrm{d}\alpha\,.$$

By using together results (20) and (21), we obtain (19). □

5. FISHER'S INFORMATION OF DEGREE q FOR AN INTERMEDIATE DENSITY BETWEEN TWO GIVEN DENSITIES

For two given densities $f(x \mid \theta)$ and $f(x \mid \theta')$, θ and $\theta' \in \Theta$, Vincze [9] used the following intermediate density;

$$h_{\theta,\theta'}(x \mid \alpha) = (1 - \alpha) f(x \mid \theta) + \alpha f(x \mid \theta') \quad 0 \leqq \alpha \leqq 1 . \tag{22}$$

in order to prove an inequality of Cramér-Rao without any regularity condition, in the case of the classical Fisher's information.

We now obtain an analogous result for Fisher's information of degree q, by considering the unbiased estimator $\hat{\alpha}$ of α defined by

$$\hat{\alpha} = \frac{T(x) - g(\theta)}{g(\theta') - g(\theta)}, \tag{23}$$

where T is also an unbiased estimator of $g(\theta)$.

Theorem 9. Let $q \in 2\mathbb{N}^*$. For two densities $f(x \mid \theta)$ and $f(x \mid \theta')$, θ and $\theta' \in \Theta$, the Fisher's information of degree q of the random variable X with density function $h_{\theta,\theta'}(x \mid \alpha)$, depending on α, defined in (22), satisfies for the estimator $\hat{\alpha}$:

$$(\mathsf{E}_\alpha |\hat{\alpha} - \alpha|^r)^{1/r} \geqq \frac{1}{I_X^q(\alpha)^{1/q}}, \tag{24}$$

with $r > 0$, $1/q + 1/r = 1$, where E_α denotes the expectation with regard to the density $h_{\theta,\theta'}(x \mid \alpha)$

Proof. From Hölder's inequality, we get $(\mathsf{E}_\alpha |\hat{\alpha} - \alpha|^r)^{1/r} . I_X^q(\alpha)^{1/q} =$

$$= \left(\int_\Omega |\hat{\alpha} - \alpha|^r \, h_{\theta,\theta'}(x \mid \alpha) \, \mathrm{d}x \right)^{1/r} \left(\int_\Omega \left(\frac{\frac{\partial h_{\theta,\theta'}(x \mid \alpha)}{\partial \alpha}}{h_{\theta,\theta'}(x \mid \alpha)} \right)^q h_{\theta,\theta'}(x \mid \alpha) \, \mathrm{d}x \right)^{1/q} \leqq$$

$$\leqq \int_\Omega (\hat{\alpha} - \alpha) \, h_{\theta,\theta'}(x \mid \alpha)^{1/r} \frac{\frac{\partial h_{\theta,\theta'}(x \mid \alpha)}{\partial \alpha}}{h_{\theta,\theta'}(x \mid \alpha)} \, h_{\theta,\theta'}(x \mid \alpha)^{1/q} \, \mathrm{d}x = 1$$

Hence, inequality (24) follows. □

By using Minkowski's inequality, we find an upper bound of $\mathsf{E}_\alpha |\hat{\alpha} - \alpha|^r$ involving the rth absolute central moments of T:

$$\mathsf{E}_\alpha |\hat{\alpha} - \alpha|^r = \int_\Omega \left| \frac{T(x) - g(\theta)}{g(\theta') - g(\theta)} - \alpha \right|^r (1 - \alpha) f(x \mid \theta) \, \mathrm{d}x +$$

$$+ \int_\Omega \left| \frac{T(x) - g(\theta')}{g(\theta') - g(\theta)} + 1 - \alpha \right|^r \alpha f(x \mid \theta') \, \mathrm{d}x$$

Thus:

$$(25)\qquad (\mathsf{E}_\alpha|\hat{\alpha} - \alpha|^r)^{1/r} \leqq \left(\int_\Omega \left|\frac{T(x) - g(\theta)}{g(\theta') - g(\theta)} - \alpha\right|^r (1 - \alpha) f(x \mid \theta)\,\mathrm{d}x\right)^{1/r} +$$

$$+ \left(\int_\Omega \left|\frac{T(x) - g(\theta')}{g(\theta') - g(\theta)} + 1 - \alpha\right|^r \alpha f(x \mid \theta')\,\mathrm{d}x\right)^{1/r} \leqq$$

$$\leqq \left(\int_\Omega \left|\frac{T(x) - g(\theta)}{g(\theta') - g(\theta)}\right|^r (1 - \alpha) f(x \mid \theta)\,\mathrm{d}x\right)^{1/r} + \left(\int_\Omega \alpha^r (1 - \alpha) f(x \mid \theta)\,\mathrm{d}x\right)^{1/r} +$$

$$+ \left(\int_\Omega \left|\frac{T(x) - g(\theta')}{g(\theta') - g(\theta)}\right|^r \alpha f(x \mid \theta')\,\mathrm{d}x\right|^{1/r} + \left(\int_\Omega (1 - \alpha)^r \alpha f(x \mid \theta')\,\mathrm{d}x\right)^{1/r} \leqq$$

$$\leqq \frac{1}{|g(\theta') - g(\theta)|}\left[(1 - \alpha)^{1/r}\sigma^r(\theta)^{1/r} + \alpha^{1/r}\sigma^r(\theta')^{1/r}\right] + \alpha(1 - \alpha)^{1/r} + (1 - \alpha)\alpha^{1/r}$$

where $\sigma^r(\theta) = \int_\Omega |T - g(\theta)|^r f(x \mid \theta)\,\mathrm{d}x$ and $\sigma^r(\theta') = \int_\Omega |T - g(\theta')|^r f(x \mid \theta')\,\mathrm{d}x$.

In the case where the rth absolute moment of T is independent of θ and equals $\sigma^r(\theta) = \sigma^r(\theta') = \sigma_r$, it yields the following lower bound

$$(26)\qquad \sigma_r^{1/r} \geqq \sup_\alpha \sup_{\theta'} \left[\frac{1}{I_X^q(\alpha)^{1/q}} - \alpha(1 - \alpha)^{1/r} - (1 - \alpha)\,\alpha^{1/r}\right] \frac{|g(\theta') - g(\theta)|}{(1 - \alpha)^{1/r} + \alpha^{1/r}}$$

6. FISHER'S INFORMATION OF DEGREE q AND φ-DIVERGENCE

We consider the φ-divergence for two densities $f(x \mid \theta)$ and $f(x \mid \theta')$, θ and $\theta' \in \Theta$, introduced by Csiszár and Perez (cf. [3], [6]) with the help of a real-valued convex function φ, which possesses derivatives until the $(q + 1)$th order:

$$(27)\qquad G_\varphi(\theta' \| \theta) = \int_\Omega \varphi\left(\frac{f(x \mid \theta')}{f(x \mid \theta)}\right) f(x \mid \theta)\,\mathrm{d}x\,.$$

6.1. Approximation of Fisher's information of degree q

We prove:

Theorem 10. Let $q \in 2\mathbb{N}^*$. If φ is a real-valued function, convex and which possesses derivatives until the $(q + 1)$th order, the $(q - 1)$ first being null in $x = 1$, the qth being non-null, and if the density f is differentiable with respect to θ, then the Fisher's information of degree q of the random variable X satisfies

$$(28)\qquad I_X^q(\theta) = \frac{q!}{\varphi^{(q)}(1)} \lim_{\theta' \to \theta} \left[\frac{1}{(\theta' - \theta)^q} \left(G_\varphi(\theta' \| \theta) - \varphi(1)\right)\right].$$

Proof. We note $\theta' - \theta = \Delta\theta$ and we use a Taylor's expansion of order 1 of function f in a neighbourhood of θ:

$$G_\varphi(\theta + \Delta\theta \mid \theta) = \int_\Omega \varphi\left(1 + \Delta\theta \frac{f'_\theta(x \mid \theta)}{f(x \mid \theta)} + 0(\theta^2)\right) f(x \mid \theta)\,\mathrm{d}x\,,$$

where f'_θ denotes the derivative of f with regard to θ.

We now expand the function φ up to the order q in the neighbourhood of 1:

$$G_\varphi(\theta + \Delta\theta \mid \theta) = \int_\Omega \left(\varphi(1) + u\,\varphi'(1) + \ldots + \frac{u^q}{q!}\varphi^{(q)}(1) + 0(\Delta\theta^{q+1})\right) f(x \mid \theta)\,\mathrm{d}x\,,$$

with $u = \Delta\theta\,[f'_\theta(x \mid \theta)/f(x \mid \theta)] + 0(\Delta\theta)^2$. Since the $(q-1)$ first derivatives of φ equal zero at the point $x = 1$, it comes

$$\begin{aligned} G_\varphi(\theta + \Delta\theta \mid \theta) &= \varphi(1) + \frac{1}{q!}\int_\Omega \left(\left(\Delta\theta \frac{f'_\theta(x \mid \theta)}{f(x \mid \theta)} + 0(\Delta\theta^2)\right)^q \varphi^{(q)}(1) + 0(\Delta\theta^{q+1})\right) f(x \mid \theta)\mathrm{d}x \\ &= \varphi(1) + \frac{(\Delta\theta)^q}{q!}\varphi^{(q)}(1)\, I^q(X) + 0(\Delta\theta)^{q+1}\,. \end{aligned}$$

We deduce (28) immediately,

Examples of functions φ allowing to approach Fisher's information of degree q are the following:

$$\varphi_1(u) = (u-1)^q \quad u \in \mathbb{R}_+$$

$$\varphi_2(u) = \mathrm{e}^{(u-1)^q}\,, \quad u \in \mathbb{R}_+$$

$$\varphi^q_\alpha(u) = \frac{(u-1)^q}{((1-\alpha) + \alpha u)^{q-1}}\,, \quad u \in \mathbb{R}_+\,, \quad \text{for} \quad \alpha \in [0, 1]\,.$$

It is easy to see that:

$$G_{\varphi^q_\alpha}(\theta' \mid \theta) = I^q(X_\alpha)\,,$$

whatever may be θ and $\theta' \in \Theta$, $\alpha \in [0, 1]$, $q \in 2\mathbb{N}^*$.

6.2. Lower bound of the rth absolute central moment of an estimator

In the case where $\alpha \to 0$, inequalities (24) and (25) give the following result, concerning the rth absolute central moment of the unbiased estimator T of $g(\theta)$ introduced in Section 5:

$$\sigma^r(\theta) \geqq \sup_{\theta'} \frac{|g(\theta') - g(\theta)|^r}{\lim\limits_{\alpha \to 0} I^q_X(\alpha)^{r-1}}\,, \tag{29}$$

If $G_{\varphi_1}(\theta' | |\theta)$ and $G_{\varphi_1}(\theta| |\theta')$ are finite, it is easy to see that

$$\lim_{\alpha \to 0} I_X^q(\alpha) = \int_\Omega \left(\frac{f(x \mid \theta')}{f(x \mid \theta)} - 1 \right)^q f(x \mid \theta) \, dx \, . = G_{\varphi_1}(\theta' | |\theta) \, .$$

Thus, we obtain a lower bound of the rth absolute central moment of T, as a function of a φ_1-divergence;

Theorem 11. If $G_{\varphi_1}(\theta| |\theta')$ and $G_{\varphi_1}(\theta'| |\theta)$ are finite, the rth absolute central moment of the unbiased estimator T of $g(\theta)$ satisfies the inequality

$$\sigma^r(\theta) \geqq \sup_{\theta'} \frac{|g(\theta') - g(\theta)|^r}{G_{\varphi_1}(\theta'| |\theta)^{r-1}} \, ,$$

where X is the random variable of density $f(x \mid \theta)$.

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