KYBERNETIKA — VOLUME 21 (1985), NUMBER 2

VARIABLE METRIC METHODS FOR A CLASS OF EXTENDED CONIC FUNCTIONS

LADISLAV LUKŠAN

The paper contains a description and an analysis of two variable metric algorithms for unconstrained minimization which find a minimum of an extended conic function after a finite number of steps provided it is possible to compute the derivatives of the model function at an arbitrary point $x \in R_n$. Moreover, the developed theory is applied to a special class of the exponenttial type of extended conic functions.

1. INTRODUCTION

Consider the objective function of the form

$$F(x) = \varphi(\tilde{F}(x), l(x)) \tag{1.1}$$

where $\tilde{F}$: $R_n \to R$ is a quadratic function with the constant positive definite Hessian matrix $\tilde{G}$ and $l : R_n \to R$ is a linear function with the constant gradient c, both defined in the n-dimensional vector space R_n. Define

$$\begin{cases} \sigma(x) = \dfrac{\partial\varphi(\tilde{F}(x), l(x))}{\partial\tilde{F}} \\ \tau(x) = \dfrac{\partial\varphi(\tilde{F}(x), l(x))}{\partial l} \end{cases} \tag{1.2}$$

and suppose that $\sigma(x) > 0$ for all $x \in R_n$. The function (1.1) generalizes a class of so called conic functions which were introduced by Bjørstad and Nocedal [1] for line search and by Davidon [3] and Sorensen [6] who used them for the construction of a new class of the variable metric methods for unconstrained minimization.

In this paper, we are proposing new modifications of the variable metric methods which minimize extended conic functions of the form (1.1) after a finite number of steps provided it is possible to compute the values $\sigma(x)$ and $\tau(x)$ at an arbitrary point $x \in R_n$. We use the single linear constraint technique instead of the collinear scaling which has been used in [3] and [6]. Section 2 is devoted to the derivation and analysis of the basic variable metric methods for extended conic functions. It also contains a detailed description of a new algorithm. Section 3 is devoted to the investigation of modification of the quasi-Newton methods without projections which were introduced in [4]. Finally, Section 4 concerns with the special class of extended conic functions for which the values $\sigma(x)$ and $\tau(x)$ can be computed at an arbitrary point $x \in R_n$.

We suppose throughout this paper that the function (1.1) has a unique critical point which is its minimizer. Some details about this problem are studied in Section 4.

2. BASIC VARIABLE METRIC METHODS

Consider the extended conic function (1.1). In order to simplify the notation, we omit the parameter x. We denote by F, g, G and $\tilde{F}$, $\tilde{g}$, $\tilde{G}$ the value, the gradient and the Hessian matrix of the function $F(x)$ and $\tilde{F}(x)$ respectively at the point $x \in R_n$. Furthermore we denote by l and c the value and the gradient of the function $l(x)$.

Using (1.1) we get the following formulae

$$\text{(2.1)} \qquad \begin{cases} F = \varphi(\tilde{F}, l) \\ g = \sigma\tilde{g} + \tau c \end{cases}$$

where $\sigma = \partial\varphi/\partial\tilde{F}$ and $\tau = \partial\varphi/\partial l$ with $\sigma > 0$. The vector c that appears in (2.1) can be determined from the values F, F_1, F_2 and the gradients g, g_1, g_2 computed at three different points x, $x_1 = x + \alpha_1 s$, $x_2 = x + \alpha_2 s$ by the formula

$$\text{(2.2)} \qquad c = \frac{\left(\frac{g_2}{\sigma_2} - \frac{g}{\sigma}\right)\alpha_1 - \left(\frac{g_1}{\sigma_1} - \frac{g}{\sigma}\right)\alpha_2}{\left(\frac{\tau_2}{\sigma_2} - \frac{\tau}{\sigma}\right)\alpha_1 - \left(\frac{\tau_1}{\sigma_1} - \frac{\tau}{\sigma}\right)\alpha_2}$$

which has been proved in [5].

The variable metric methods for minimizing extended conic functions are based on the iterative scheme

$$\text{(2.3)} \qquad x_{i+1} = x_i + \alpha_i s_i ,$$

$i \in N = \{1, 2, \ldots\}$, where s_i is a direction vector and α_i is a steplength. We assume, in this section, that the steplengths are chosen by the perfect line searches, so that

$$\text{(2.4)} \qquad s_i^{\mathrm{T}} g_{i+1} = 0$$

for $i \in N$. The following lemma is essential for conjugate direction methods.

Lemma 2.1. Let F: $R_n \to R$ be an extended conic function. Consider the iterative scheme (2.3) and (2.4). Let the direction vectors satisfy the conditions $s_i^{\mathrm{T}}\tilde{G}s_j = 0$

for $1 \leqq i < j \leqq k$ and $s_i^T c = 0$ for $1 \leqq i < k$ with $k \leqq n$. Then

(2.5) $$s_i^T g_{k+1} = 0$$

for $1 \leqq i \leqq k$.

Proof. See [5], proof of Lemma 3.1. □

Lemma 2.1 shows that the conjugate directions have to be generated in such a way that first $n - 1$ of them lie in the subspace which is orthogonal to the vector c. Then $s_i^T g_{n+1} = 0$ for $1 \leqq i \leqq n$. If, in addition, $s_i \neq 0$ for $1 \leqq i \leqq n$ then $g_{n+1} = 0$ and, consequently, x_{n+1} is a minimizer of the extended conic function $F(x)$.

The next theorem gives the possibility of determining a set of mutually conjugate directions with desired properties.

Theorem 2.1. Let $F: R_n \to R$ be an extended conic function. Consider the iterative scheme (2.3) and (2.4) where

(2.6) $$\begin{cases} s_i = -\dfrac{1}{\sigma_i}\left(H_i - \dfrac{H_i c c^T H_i}{c^T H_i c}\right) g_i \\ \text{for } 1 \leqq i < n \text{ and} \\ s_n = -\dfrac{1}{\sigma_n} H_n g_n \,. \end{cases}$$

Define

(2.7) $$\begin{cases} d_i = x_{i+1} - x_i = \alpha_i s_i \\ \text{and} \\ \tilde{y}_i = \tilde{g}_{i+1} - \tilde{g}_i = \dfrac{g_{i+1}}{\sigma_{i+1}} - \dfrac{g_i}{\sigma_i} - \left(\dfrac{\tau_{i+1}}{\sigma_{i+1}} - \dfrac{\tau_i}{\sigma_i}\right) c \end{cases}$$

for $1 \leqq i < n$. Let H_1 be an arbitrary symmetric positive definite matrix of order n and let

(2.8) $$H_{i+1} = H_i + U_i A_i U_i^T$$

for $1 \leqq i < n$, where U_i is the $n \times 2$ matrix which has the columns d_i and $H_i \tilde{y}_i$ and A_i is a 2×2 symmetric matrix which is chosen in such a way that H_{i+1} is positive definite and

(2.9) $$H_{i+1} \tilde{y}_i = d_i \,.$$

Then the direction vectors s_i, $1 \leqq i \leqq n$ are nonzero and mutually conjugate provided g_i is not parallel to the vector c for $1 \leqq i < n$ and $g_n \neq 0$ (regular case). Moreover, $d_i^T c = 0$ and $H_n \tilde{y}_i = d_i$ for $1 \leqq i < n$ in the regular case.

Proof. We prove this theorem by induction. Suppose that $d_k \neq 0$ and $d_k^T c = 0$ and, moreover, $d_i^T \tilde{G} d_k = 0$ and $H_k \tilde{y}_i = d_i$ for $1 \leqq i < k$ where $k < n$. It certainly holds for $k = 1$ provided g_1 is not parallel to the vector c.

(a) The equality $H_{k+1} \tilde{y}_k = d_k$ follows from (2.9). Furthermore

$$H_{k+1} \tilde{y}_i = H_k \tilde{y}_i + U_k A_k U_k^T \tilde{y}_i = H_k \tilde{y}_i = d_i$$

for $1 \leqq i < k$ since $H_k\tilde{y}_i = d_i$, $d_k^T\tilde{y}_i = d_k^T\tilde{G}d_i = 0$ and $\tilde{y}_k^T H_k \tilde{y}_i = \tilde{y}_k^T d_i = d_k^T \tilde{G} d_i = 0$ for $1 \leqq i < k$ by the assumption.

(b) The conditions $d_i^T g_{k+1} = 0$, $1 \leqq i \leqq k$ follow from Lemma 2.1.

(c) If $k + 1 < n$ then using (2.6) and (a), (b) we get

$$\sigma_{k+1} s_{k+1}^T c = \frac{g_{k+1}^T H_{k+1} c}{c^T H_{k+1} c} c^T H_{k+1} c - g_{k+1}^T H_{k+1} c = 0$$

and

$$\sigma_{k+1} s_{k+1}^T \tilde{y}_i = \frac{g_{k+1}^T H_{k+1} c}{c^T H_{k+1} c} c^T H_{k+1} \tilde{y}_i - g_{k+1}^T H_{k+1} \tilde{y}_i = \frac{g_{k+1}^T H_{k+1} c}{c^T H_{k+1} c} c^T d_i - g_{k+1}^T d_i = 0$$

for $1 \leqq i \leqq k$ so that $d_{k+1}^T c = 0$ and $d_{k+1}^T \tilde{G} d_i = d_{k+1}^T \tilde{y}_i = 0$ for $1 \leqq i \leqq k$. Moreover, $s_{k+1} \neq 0$ provided g_{k+1} is not parallel to the vector c. But $s_{k+1} \neq 0$ implies $g_{k+1}^T s_{k+1} \neq 0$ by (2.6) so that $\alpha_{k+1} \neq 0$ and, therefore, $d_{k+1} \neq 0$ in the regular case.

(d) If $k + 1 = n$ then g_{k+1} is parallel to the vector c and, therefore, $s_{k+1}^T c \neq 0$ for $g_{k+1} \neq 0$. Again $s_{k+1}^T \tilde{y}_i = 0$ for $1 \leqq i \leqq k$ as well as in the case (c). Also $g_{k+1}^T s_{k+1} \neq 0$ for $g_{k+1} \neq 0$ so that $\alpha_{k+1} \neq 0$ and $d_{k+1} \neq 0$ in the regular case. □

Combining (2.8) and (2.9) we get a one-parameter class of variable metric methods which was introduced by Broyden [2]. In this case

$$H_{i+1} = H_i + \frac{d_i d_i^T}{\tilde{y}_i^T d_i} - \frac{H_i \tilde{y}_i (H_i \tilde{y}_i)^T}{\tilde{y}_i^T H_i \tilde{y}_i} + \frac{\vartheta_i}{\tilde{y}_i^T H_i \tilde{y}_i} \left(\frac{\tilde{y}_i^T H_i \tilde{y}_i}{\tilde{y}_i^T d_i} d_i - H_i \tilde{y}_i \right) \left(\frac{\tilde{y}_i^T H_i \tilde{y}_i}{\tilde{y}_i^T d_i} d_i - H_i \tilde{y}_i \right)^T \tag{2.10}$$

for $1 \leqq i < n$, where ϑ_i is a free parameter. Most frequently used formulae use the values $\vartheta_i = 0$ (DFP method) or $\vartheta_i = 1$ (BFGS method). Note that

$$d_i^T \tilde{y}_i = d_i^T \left(\frac{g_{i+1}}{\sigma_{i+1}} - \frac{g_i}{\sigma_i} - \left(\frac{\tau_{i+1}}{\sigma_{i+1}} - \frac{\tau_i}{\sigma_i} \right) c \right) = \frac{\alpha_i}{\sigma_i^2} g_i^T \left(H_i - \frac{H_i c c^T H_i}{c^T H_i c} \right) g_i > 0$$

for $1 \leqq i < n$, in the regular case, which is a necessary assumption for the positive definiteness of the matrix (2.10).

The following algorithm summarizes our results.

Algorithm 2.1.

Step 1: Determine an initial point x and compute the value $F := F(x)$ and the gradient $g := g(x)$. Compute the values $\sigma := \sigma(x)$ and $\tau := \tau(x)$ defined by (1.2). Determine an initial symmetric positive definite matrix H of order n (usually set $H := I$, where I is the unit matrix of order n). Set $k := 0$.

Step 2: If the termination criteria are satisfied (for example if $\|g\|$ is sufficiently small) then stop.

Step 3: If $k = 0$ then determine the vector c by (2.2) where x, x_1 and x_2 are three different points lying on a line.

Step 4: Set $k := k + 1$. If $k < n$ then set

$$s := -H(g/\sigma) + \frac{c^T H(g/\sigma)}{c^T H c} Hc$$

else set $k := 0$ and $s := -H(g/\sigma)$.

Step 5: Use a perfect line search procedure to determine the point $x_2 := x + \alpha_2 s$ such that $s^T g(x_2) = 0$. Compute the value $F_2 := F(x_2)$ and the gradient $g_2 := g(x_2)$. Compute the values $\sigma_2 := \sigma(x_2)$ and $\tau_2 := \tau(x_2)$ defined by (1.2).

Step 6: If $k \neq 0$ then set

$$d := x_2 - x$$

$$\tilde{y} := \frac{g_2}{\sigma_2} - \frac{g}{\sigma} - \left(\frac{\tau_2}{\sigma_2} - \frac{\tau}{\sigma}\right) c$$

and compute the matrix

$$H_2 := H + \frac{dd^T}{\tilde{y}^T d} - \frac{H\tilde{y}(H\tilde{y})^T}{\tilde{y}^T H \tilde{y}} + \frac{\vartheta}{\tilde{y}^T H \tilde{y}} \left(\frac{\tilde{y}^T H \tilde{y}}{\tilde{y}^T d} d - H\tilde{y}\right)\left(\frac{\tilde{y}^T H \tilde{y}}{\tilde{y}^T d} d - H\tilde{y}\right)^T$$

for a given value of the parameter ϑ.

Step 7: Set $x := x_2$, $F := F_2$, $g := g_2$, $\sigma := \sigma_2$, $\tau := \tau_2$, $H := H_2$ and go to Step 2.

Theorem 2.1 shows that Algorithm 2.1 finds a minimum of the extended conic function $F: R_n \to R$ after n perfect steps in the regular case. Now we are analyzing the singular case when g_i is parallel to the vector c for some index $i < n$. If this is the case then x_i is a minimizer of the extended conic function $F(x)$ with the constraint $l(x) = l_i$. Therefore, it is also a minimizer of the quadratic function $\tilde{F}(x)$ with the same constraint and we can use the following lemma.

Lemma 2.2. Let $\tilde{F}(x)$ be a quadratic function with positive definite Hessian matrix $\tilde{G}$. Let $\tilde{g}_i = \tilde{g}(x_i)$, $1 \leq i \leq 3$, be the gradients of the function $\tilde{F}(x)$ at the points $x_i \in R_n$, $1 \leq i \leq 3$. Then $\tilde{g}_i$, $1 \leq i \leq 3$, are parallel only if x_i, $1 \leq i \leq 3$ lie on a line.

Proof. See [5], proof of Lemma 3.2. □

Lemma 2.2 can be used in the singular case. Let $g_1 = \lambda_1 c$ and $g_2 = \lambda_2 c$ hold at two different points x_1 and x_2 respectively. Then also $\tilde{g}_1 = \tilde{\lambda}_1 c$ and $\tilde{g}_2 = \tilde{\lambda}_2 c$ hold for some coefficients $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ respectively (see (2.1)). Let x_3 be a minimizer of the extended conic function $F: R_n \to R$. Then $g_3 = 0$ and, consequently, $\tilde{g}_3 = \tilde{\lambda}_3 c$ by (2.1). Therefore, using Lemma 2.2, we can write

$$x_3 = x_2 + \alpha(x_2 - x_1) \tag{2.11}$$

for some steplength α. The points x_1 and x_2 such that $g_1 = \lambda_1 c$ and $g_2 = \lambda_2 c$ hold can be obtained in two immediately consecutive cycles of Algorithm 2.1. Therefore we can find a minimizer of the conic function in the second cycle by the special step (2.11).

3. QUASI-NEWTON METHODS WITHOUT PROJECTIONS

The variable metric methods described in the previous section minimize an extended conic function after a finite number of steps in case a perfect line search procedure is used in all steps. In this section, we generalize the quasi-Newton methods without projections, which were introduced in [4], in such a way that they find a minimum of an extended conic function after a finite number of steps using the perfect line search procedure in the last step only.

Theorem 3.1. Let $F\colon R_n \to R$ be an extended conic function. Consider the iterative scheme (2.3) where

$$s_i = -\frac{1}{\sigma_i}\left(H_i - \frac{H_i c c^T H_i}{c^T H_i c}\right) g_i \tag{3.1}$$

for $1 \leqq i < n$. Define

$$v_i = d_i - H_i \tilde{y}_i \tag{3.2}$$

for $1 \leqq i < n$, where d_i and $\tilde{y}_i$ are vectors given by (2.7). Let H_1 be an arbitrary symmetric positive definite matrix of order n and $u_1 = (1/\sigma_1) H_1 g_1$. Let

$$H_{i+1} = H_i + V_i B_i V_i^T \tag{3.3}$$

and

$$u_{i+1} = (\tilde{y}_i^T v_i) u_i - (\tilde{y}_i^T u_i) v_i \tag{3.4}$$

for $1 \leqq i < n$, where V_i is the $n \times 2$ matrix which has the columns u_i and v_i and B_i is a 2×2 symmetric matrix which is chosen in such a way that H_{i+1} is positive definite and

$$H_{i+1} \tilde{y}_i = d_i . \tag{3.5}$$

Then the direction vectors s_i, $1 \leqq i < n$ are nonzero and linearly independent provided g_i is not parallel to the vectors c and $v_i \neq 0$ for $1 \leqq i < n$ (regular case). Moreover, $d_i^T c = 0$, $u_n^T \tilde{y}_i = 0$ and $H_n \tilde{y}_i = d_i$ for $1 \leqq i < n$ in the regular case. If u_i is not parallel to v_i for $1 \leqq i < n$ then $u_n \neq 0$.

Proof. We prove this theorem by induction. Suppose that $u_k \neq 0$, $d_k \neq 0$ and $d_k^T c = 0$ and, moreover, $u_k^T \tilde{y}_i = 0$ and $H_k \tilde{y}_i = d_i$ for $1 \leqq i < k$ where $k < n$. It certainly holds for $k = 1$ provided g_1 is not parallel to the vector c and $v_1 \neq 0$.

(a) We show that d_k is not a linear combination of d_i, $1 \leqq i < k$. Suppose, on the contrary, that

$$d_k = \sum_{i=1}^{k-1} \lambda_i d_i$$

for some λ_i, $1 \leqq i < k - 1$. Then

$$\tilde{y}_k = \tilde{G}d_k = \sum_{i=1}^{k-1} \lambda_i \tilde{G} d_i = \sum_{i=1}^{k-1} \lambda_i \tilde{y}_i$$

and, therefore,

$$v_k = d_k - H_k \tilde{y}_k = \sum_{i=1}^{k-1} \lambda_i (d_i - H_k \tilde{y}_i) = 0$$

since $H_k \tilde{y}_i = d_i$ for $1 \leqq i < k$ by assumption. But it is in contradiction to the assumption $v_k \neq 0$.

(b) The equality $H_{k+1}\tilde{y}_k = d_k$ follows from (3.5). Furthermore

$$H_{k+1}\tilde{y}_i = H_k \tilde{y}_i + V_k B_k V_k^{\mathrm{T}} \tilde{y}_i = H_k \tilde{y}_i = d_i$$

for $1 \leqq i < k$ since $H_k \tilde{y}_i = d_i$, $u_k^{\mathrm{T}} \tilde{y}_i = 0$ and $v_k^{\mathrm{T}} \tilde{y}_i = d_k^{\mathrm{T}} \tilde{y}_i - \tilde{y}_k^{\mathrm{T}} H_k \tilde{y}_i = d_k^{\mathrm{T}} \tilde{y}_i - \tilde{y}_k^{\mathrm{T}} d_i = 0$ for $1 \leqq i < k$ by the assumption.

(c) The equality $u_{k+1}^{\mathrm{T}} \tilde{y}_k = 0$ follows from (3.4). Furthermore

$$u_{k+1}^{\mathrm{T}} \tilde{y}_i = (\tilde{y}_k^{\mathrm{T}} v_k)\, u_k^{\mathrm{T}} \tilde{y}_i - (\tilde{y}_k^{\mathrm{T}} u_k)\, v_k^{\mathrm{T}} \tilde{y}_i = 0$$

for $1 \leqq i < k$ since $H_k \tilde{y}_i = d_i$, $u_k^{\mathrm{T}} \tilde{y}_i = 0$ and $v_k^{\mathrm{T}} \tilde{y}_i = d_k^{\mathrm{T}} \tilde{y}_i - \tilde{y}_k^{\mathrm{T}} H_k \tilde{y}_i = d_k^{\mathrm{T}} \tilde{y}_i - \tilde{y}_k^{\mathrm{T}} d_i = 0$ for $1 \leqq i < k$ by the assumption.

(d) If $k + 1 < n$ then using (2.6) we get

$$\sigma_{k+1} s_{k+1}^{\mathrm{T}} c = \frac{g_{k+1}^{\mathrm{T}} H_{k+1} c}{c^{\mathrm{T}} H_{k+1} c}\, c^{\mathrm{T}} H_{k+1} c - g_{k+1}^{\mathrm{T}} H_{k+1} c = 0$$

Moreover $s_{k+1} \neq 0$ and also $d_{k+1} \neq 0$ if g_{k+1} is not parallel to the vector c and $v_{k+1} \neq 0$. □

Theorem 3.1 shows that the vectors s_i, $1 \leqq i < n$, generated by the formula (3.1), are nonzero and linearly independent in the regular case. Moreover,

$$H_n - \frac{H_n c c^{\mathrm{T}} H_n}{c^{\mathrm{T}} H_n c} = \tilde{G}^{-1} - \frac{\tilde{G}^{-1} c c^{\mathrm{T}} \tilde{G}^{-1}}{c^{\mathrm{T}} \tilde{G}^{-1} c}\,. \tag{3.6}$$

This equality can be easily verified by multiplying it by the linearly independent vectors $\tilde{G}s_i$, $1 \leqq i \leqq n - 1$ and c. Using (3.6), we can find a minimizer of both the quadratic function $\tilde{F}(x)$ and the extended conic function $F(x)$ subject to the linear constraint $l(x) = l_n$. It is given by the formula

$$x_{n+1} = x_n + s_n \tag{3.7}$$

where

$$s_n = -\left(H_n - \frac{H_n c c^{\mathrm{T}} H_n}{c^{\mathrm{T}} H_n c}\right) \tilde{g}_n = -\frac{1}{\sigma_n}\left(H_n - \frac{H_n c c^{\mathrm{T}} H_n}{c^{\mathrm{T}} H_n c}\right) g_n \tag{3.8}$$

by (3.6) and (2.1).

Since x_{n+1} is a minimizer of the quadratic function $\tilde{F}(x)$ subject to the linear constraint $l(x) = l_n$, we can write

$$d_i^{\mathrm{T}} \tilde{g}_{n+1} = 0 \tag{3.9}$$

for $1 \leqq i < n$. Suppose now that $u_n \neq 0$ and set

(3.10) $$x_{n+2} = x_{n+1} + \alpha_{n+1} s_{n+1}$$

where

(3.11) $$s_{n+1} = u_n$$

and where the steplength α_{n+1} is chosen by the perfect line search procedure so that

(3.12) $$u_n^T g_{n+2} = 0 .$$

Then

(3.13) $$d_i^T g_{n+2} = \sigma_{n+2} d_i^T \tilde{g}_{n+2} = \sigma_{n+2}(d_i^T \tilde{g}_{n+1} + d_i^T \tilde{y}_{n+1}) = \\ = \sigma_{n+2}(d_i^T \tilde{g}_{n+1} + \alpha_{n+1} \tilde{y}_i^T u_n) = 0$$

for $1 \leqq i < n$ since $d_i^T \tilde{g}_{n+1} = 0$ by (3.9) and $\tilde{y}_i^T u_n = 0$ by Theorem 3.1. Using both (3.12) and (3.13) we get $g_{n+2} = 0$ since the vectors d_i, $1 \leqq i \leqq n - 1$, and u_n are linearly independent. Therefore, x_{n+1} is a minimizer of the extended conic function $F(x)$.

Combining (3.3), (3.4) and (3.5) we get a one parameter class of quasi-Newton methods without projections. In this case

(3.14) $$H_{i+1} = H_i + \frac{1}{\tilde{y}_i^T v_i} (v_i v_i^T - \varphi_i u_{i+1} u_{i+1}^T)$$

for $1 \leqq i < n$, where φ_i is a free parameter. Setting $\varphi_i = 0$, we get the rank-one formula. More details about the choice of the parameter φ are given in [4]. Note that (3.14) is defined only in the case when $\tilde{y}_i^T v_i \neq 0$. This is a stronger requirement than $v_i \neq 0$ which has been used in Theorem 4.1.

The following algorithm summarizes above results.

Algorithm 3.1.

Step 1: Determine an initial point x and compute the value $F := F(x)$ and the gradient $g := g(x)$. Compute the values $\sigma := \sigma(x)$ and $\tau := \tau(x)$ defined by (1.2) Determine an initial symmetric positive definite matrix H of order n (usually set $H := I$, where I is the unit matrix of order n) and set $u := H(g/\sigma)$. Determine the vector c by (2.2) where x, x_1 and x_2 are three different points lying on a line. Set $k := 0$.

Step 2: If the termination criteria are satisfied (for example if $\|g\|$ is sufficiently small) then stop.

Step 3: Set $k := k + 1$. If $k \leqq n$ then set

$$s := - H(g/\sigma) + \frac{c^T H(g/\sigma)}{c^T Hc} Hc$$

and go to Step 4 else set $k := 0$, $s := -\operatorname{sgn}(g^T u)\, u$, $u := H(g/\sigma)$ and go to Step 6.

Step 4: Use an imperfect line search procedure to determine the point $x_2 := x + \alpha_2 s$ such that $F(x_2) < F(x)$. Compute the value $F_2 := F(x_2)$ and the gradient $g_2 := g(x_2)$. Compute the values $\sigma_2 := \sigma(x_2)$ and $\tau_2 := \tau(x_2)$ defined by (1.2).

Step 5: If $k < n$ then set

$$v := x_2 - x - H\tilde{y}$$

$$\tilde{y} := \frac{g_2}{\sigma_2} - \frac{g}{\sigma} - \left(\frac{\tau_2}{\sigma_2} - \frac{\tau}{\sigma}\right) c$$

and compute

$$u_2 := (\tilde{y}^T v)\, u - (\tilde{y}^T u)\, v$$

$$H_2 := H + \frac{1}{\tilde{y}^T v}\,(v v^T - \varphi u_2 u_2^T)$$

for a given value of the parameter φ. Go to Step 7.

Step 6: Use a perfect line search procedure to determine two points $x_1 := x + \alpha_1 s$ and $x_2 := x + \alpha_2 s$ such that $s^T g(x_2) = 0$. Compute the values $F_1 :=$ $:= F(x_1), F_2 := F(x_2)$ and the gradients $g_1 := g(x_1)$, $g_2 := g(x_2)$. Compute the values $\sigma_1 := \sigma(x_1), \sigma_2 := \sigma(x_2)$ and $\tau_1 := \tau(x_1), \tau_2 := \tau(x_2)$ defined by (1.2). Determine the vector c by (2.2).

Step 7: Set $x := x_2$, $F := F_2$, $g := g_2$, $\sigma := \sigma_2$, $\tau := \tau_2$, $u := u_2$, $H := H_2$ and go to Step 2.

Theorem 3.1 shows that Algorithm 3.1 finds a minimum of the extended conic function $F\colon R_n \to R$ after n imperfect steps and one perfect step in the regular case. Note that the condition for positive definiteness of the matrix (3.14) is not satisfied in general (see [4]). Therefore the statement $s := -\operatorname{sgn}(g^T s)\, s$ could be added to Step 4 of the algorithm.

4. A SPECIAL CLASS OF EXTENDED CONIC FUNCTIONS

The most complicated problem associated with the extended conic functions of the form (1.1) is the determination of the values $\sigma_2 = \sigma(x + \alpha_2 s)$ and $\tau_2 = \tau(x + \alpha_2 s)$ from the values $\sigma = \sigma(x)$ and $\tau = \tau(x)$ respectively. In [5], it has been shown that considering the special class of extended conic functions, namely

$$F(x) = \tilde{F}(x)\, l^p(x)\,, \tag{4.1}$$

we can set $\sigma_2/\sigma = (l_2/l)^p$ and $\tau_2/\tau = (F_2/F)/(l_2/l)$ where the ratio l_2/l is determined by solving the equation

$$pF\left(\frac{l_2}{l}\right)^{p+2} - ((2+p)\,F + \alpha_2 g^T s)\left(\frac{l_2}{l}\right)^{p+1} + ((2+p)\,F_2 - \alpha_2 g_2^T s)\left(\frac{l_2}{l}\right) - pF_2 = 0 \tag{4.2}$$

(we set $l = 1$ initially, which gives the initial values $\sigma = 1$ and $\tau = kF$).

Now we are considering the objective function of the form

$$F(x) = \tilde{F}(x) \exp(l(x)) \tag{4.3}$$

defined in all R_n. Using (4.3) we get the following formulae

$$\left\{ \begin{aligned} F &= \tilde{F} \exp(l) \\ g &= \tilde{g} \exp(l) + Fc \end{aligned} \right. \tag{4.4}$$

so that $\sigma_2/\sigma = \exp(l_2 - l)$ and $\tau_2/\tau = F_2/F$. Note that we can set $l = 0$ initially, which gives the initial values $\sigma = 1$ and $\tau = F$. The following lemma gives the possibility of determining the difference $l_2 - l$ from the values F and F_2 and the gradients g and g_2 computed at two different points x and x_2.

Lemma 4.1. Let $x \in R_n$ and $x_2 = x + \alpha_2 s \in R_n$ be two different points. Then the difference $l_2 - l$ is a solution of the equation

(4.5)

$$F(l_2 - l) \exp(l_2 - l) - (g^{\mathrm{T}}d + 2F) \exp(l_2 - l) + F_2(l_2 - l) - (g_2^{\mathrm{T}}d - 2F_2) = 0$$

where $d = x_2 - x = \alpha_2 s$.

Proof. Using (4.4) we get

$$\tilde{F} = \frac{1}{\exp(l)} F$$

$$\tilde{g} = \frac{1}{\exp(l)} (g - Fc)$$

Since the quadratic function has to satisfy the equality

$$2(\tilde{F}_2 - \tilde{F}) = \tilde{g}_2^{\mathrm{T}}d + \tilde{g}^{\mathrm{T}}d$$

and since $c^{\mathrm{T}}d = l_2 - l$, we get after substitution

$$\frac{2F_2}{\exp(l_2)} - \frac{2F}{\exp(l)} = \frac{g_2^{\mathrm{T}}d}{\exp(l_2)} + \frac{g^{\mathrm{T}}d}{\exp(l)} - \frac{F_2}{\exp(l_2)} (l_2 - l) - \frac{F}{\exp(l)} (l_2 - l)$$

which gives (4.5) after rearrangements. □

Note that the equation (4.5) has a real solution $l_2 - l$ if $FF_2 > 0$, which is usually satisfied for x_2 sufficiently close to x.

So far we have assumed that the extended conic function has a unique critical point which is its minimizer. Now, we are considering the case when the extended conic function has several critical points. Using Lemma 2.2, we can see that all critical points lie on a line. It is exactly the line which is determined both in Algorithm 2.1 (Step 4 for $k = n$) and in Algorithm 3.1 (Step 3 for $k = n + 1$). If we use the global line search procedure in this case, we can find a global minimizer of the extended conic function.

The following lemma shows some properties of critical points of the special extended conic function (4.3).

Lemma 4.2. Let $x \in R_n$ be a critical point of the function (4.3) and let G be the Hessian matrix of this function at the point $x \in R_n$. Then

$$G = \tilde{G} \exp(l) - Fcc^T. \tag{4.6}$$

Let $x_1 \in R_n$ and $x_2 \in R_n$ be two different critical points of the function (4.3). Then

$$\frac{F_2}{F_1} = \frac{2 - l_2 + l_1}{2 + l_2 - l_1} \exp(l_2 - l_1). \tag{4.7}$$

Proof. Let $x \in R_n$ be a critical point of the function (4.3). Using (4.4) we get

$$g = \tilde{g} \exp(l) + Fc = 0$$

and

$$G = \tilde{G} \exp(l) + (\tilde{g}c^T + c\tilde{g}^T) \exp(l) + Fcc^T.$$

Therefore

$$\tilde{g} = -\frac{F}{\exp(l)} c$$

so that

$$G = \tilde{G} \exp(l) + \left(-2 \frac{F}{\exp(l)} \exp(l) + F\right) cc^T = \tilde{G} \exp(l) - Fcc^T.$$

Let $x_1 \in R_n$ and $x_2 \in R_n$ be two different critical points of the function (4.3). Denote $d = x_2 - x_1$. Then $c^T d = l_2 - l_1$ and, using (4.4), we get

$$g_1^T d = \tilde{g}_1^T d \exp(l_1) + F_1(l_2 - l_1) = 0$$
$$g_2^T d = \tilde{g}_2^T d \exp(l_2) + F_2(l_2 - l_1) = 0$$

Therefore

$$\tilde{g}_1^T d = -\tilde{F}_1(l_2 - l_1),$$
$$\tilde{g}_2^T d = -\tilde{F}_2(l_2 - l_1).$$

Since the quadratic function has to satisfy the equality

$$2(\tilde{F}_2 - \tilde{F}_1) = \tilde{g}_2^T d + \tilde{g}_1^T d$$

we get

$$2\tilde{F}_2 - 2\tilde{F}_1 = -\tilde{F}_2(l_2 - l_1) - \tilde{F}_1(l_2 - l_1)$$

which implies

$$\frac{F_2}{F_1} = \frac{\tilde{F}_2}{\tilde{F}_1} \exp(l_2 - l_1) = \frac{2 - l_2 + l_1}{2 + l_2 - l_1} \exp(l_2 - l_1)$$

and the lemma is proved. □

The same considerations as above can be applied to the function of the form (4.1). In this case we obtain

$$G = \tilde{G} l^p - \frac{p(p+1) F}{l^2} cc^T \tag{4.8}$$

and

$$\frac{F_2}{F_1} = \left(\frac{l_2}{l_1}\right)^p \frac{2 + p - p(l_2/l_1)}{2 + p - p(l_1/l_2)} \tag{4.9}$$

instead of (4.6) and (4.7). Note that the expressions (4.6) and (4.8) indicate that functions (4.1) and (4.3) are useful especially when the minimal function value is less then zero. Note also that (4.9) implies $F_2/F_1 = 1$ in case $p = -2$. This is the result given in [3].

(Received January 31, 1984.)

REFERENCES

[1] P. Bjørstad and J. Nocedal: Analysis of a new algorithm for one-dimensional minimization. Computing *22* (1979), 2, 93—100.

[2] C. G. Broyden: Quasi-Newton methods and their application to function minimization. Math. Comp. *21* (1967), 99, 368—381.

[3] W. C. Davidon: Conic approximations and collinear scalings for optimizers. SIAM J. Numer. Anal. *17* (1980), 2, 268—281.

[4] L. Lukšan: Quasi-Newton methods without projections for unconstrained minimization. Kybernetika *18* (1982), 4, 290—306.

[5] L. Lukšan: Conjugate gradient algorithms for conic functions. Computing (submitted for publication).

[6] D. C. Sorensen: The Q-superlinear convergence of a collinear scaling algorithm for unconstrained optimization. SIAM J. Numer. Anal. *17* (1980), 1, 84—114.

Ing. Ladislav Lukšan, CSc., Středisko výpočetní techniky ČSAV (General Computing Centre — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 07 Praha 8. Czechoslovakia.