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INFORMATION CHANNELS COMPOSED OF MEMORYLESS COMPONENTS

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Explicit bounds for the maximum length of n-dimensional codes at any admitted level of the probability of error are derived, valid for all n, in case that the channels considered are composed of a finite number of memoryless components. The special case studied by the author in [2], is discussed in this more general frame.

BASIC NOTATIONS

Given a finite non-empty set M, the symbol W_M means the class of all shift-invariant probability measures in the space M^I, where I denotes the set of all integers. A measure $m \in W_M$ (satisfying the relation $m \circ T_M^{-1} = m$) is defined on the σ-algebra F_M of Borel sets in M^I which may be generated by the class of "elementary" cylinders (a base of the topology) of the form $T_M^i[z]$, $i \in I$,

$$[z] = \bigcap_{0 \leqq i < n} \{\zeta \in M^I : \zeta_i = z_i\} \quad \text{for} \quad z = (z_i, 0 \leqq i < n) \in M^n ;$$

here T_M is the shift (defined by $(T_M\zeta)_i = \zeta_{i+1}$). Define

$$\tilde{W}_M = \{m \in W_M : m \text{ is ergodic w.r. to } T_M\} .$$

All the information channels considered here are supposed to have finite alphabets A, B (card $A \geqq 2$, card $B \geqq 2$), A the output alphabet, B the input alphabet. Denoting by xy the element in $(A \times B)^n$ for $x \in A^n$, $y \in B^n$ given by $(xy)_i = (x_i, y_i)$, $0 \leqq i < n$, associate with any measure ω in $W_{A \times B}$ the information rate (convention: $\log = \log_2$)

$$J(\omega) = \lim_n (1/n) \sum_{x \in A^n, y \in B^n} \omega[xy] \log \frac{\omega[xy]}{\omega^A[x]\, \omega^B[y]} ,$$

where $\omega^A \in W_A$, $\omega^B \in W_B$ are the marginal measures determined by the conditions

$$\omega^A[x] = \sum_y \omega[xy] , \quad \omega^B[y] = \sum_x \omega[xy] .$$

A point $\zeta \in Z = (A \times B)^I$ is called regular if there is a (uniquely determined) measure $m_\zeta \in \tilde{W}_{A\times B}$ such that

$$m_\zeta[z] = \lim_n (1/n) \sum_{i=0}^{n-1} \chi_{[z]}(T^i_{A\times B}\zeta) , \quad z \in \bigcup_n (A \times B)^n ;$$

χ_E designates the characteristic function of $E \subset Z$. The set of all regular points in Z will be denoted by R. Since $\omega(R) = 1$, we may define, for $\omega \in W_{A\times B}$,

$$q(\Theta, \omega) = \min \{t \geqq 0: \omega\{z \in R: J(m_\zeta) \leqq t\} \geqq \Theta\} , \quad 0 < \Theta \leqq 1 ;$$

$$\bar{q}(\Theta, \omega) = \max \{t \geqq 0: \omega\{z \in R: J(m_\zeta) \geqq t\} \geqq 1 - \Theta\} , \quad 0 \leqq \Theta < 1 .$$

The latter quantities are the lower and the upper Θ-quantiles of the random variable $(J(m_\zeta), \zeta \in R)$ w.r. to ω.

In the entire paper a channel (a discrete information channel, stationary and of zero past history; cf. [1]) is defined as a family $\nu = (\nu_\eta, \eta \in B^I)$ of probability measures ν_η on F_A satisfying the relations

$$\nu_{\eta'}(T^i_A[x]) = \nu_\eta[x] \quad \text{for} \quad \eta' \in T^i_B[y] , \quad \eta \in [y] , \quad x \in A^n , \quad y \in B^n , \quad n = 1, 2 \ldots$$

Since $\nu_\eta[x]$ is constant for $\eta \in [y]$, define for $E \subset A^n$

$$\nu[E \mid y] = \sum_{x \in E} \nu[x \mid y] , \quad \nu[x \mid y] = \nu_\eta[x] , \quad \eta \in [y] .$$

If $\mu \in W_B$ then $\nu\mu$ will denote the measure in $W_{A\times B}$ satisfying

$$\nu\mu[xy] = \nu[x \mid y] \, \mu[y] , \quad x \in A^n , \quad y \in B^n , \quad n = 1, 2, \ldots.$$

The quantile function q_ν of a channel ν (cf. [5]) is defined by

$$q_\nu(\Theta) = \sup \{q(\Theta, \nu\mu): \mu \in \tilde{W}_B\} , \quad 0 < \Theta \leqq 1 .$$

As an auxiliary function we define

$$\bar{q}_\nu(\Theta) = \sup \{\bar{q}(\Theta, \nu\mu): \mu \in \tilde{W}_B\} , \quad 0 \leqq \Theta < 1 .$$

The quantile functions q_ν and $\bar{q}_\nu$ of a memoryless channel ν are constant, both identically equal to the transmission-rate capacity of the channel (cf. [1]); recall that ν is memoryless iff

$$\nu[x \mid y] = \prod_{0 \leqq i < n} \nu[x_i \mid y_i] .$$

If $0 < \varepsilon < 1$, $Y \subset B^n$ then a family $Q = (Q(y), y \in Y)$ of mutually disjoint sets $Q(y) \subset A^n$ is, by definition, an n-dimensional ε-code for a channel ν of length $l_Q =$ = card Y iff $\nu[Q(y) \mid y] > 1 - \varepsilon$ for all $y \in Y$. The maximum length of n-dimensional ε-codes will be denoted by $S_n(\varepsilon, \nu)$; in symbols:

$$S_n(\varepsilon, \nu) = \max \{l_Q: Q \text{ is an } n\text{-dimensional } \varepsilon\text{-code for } \nu\} .$$

The set of all families $p = (p(b), b \in B)$ of non-negative real numbers which add to one, will be denoted by P (the set of probability vectors on alphabet B). Let μ^p

be the measure in W_B satisfying the condition

$$\mu^p[y] = \prod_{0 \leqq i < n} p(y_i), \quad y \in B^n(p \in P).$$

If $N(b \mid y) = \{i: y_i = b(0 \leqq i < n)\}$, $s_p(b) = (p(b)(1 - p(b))^{1/2}$,

$$d = \max(\text{card } A, \text{card } B),$$

define

$$F_n(p) = \bigcap_{b \in B} \{y \in B^n: |N(b \mid y) - n\, p(b)| \leqq 2 s_p(b)(nd)^{1/2}\}, \quad p \in P.$$

An n-dimensional ε-code $Q = (Q(y), y \in Y)$ for a channel ν is said to be a (p, ε)-code for p in P iff $Y \subset F_n(p)$. Define

$$S_n^*(\varepsilon, \nu, \mu^p) = \max\{l_Q: Q \text{ is an } n\text{-dimensional } (p, \varepsilon)\text{-code for } \nu\}.$$

The behaviour of the latter auxiliary quantity will be studied by means of the n-dimensional information density

$$I_n(xy; \nu\mu^p) = (1/n) \log(\nu[x \mid y]/(\nu\mu^p)^A[x]).$$

COMPOSED CHANNEL

We shall make the following assumptions: k is a natural number, $(\nu^\alpha, \alpha \in \mathscr{K})$ for $\mathscr{K} = \{1, \ldots, k\}$ is a family of memoryless channels, $\xi = (\xi_1, \ldots, \xi_k)$ is a probability vector satisfying the condition

$$\xi_0 = \min\{\xi_\alpha: \alpha \in \mathscr{K}\} > 0,$$

and ν is the composed channel defined by

$$\nu_\eta(E) = \sum_{\alpha=1}^{k} \xi_\alpha \nu_\eta^\alpha(E), \quad \eta \in B^I, \quad E \in F_A;$$

the latter relation will be written as

$$\nu = \sum_{\alpha \in \mathscr{K}} \xi_\alpha \nu^\alpha.$$

Together with the given channel ν we shall consider its "subchannel" $\nu^{\mathscr{A}}$ for $\mathscr{A} \subset \mathscr{K}$ ($\mathscr{A}$ non-empty) defined by

$$\nu^{\mathscr{A}} = \sum_{\alpha \in \mathscr{A}} (\xi_\alpha/\xi(\mathscr{A}))\, \nu^\alpha, \quad \xi(\mathscr{A}) = \sum_{\alpha \in \mathscr{A}} \xi_\alpha,$$

particularly in case that $\mathscr{A}$ belongs to the class

$$\mathbf{A}(\Theta) = \{\mathscr{A} : \mathscr{A} \subset \mathscr{K}, \quad \xi(\mathscr{A}) \geqq \Theta\}, \quad 0 < \Theta \leqq 1.$$

Let $\mathscr{R}_\alpha(p) = \sum\{p(b)\, \nu^\alpha[a \mid b]\, I_1(ab; \nu^\alpha\mu^p): a \in A, b \in B\}$,

$$r_p(\Theta, \xi) = \min_{\mathscr{A} \in \mathbf{A}(\Theta)} \max_{\alpha \in \mathscr{A}} \mathscr{R}_\alpha(p), \quad 0 < \Theta \leqq 1,$$

$$r'_p(\Theta, \xi) = \max_{\mathscr{A} \in \mathbf{A}(1-\Theta)} \min_{\alpha \in \mathscr{A}} \mathscr{R}_\alpha(p), \quad 0 \leqq \Theta < 1, \quad p \in P.$$

Then the quantile function of the composed channel is expressed by

$$q_\nu(\Theta) = \max_{p \in P} r_p(\Theta, \xi) = \lim_{\lambda \downarrow 0} \bar{q}_\nu(\Theta - \lambda)\,, \quad 0 < \Theta \leqq 1$$

(cf. Theorem 4 in [4] and Lemma 3 in [2]), analogously

$$\bar{q}_\nu(\Theta) = \max_{p \in P} r'_p(\Theta, \xi) = \lim_{\lambda \downarrow 0} q_\nu(\Theta + \lambda)\,, \quad 0 \leqq \Theta < 1\,,$$

and both $q_\nu, \bar{q}_\nu$ are monotonically increasing, having the same set $\mathscr{D}_\nu$ of discontinuity points in the open interval $(0, 1)$ satisfying the relations

$$\mathscr{D}_\nu = \{\Theta: 0 < \Theta < 1\,, \quad q_\nu(\Theta) < \bar{q}_\nu(\Theta)\}\,, \quad \mathscr{D}_\nu \subset \{\xi(\mathscr{A}): \mathscr{A} \subset \mathscr{K}\}\,.$$

In every open subinterval (Θ_1, Θ_2) not containing any discontinuity point from $\mathscr{D}_\nu$, q_ν and $\bar{q}_\nu$ are constant and equal to each other (cf. Theorem 4 in [3]).

Hence it follows that the set

$$P(\Theta) = \{p \in P: r'_p(\Theta, \xi) = \bar{q}_\nu(\Theta)\} \quad \text{for} \quad 0 \leqq \Theta < 1$$

is non-empty. Let

$$w_0 = \min\{\nu^\alpha[a \mid b]: \alpha \in \mathscr{K}\,, \quad a \in A\,, \quad b \in B\,, \quad \nu^\alpha[a \mid b] > 0\}\,,$$

$$p_0 = \min\{p(b): b \in B\,, \quad p(b) > 0\}\,, \quad p \in P\,,$$

$$\delta_\varepsilon = \tfrac{1}{4}\min\{|\xi(\mathscr{A}) - \varepsilon|: \mathscr{A} \subset \mathscr{K}\,, \quad \xi(\mathscr{A}) \neq \varepsilon\}\,,$$

$$K_\varepsilon = (2d)^3 \sqrt{(d)}\,(w_0\delta_\varepsilon)^{-(1/2)}\,, \quad p_\varepsilon = \sup\{p_0: p \in P(\varepsilon)\}\,, \quad 0 < \varepsilon < 1\,.$$

The composed channel ν is, by definition (cf. [2]), non-singular if $\nu^\alpha[a \mid b] > 0$ for all $a \in \mathscr{K}$, $a \in A$, $b \in B$.

Theorem 1. If ε is a continuity point of the quantile function q_ν (i.e., $0 < \varepsilon < 1$, $\varepsilon \notin \mathscr{D}_\nu$), then the maximum length of n-dimensional ε-codes for the channel ν satisfies the inequalities

$$\log S_n(\varepsilon, \nu) < n\, q_\nu(\varepsilon) + \log(2\delta_\varepsilon\xi_0)^{-1} + K_\varepsilon \sqrt{n}\,,$$

$$\log S_n(\varepsilon, \nu) > n\, q_\nu(\varepsilon) - \log(2\delta_\varepsilon\xi_0)^{-1} - K_\varepsilon(p_\varepsilon)^{-1} \sqrt{n}\,,$$

for $n = 1, 2, \ldots$; if ν is non-singular, then

$$|\log S_n(\varepsilon, \nu) - n\, q_\nu(\varepsilon)| < \log(2\delta_\varepsilon\xi_0)^{-1} + K_\varepsilon \sqrt{n}\,.$$

Remark. The non-singular case was treated in [2], but the method used there for finding the bounds must be modified for the case considered here, as will be seen below. On the other hand, Theorem 4 in [3] guarantees only the existence of a constant c_ε such that $|\log S_n(\varepsilon, \nu) - nq_\nu(\varepsilon)| < c_\varepsilon \sqrt{n}$, but yields no direct method for its computing.

LEMMATA

To show the validity of Theorem 1, we shall first prove two lemmas under the assumption that we are given a real number $\varepsilon' (0 < \varepsilon' < 1)$, a probability vector p in P, and a non-empty set $\mathscr{A} \subset \mathscr{K}$. Let $\omega^\alpha = (\nu^\alpha \mu^p)^A$ for $\alpha \in \mathscr{A}$, $L(t) = t \log t^{-1}$ $(0 < t \leqq 1)$, $L(0) = 0$,

$$L_0 = \max\{L(t): 0 \leqq t \leqq 1\} = e^{-1} \log e\,,$$
$$K' = K'(\varepsilon') = d(\varepsilon')^{-(1/2)}\,,$$
$$\bar{K} = \bar{K}(\varepsilon') = L_0\, d^2(5\sqrt{(d)} + 2K' + 4K'(1 + 2\sqrt{(d)})^{1/2})\,,$$
$$K_0 = K_0(\varepsilon') = K' L_0\, d(d+1)(w_0)^{-(1/2)}\,,$$
$$K_0' = K_0'(\varepsilon') = (K' + 2\sqrt{(d)})\, L_0(d^2 + 1)(w_0)^{-(1/2)}\,,$$
$$\lambda_n = \lambda_n(\varepsilon') = (1/n)(\log(\xi_0)^{-1} + (\bar{K} + K_0)\sqrt{n})\,,$$
$$\lambda_n' = \lambda_n'(\varepsilon', p_0) = (1/n)(\log(\xi_0)^{-1} + (\bar{K} + K_0')(p_0)^{-1}\sqrt{n})\,,$$
$$\lambda_n'' = \lambda_n''(\varepsilon') = (1/n)(\log(\xi_0)^{-1} + (\bar{K} + K_0')\sqrt{n})\,.$$

If $N(a, b \mid x, y) = \{i: x_i = a,\ y_i = b (0 \leqq i < n)\}$, $s_\alpha(a \mid b) = (\nu^\alpha[a \mid b]\ (1 - \nu^\alpha[a \mid b]))^{1/2}$, define for $\alpha \in \mathscr{K}$, $y \in B^n$,

$$\Gamma_n^\alpha(y; a, b) = \{x \in A^n: |N(a, b \mid x, y) - N(b \mid y)\, \nu^\alpha[a \mid b]| \leqq K'\, s_\alpha(a \mid b) \sqrt{[N(b \mid y)]}\}\,,$$
$$\Gamma_n^\alpha(y) = \bigcap_{a \in A, b \in B} \Gamma_n^\alpha(y; a, b)\,, \quad \Gamma_n^{\mathscr{A}}(y) = \bigcup_{\alpha \in \mathscr{A}} \Gamma_n^\alpha(y)\,.$$

Lemma 1. If $x \in \Gamma_n^{\mathscr{A}}(y)$ for $y \in F_n(p)$ then

$$\min_{\alpha \in \mathscr{A}} \mathscr{R}_\alpha(p) - \lambda_n' < I_n(xy; \nu^{\mathscr{A}} \mu^p) < \max_{\alpha \in \mathscr{A}} \mathscr{R}_\alpha(p) + \lambda_n\,.$$

Moreover, if ν is non-singular, $I_n(xy; \nu^{\mathscr{A}} \mu^p) + \lambda_n'' > \min_{\alpha \in \mathscr{A}} \mathscr{R}_\alpha(p)$.

Proof. I. Given $\alpha \in \mathscr{A}$, suppose that $x \in \Gamma_n^\alpha(y)$; then $\nu^\alpha[x \mid y] \geqq w_0^n$, $\omega^\alpha[x] \geqq (p_0 w_0)^n$ because of $\mu^p[y] \geqq (p_0)^n$, and if $\nu^\alpha[a \mid b] > 0$ and $\omega^\alpha[a] > 0$, respectively, then

$$|N(a, b \mid x, y) - n\, p(b)\, \nu^\alpha[a \mid b]| <$$
$$< \sqrt{(n)}\,(\sqrt{(d)}\, \nu^\alpha[a \mid b] + K'(\nu^\alpha[a \mid b])^{1/2})\,,$$
$$|N(a \mid x) - n\, \omega_\alpha[x]| <$$
$$< \sqrt{(n)}\, d(2\sqrt{(d)}\,(\omega^\alpha[a])^{1/2} + \bar{K}'(\omega^\alpha[a])^{1/4})$$

where $N(a \mid x) = \sum\{N(a, b \mid x, y): b \in B\}$, $\bar{K}' = K'(1 + 2\sqrt{(d)})^{1/2}$; cf. [6], Chapter 2, and notice that $s_p(b) \leqq \frac{1}{2}$. Similarly as in [6], loc. cit., we have

$$\log \frac{\nu^\alpha[x \mid y]}{\omega^\alpha[x]} = \sum_{a,b} N(a, b \mid x, y) \log \nu^\alpha[a \mid b] - \sum_a N(a \mid x) \log \omega^\alpha[a] \lesseqgtr$$
$$\lesseqgtr n\, \mathscr{R}_\alpha(p) \pm L_0\, d^2(\sqrt{(d)} + 2K' + 4\sqrt{(d)} + 4\bar{K}')\sqrt{(n)}\,.$$

Thus it is proved under the above assumptions that

(1) $$|I_n(xy; \nu^\alpha \mu^p) - \mathscr{R}_\alpha(p)| < \bar{K} \cdot n^{-(1/2)} .$$

II. Given α, β in $\mathscr{A}$, suppose that $x \in \Gamma_n^\alpha(y)$, and that $\nu^\beta[x \mid y] > 0$; then $\nu^\beta[a \mid b] = 0$ implies that $N(a, b \mid x, y) = 0$. Let

$$A_b = \{a \in A: \nu^\alpha[a \mid b] > 0,\ \nu^\beta[a \mid b] > 0\} ,$$

$$V_b = \sum_{a \in A_b} N(a, b \mid x, y) \log \frac{\nu^\beta[a \mid b]}{\nu^\alpha[a \mid b]} \quad \text{for} \quad b \in B_y ,$$

where $B_y = \{b \in B: N(b \mid y) > 0\}$. Since (cf. [2], (2.14))

$$\sum_{a \in A_b} \nu^\alpha[a \mid b] \log \frac{\nu^\beta[a \mid b]}{\nu^\alpha[a \mid b]} \leqq L(\nu^\alpha[A_b \mid b]) ,$$

$$|N(a, b \mid x, y) - N(b \mid y)\, \nu^\alpha[a \mid b]| < \tfrac{1}{2} K' \sqrt{[N(b \mid y)]} \text{ for } b \in B_y ,$$

we obtain the inequality

$$V_b < N(b \mid y)\, L(\nu^\alpha[A_b \mid b]) + \tfrac{1}{2} K' \log (w_0)^{-1} \cdot d \sqrt{[N(b \mid y)]} , \quad b \in B_y .$$

Case 1: If, for all a in A, $\nu^\beta[a \mid b] = 0$ implies that $\nu^\alpha[a \mid b] = 0$, then $\nu^\alpha[A_b \mid b] = 1$, and

$$V_b < \sqrt{[N(b \mid y)]}\, K' d \log (w_0)^{-(1/2)} \quad \text{for} \quad b \in B_y .$$

Case 2: Let $\nu^\beta[a_b \mid b] = 0$ and $\nu^\alpha[a_b \mid b] > 0$, $a_b \in A$, $b \in B_y$; then

$$N(a_b, b \mid x, y) = 0 > N(b \mid y)\, \nu^\alpha[a_b \mid b] - K'(N(b \mid y)\, \nu^\alpha[a \mid b])^{1/2}$$

so that $\sqrt{[N(b \mid y)]} < K'(w_0)^{-(1/2)}$. Thus in both cases

$$V_b < \sqrt{[N(b \mid y)]}\, (K'(w_0)^{-(1/2)} L_0 + K' d \log (w_0)^{-(1/2)}) \quad \text{for} \quad b \in B_y .$$

From here it follows that

(2) $$\sum_{b \in B_y} V_b = \log \frac{\nu^\beta[x \mid y]}{\nu^\alpha[x \mid y]} < \sqrt{(n)}\, K_0 .$$

III. Given α, β in $\mathscr{A}$, suppose that $x \in \Gamma_n^\alpha(y)$, and that $\omega^\beta[x] > 0$; then $\omega^\beta[a] = 0$ implies that $N(a \mid x) = 0$. Let

$$A_0 = \{a \in A: \omega^\alpha[a] > 0 ,\quad \omega^\beta[a] > 0\} ,$$

$$V_0 = \log \frac{\omega^\beta[x]}{\omega^\alpha[x]} = \sum_{a \in A_0} N(a \mid x) \log \frac{\omega^\beta[a]}{\omega^\alpha[a]} .$$

Since $|N(a \mid x) - n\, \omega^\alpha[a]| < \tfrac{1}{2}(K' + 2 \sqrt{(d)})\, d \sqrt{(n)}$, we find that

$$V_0 < n\, L(\omega^\alpha[A_0]) + \tfrac{1}{2}(K' + 2 \sqrt{(d)})\, d^2 \log (p_0 w_0)^{-1} ,$$

because under the assumptions made, if $\omega^\alpha[a] > 0$ and $\omega^\beta[a] > 0$, respectively, then $\omega^\alpha[a] \geqq p_0 w_0$, $\omega^\beta[a] \geqq p_0 w_0$.

Case 1: If, for all a in A, $\omega^\beta[a] = 0$ implies that $\omega^\alpha[a] = 0$, then $\omega^\alpha[A_0] = 1$, and

$$V_0 < \sqrt{(n)} \cdot \tfrac{1}{2}(K' + 2\sqrt{(d)})\, d^2 \log (p_0 w_0)^{-1} .$$

Case 2: Let $\omega^\beta[a_0] = 0$ and $\omega^\alpha[a_0] > 0$, $a_0 \in A$; then $N(a_0 \mid x) = 0$. Choose b_0 in B such that

$$p(b_0)\, v^\alpha[a_0 \mid b_0] > 0 ;$$

then

$$N(a_0, b_0 \mid x, y) = 0 > n\, p(b_0)\, v^\alpha[a_0 \mid b_0] - K''(n\, p(b_0)\, v^\alpha[a \mid b])^{1/2} - \\ - K'(v^\alpha[a \mid b]\, N(b_0 \mid y))^{1/2}$$

where $K'' = 2\sqrt{(d)}$. Thus $\sqrt{(n)} < (K' + K'')(p_0)^{-1}(w_0)^{-(1/2)}$ so that

$$V_0 < \sqrt{(n)} \frac{K' + 2\sqrt{(d)}}{p_0 \sqrt{(w_0)}} (L_0 + d^2 L(p_0 \sqrt{(w_0)})) .$$

In both cases we obtain that

$$\log \frac{\omega^\beta[x]}{\omega^\alpha[x]} < \sqrt{(n)}\, K_0' \frac{1}{p_0} . \tag{3}$$

If v is non-singular, then $\omega^\alpha[a] \geqq w_0$ for $\omega^\alpha[a] > 0$, and $\omega^\beta[a] \geqq w_0$ for $\omega^\beta[a] > 0$, which yields (cf. Case 1) the inequalities

$$\log \frac{\omega^\beta[x]}{\omega^\alpha[x]} < \sqrt{(n)}\,(K' + 2\sqrt{(d)})\, d^2 \log (w_0)^{-(1/2)} < \sqrt{(n)}\, K_0' . \tag{4}$$

IV. Let $\mathscr{A}^* = \{\alpha \in \mathscr{A} : x \in \Gamma_n^\alpha(y)\}$; by assumption, $\mathscr{A}^*$ is non-empty. Write

$$I_n = I_n(xy;\, v^{\mathscr{A}}\mu^p), \quad I_n^* = I_n(xy;\, v^{\mathscr{A}^*}\mu^p),$$

$$\mathscr{B} = \{\beta \in \mathscr{A} : \beta \notin \mathscr{A}^*,\ v^\beta[x \mid y] > 0\}, \quad \mathscr{B}' = \{\beta \in \mathscr{A} : \beta \notin \mathscr{A}^*,\ \omega^\beta[x] > 0\},$$

$$r = \max_{\alpha \in \mathscr{A}} \mathscr{R}_\alpha(p), \quad r' = \min_{\alpha \in \mathscr{A}} \mathscr{R}_\alpha(p),$$

$$\varrho = (1/n)(r + \bar{K}\sqrt{n}), \quad \varrho' = (1/n)(r' - \bar{K}\sqrt{n}),$$

$$V = \sum_{\beta \in \mathscr{B}} \xi_\beta \left(\sum_{\alpha \in \mathscr{A}^*} \xi_\alpha \frac{v^\alpha[x \mid y]}{v^\beta[x \mid y]} \right)^{-1},$$

$$V' = \sum_{\beta \in \mathscr{B}'} \xi_\beta \left(\sum_{\alpha \in \mathscr{A}^*} \xi_\alpha \frac{\omega^\alpha[x]}{\omega^\beta[x]} \right)^{-1}.$$

If $\alpha \in \mathscr{A}^*$ then, according to (1),

$$2^{n\varrho'}\, \omega^\alpha[x] < v^\alpha[x \mid y] < \omega^\alpha[x]\, 2^{n\varrho}$$

so that $\varrho' < I_n^* < \varrho$. Applying the inequality (2), we obtain

$$\exp_2(nI_n) = \exp_2(nI_n^*) \cdot (1 + V)(1 + V')^{-1} < 2^{n\varrho}(1 + (1 - \xi_0)(\xi_0)^{-1} \exp(K_0\sqrt{n}))$$

so that $I_n < r + \lambda_n$. Analogously, making use of (3), we get $I_n > r' - \lambda_n'$. If v is non-singular, then $I_n > r' - \lambda_n''$ by (4). □

Lemma 2. If $0 < \varepsilon < 1 - \varepsilon'$ then

$$\log S_n^*(\varepsilon, \nu^{\mathscr{A}}\mu^p) < n(\max_{\alpha\in\mathscr{A}} \mathscr{R}_\alpha(p) + \lambda_n(\varepsilon')) + \log(1 - \varepsilon - \varepsilon')^{-1};$$

if $\varepsilon' < \varepsilon < 1$ then

$$\log S_n^*(\varepsilon, \nu^{\mathscr{A}}\mu^p) > n(\min_{\alpha\in\mathscr{A}} \mathscr{R}_\alpha(p) - \lambda_n'(\varepsilon', p_0)) - \log(\tfrac{4}{3}(\varepsilon - \varepsilon')^{-1}).$$

If ν is non-singular, then it holds for $\varepsilon' < \varepsilon < 1$ that

$$n(\min_{\alpha\in\mathscr{A}} \mathscr{R}_\alpha(p) - \lambda_n''(\varepsilon')) < \log S_n^*(\varepsilon, \nu^\alpha\mu^p) + \log(\tfrac{4}{3}(\varepsilon - \varepsilon')^{-1}).$$

The proof is the same as that of Lemma 2 in [2], performed by making use of the preceding Lemma 1.

Theorem 2. Let $K_1(\delta) = \bar{K}(\delta) + K_0(\delta) + d \log e$, $K_2(\delta) = \bar{K}(\delta) + K_0'(\delta)$, $0 < \delta < \frac{1}{4}$. Then the maximum length of n-dimensional ε-codes $(0 < \varepsilon < 1)$ for the composed channel ν satisfies the inequalities

$$\log S_n(\varepsilon, \nu) < n\, q_\nu(\varepsilon + 4\delta) + \log(2\delta\xi_0)^{-1} + \sqrt{(n)}\, K_1(\delta)$$

for $4\delta \leqq 1 - \varepsilon$, and

$$\log S_n(\varepsilon, \nu) > n\, \bar{q}_\nu(\varepsilon - 4\delta) - \log(2\delta\xi_0)^{-1} - \sqrt{(n)}\,(p_0)^{-1} K_2(\delta)$$

for $\varepsilon \geqq 4\delta$ and any $p \in P(\varepsilon - 4\delta)$. If ν is non-singular, then

$$\log S_n(\varepsilon, \nu) > n\, \bar{q}_\nu(\varepsilon - 4\delta) - \log(2\delta\xi_0)^{-1} - \sqrt{(n)}\, K_2(\delta), \quad \varepsilon \geqq 4\delta.$$

Proof. Repeating the proof of Theorem 1 in [2] with the aid of the preceding lemma, we obtain that

$$\log S_n(\varepsilon, \nu) < n\, q_\nu(\Theta) + n\, \lambda_n(\varepsilon') + \log(1 - \varepsilon\,.\,\Theta^{-1} - \varepsilon')^{-1} + d \log(n + 1)$$

for $\varepsilon < \Theta \leqq 1$, $\varepsilon' > 0$, $\varepsilon' + \varepsilon\,.\,\Theta^{-1} < 1$. Since $1 - \varepsilon\,.\,\Theta^{-1} - \varepsilon' \geqq \frac{1}{2}(\Theta - \varepsilon)$ for $\varepsilon' \leqq \frac{1}{2}(\Theta - \varepsilon)$, $\log(n + 1) < \sqrt{(n)} \log e$, the first inequality follows from the preceding one by setting $\delta = \varepsilon' = \frac{1}{4}(\Theta - \varepsilon)$. Taking p in $P(\Theta')$ and using the second inequality of Lemma 2 above in the proof of Theorem 2 in [2], we derive the inequality

$$\log S_n(\varepsilon, \nu) > n\, \bar{q}_\nu(\Theta') - n\, \lambda'(\varepsilon', p_0) - \log(\tfrac{4}{3}(\varepsilon - \varepsilon')^{-1})$$

for $0 \leqq \Theta' < \varepsilon$, $0 < \varepsilon' < \varepsilon - \Theta'$. From here the second inequality given in the theorem follows for $\delta = \varepsilon' = \frac{1}{4}(\varepsilon - \Theta')$. The third inequality is verified analogously. □

Theorem 1 is a corollary to Theorem 2 for $\delta = \delta_\varepsilon$ because $\bar{q}_\nu(\varepsilon - 4\delta_\varepsilon) = q_\nu(\varepsilon) = q_\nu(\varepsilon + 4\delta_\varepsilon)$ for $\varepsilon \notin \mathscr{D}_\nu$, $K_1(\delta_\varepsilon) < K_\varepsilon$, $K_2(\delta_\varepsilon) < K_\varepsilon$. In verifying the latter inequalities it is necessary to use the following relations: $\delta < \frac{1}{4}$, $d \geqq 2$, $L_0 < (\frac{3}{4})^2$.

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