ON GEOMETRIC-OPTICAL PROJECTION
OF SPATIAL PARTICLE SIZE DISTRIBUTION

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A stereological model devoted to the relationship between the spatial particle size distribution and the distribution of circles gained by geometric-optical projection (not orthogonal projection) of these particles when recorded on a photography is presented.

1. INTRODUCTION

In drying process, chemical defence, meteorological office etc. we have been meeting with problem of the analysis of aerosol cloud formed by microscopic particles of spherical shape. The basic question put on the imaging of such a particle flow is to evaluate it taking into account the spacing of these particles. One way is to replace the optical memory by a holographical one which provides the transformed data on the sizes of spaced particles and in this way the possibility to obtain an actual image using a reconstruction of a hologram.

The reconstruction of a three-dimensional image from a two-dimensional photography requires to respect the fact that a two-dimensional picture records particles located in different distances from the photographic plate. In [1] the principles of manual, semi-automatic and fully-automatic reconstruction of a hologram are presented. The TV system used for manual reconstruction is in semi-automatic and fully-automatic reconstruction replaced by an image analyzer. In these three methods the basic steps of the reconstruction are the following:

a) calibrate the system,
b) find the particle and focus it,
c) measure its diameter using a graticule,
d) read off the position of the hologram and thus determine the coordinates of the particle,
e) transfer the data to a coding sheet (for manual or semiautomatic reconstruction)

or to transfer the data into a memory as elements of particle size distribution or coordinates of these particles (for fully automatic reconstruction).

In the hologram reconstruction the steps b) to e) are applied to each particle separately. After measuring the last particle, for manual and semi-automatic reconstruction, the data on the coding sheet are processed and an empirical distribution of actual sizes of particles can be gained; for fully-automatic reconstruction the stored particle diameters are arranged into an empirical distribution of actual sizes or in this form displayed. In all three cases under consideration the statistical characteristics of the size distribution can be calculated and from the corresponding data on coordinates the mean space density of particles per unit volume can be estimated.

The described aerosol sizing apparatuses are equipped by a double-pulsed ruby laser and an image analyzer; their application for routine work in industry is limited. Therefore, another way for aerosol particle sizing has been searched; a new method based on the geometric-optical theory of optical projection is suggested in [2]. It uses the information from a photography recording aerosol particles lying in a limited space the depth of which is given by the definition of the used optics. It respects the fact that most of particles registrated in the photographic plate have not their actual sizes, but sizes following the geometric-optical theory of projecting (not orthogonal projection) spatial objects onto a plane, in our case represented by a photographic plate. A probabilistic model enabling to relate the distribution of particle sizes observed on the photography and the actual size distribution of spaced particles is the subject of this paper. For processing the image information from such a photography an image analyzer can be used. It is clear that the precision of the presented method cannot be as high as that one based on the hologram reconstruction, but it seems to be fully sufficient for routine work. In fact, this method brings a stereological correction if a method of direct evaluation of a picture of aerosol particles using an image analyzer is applied.

2. PROBLEM FORMULATION

The situation presented in Fig. 1a follows from the geometric-optical theory for a thin lens. The relationship between the position of an object y and its corresponding picture y' has the form

$$(2.1) \qquad \frac{1}{a} + \frac{1}{a'} = \frac{1}{f'},$$

where f' is the focal distance; the lateral magnification m is given by the equation

$$(2.2) \qquad m = \frac{y'}{y} = -\frac{a'}{a}$$

(in our arrangement the object coordinates are negative). Without a loss of generality we can consider only the case $a' = a$ i.e. $f' = 0 \cdot 5a'$. Then the object and its picture

are of the same size. From the point of view of the geometry of holographic projection the assumption $a' = a$ means that the wavelength λ' of the reconstruction wave is equal to the wavelength λ of the coherent wave illuminating the object and forming a coherent background. The actual picture of the object lies likewise in the

Fig. 1a.
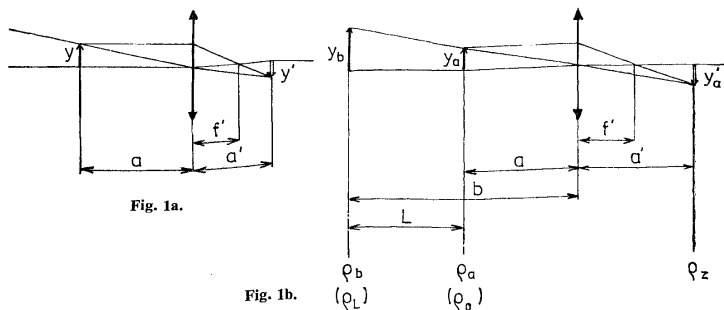
Fig. 1b.

distance a behind the plate (see the plane ϱ_z in Fig. 1b), on which the interference patterns can be observed.

Let us assume that the particles in the distance a i.e. the particles lying in the plane ϱ_a, are focused. For fixed focal distance f', all particles lying behind ϱ_a are in the plane ϱ_z of the plate linearly reduced. The reduction k $(0 < k < 1)$ is given by the ratio

$$(2.3) \qquad\qquad k = \frac{y_a}{y_b} = \frac{a}{b},$$

where y_b is the actual size of the particle in the plane ϱ_b (see Fig. 1b). Of course, this particle has in the plane ϱ_z, where it is recorded, the size $y_a' < y_b$.

The method of hologram reconstruction outlined in the Introduction is applied in such a way that the optical system is successively focused in those planes in which particular particles occur. In our model we assume that the relationship between the actual particle size in the plane ϱ_b and the reduced size of this particle in the plane ϱ_z is uniquely determined by the distance of the plane ϱ_b from the plane ϱ_a; this relationship which is a linear function of this distance is valid in the whole depth of focus of the used image-forming optical apparatus. The depth of focus L of the investigated space, bounded by the planes ϱ_a and ϱ_b, is assumed to be finite.

In construction of a corresponding stereological model we shall start from the three-dimensional Euclidean space $\mathbb{E}_3$ with axes u_1, u_2 and u_3. The random spatial structure $S_\varkappa$ will be represented by opaque spherical particles in a transparent space $\mathbb{E}_3$, the centres S_j $(j = 1, 2, \ldots)$ of the particles being located in the points of a homogeneous Poisson process with a constant density $\varkappa$ and the particle diameters Y_j being

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random variables mutually independent and identically distributed with a continuous probability density function (p.d.f.) $g(y)$, $0 < y < \infty$, having p finite moments α_i $(i = 1, 2, ..., p)$. The density $\varkappa$ is assumed to be so small that for a given value of the depth of focus L the overlapping of particles can be ignored.

Let us introduce two planes: ϱ_0 and ϱ_L defined by $u_1 = 0$ and $u_1 = L$, respectively. The planes ϱ_0 and ϱ_L form the boundaries of a subspace $\mathscr{L}$ and in fact they represent the original planes ϱ_a and ϱ_b after their linear transformation.

Now let $g_*(y_*)$ be the p.d.f. of diameters Y_* of circles, the size of which we measure on the photography i.e. on the plane ϱ_z and $_*\alpha_i$ $(i = 1, 2, ..., p)$ the corresponding moment of the i-th order. These diameters are not, in general, identical with diameters of particles lying in $\mathscr{L}$. Their sizes are reduced in agreement with the geometric-optical theory i.e. in dependence on the distance x of the particle centre from ϱ_0. It is clear that the distance x of the centre of a particular particle is a random variable; we shall denote it by X $(0 \leqq X \leqq L)$. Due to our assumption $(a = a')$ the plane ϱ_z of the photographic plate is symmetrical to ϱ_0 and therefore we can hereinafter consider only ϱ_0.

The problem is to investigate the relationship between $g_*(y_*)$ and $g(y)$ and between the corresponding moments $_*\alpha_i$ and α_i $(i = 1, 2, ..., p)$ and to analyze them.

3. DERIVATION OF BASIC RELATIONSHIPS

Consider now a fixed plane ϱ_x defined by $u_1 = x$ $(0 < x < L)$. A particular particle having its centre in ϱ_x and its diameter y is recorded on the photography as a circle with diameter y_*. From the point of view of the above introduced assumptions it is clear that the random variables X and Y are mutually independent and are related to Y_* in this way

$$Y_* = Y\left[1 + \left(\frac{1}{k} - 1\right)\frac{X}{L}\right]^{-1} \quad \text{for} \quad 0 \leqq X \leqq L.$$

For $k \in \langle 0.8; 1 \rangle$, usual in practice, we shall use the following approximation

$$(3.1) \qquad Y_* \approx Y\left[1 - (1 - k)\frac{X}{L}\right] \quad \text{for} \quad 0 \leqq X \leqq L,$$

very tightly fitting to the exact formula. Then, due to the independence of X and Y, we can write for the mean values

$$(3.2) \qquad \mathsf{E}(Y_*^i) \approx \mathsf{E}(Y^i)\,\mathsf{E}\left\{\left[1 - (1 - k)\frac{X}{L}\right]^i\right\} \quad \text{for} \quad i = 1, 2, ..., p.$$

From the introduced assumption on the centre particle spacing in $\mathbb{E}_3$ it follows that the random variable X has a uniform distribution over the interval $(0, L)$. Therefore

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we have

$$(3.3) \qquad \mathsf{E}\left\{\left[1 - (1 - k)\frac{X}{L}\right]^i\right\} = \frac{1}{1 + i}\,\frac{1 - k^{i+1}}{1 - k}\,,$$

so that we can deduce the relationship between the moments α_i and $_*\alpha_i$

$$(3.4) \qquad \alpha_i \approx C(k, i)\,_*\alpha_i$$

where

$$(3.5) \qquad C(k, i) = (1 + i)\Big(\sum_{j=0}^{i} k^j\Big)^{-1} \quad \text{for} \quad i = 1, 2, \dots, p\,.$$

In the next step we shall derive the p.d.f. $g_*(y_*)$ relevant to the random variable Y_*. For simplicity we shall put in (3.1)

$$(3.6) \qquad \left[1 - (1 - k)\frac{X}{L}\right] = X_*\,.$$

Then (3.1) becomes the form

$$(3.7) \qquad Y_* \approx YX_*\,,$$

the random variables Y and X_* being mutually independent, too. The definition region $\langle 0, L\rangle$ for the original random variable X changes to $\langle k, 1\rangle$ for the random variable X_*, the definition region for the random variable Y remains $(0, \infty)$. Therefore the distribution function (d.f.) $G_*(y_*)$ of random variable Y_* can be expressed as

$$(3.8) \quad G_*(y_*) = \mathsf{P}(Y_* < y_*) \approx \mathsf{P}(YX_* < y_*) = \frac{1}{1 - k}\int_k^1\left[\int_0^{y*/x*} g(y)\,\mathrm{d}y\right]\mathrm{d}x_*\,.$$
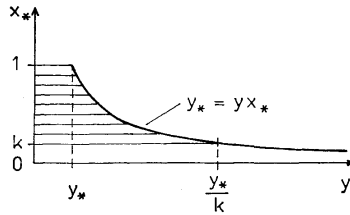

Fig. 2.

The domain of integration for which $yx_* < y_*$ holds, is presented in Fig. 2. Differentiating (3.8) with respect to y_* we find that

$$(3.9) \qquad G'_*(y_*) = g_*(y_*) \approx \frac{1}{1 - k}\int_{y*}^{y*/k} \frac{1}{y}\, g(y)\,\mathrm{d}y\,.$$

Now we shall add some notes to (3.5) and (3.9). Hereinafter we shall consider in $(3.9) =$ instead of $\approx$, assuming $k \in \langle 0.8; 1\rangle$.

From (3.9) it can be easily derived that the modus of p.d.f. $g_*(y_*)$ lies in the point $y_* = y_{*(0)}$ where $y_{*(0)}$ satisfies the equation

$$(3.10) \qquad g\left(\frac{y_*}{k}\right) = g(y_*).$$

This result will be used in the next Chapter.

Comprehensively in the limit situation (k tends to 1 in the whole subspace $\mathscr{L}$) the presented model must converge to a well known model constructed for the orthogonal projection (all particles with centres in $\mathscr{L}$ are orthogonally projected onto ϱ_0 and the diameters of particles are identical with the diameters of corresponding circles in the photography). Really, in this case we have from (3.5)

$$(3.11) \qquad \lim_{k \to 1} C(k, i) = 1$$

and after solving the relevant indetermined form in (3.9)

$$(3.12) \qquad \lim_{k \to 1} g_*(y_*) = g(y).$$

The moments $_*\alpha_i$ $(i = 1, 2, \ldots, p)$ can be estimated using data obtained after processing the photography by means of an image analyzer.

The analytical form of $g(y)$ is not usually known. The type of this size distribution follows from the physical character of the process of particle forming. On the other hand the distribution parameters characterizing the physical conditions of the investigated process (e.g. in drying process we consider film thickness, physical dimension and shape of the spraying nozzles, fluid pressure etc.) are usually unknown and the question is how to estimate them. Mostly we can express these parameters as functions of moments α_i, $i = 1, 2, \ldots, p$. From this point of view it is important that the correction coefficients $C(k, i)$ given in (3.4) are distribution-free.

In practice the problem of testing the hypothesis on $G(y)$ can be realized only as testing the hypothesis on a corresponding $G_*(y_*)$. But this procedure assumes the possibility to solve the integral involved in (3.8) or in (3.9), if we determine the relevant p.d.f. In the next chapter we shall show the solution of (3.9) for $g(y)$ following the Rayleigh and normal distribution. Both cases occurring in practice, the reduction ratio k is either known or unknown, will be considered.

4. DETERMINATION OF $g_*(y_*)$ FOR GIVEN $g(y)$

4.1. Rayleigh distribution

For the Rayleigh distribution

$$(4.1) \qquad g(y) = \frac{y}{\mu}\, e^{-\frac{y^2}{2\mu}}, \quad y > 0,$$

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(3.9) becomes the form

$$(4.2) \quad g_*(y_*) = \frac{1}{(1-k)\mu} \int_{y_*}^{y_*/k} e^{-\frac{y^2}{2\mu}} \, dy = \frac{\sqrt{(2\pi)}}{(1-k)\sqrt{(\mu)}} \left[\Phi\left(\frac{y_*}{k\sqrt{(\mu)}}\right) - \Phi\left(\frac{y_*}{\sqrt{(\mu)}}\right) \right],$$

where

$$(4.3) \qquad\qquad \Phi(t) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{t} e^{-\frac{1}{2}v^2} \, dv$$

is the distribution function of $N(0, 1)$ distribution, tabulated e.g. in [3], [4].

The modus $y_{(0)}$ of $g(y)$ has its coordinate located in $\sqrt{\mu}$. The modus $y_{*(0)}$ of $g_*(y_*)$ can be gained as a solution of (3.10). For $g(y)$ following (4.1) we have after some algebra

$$y_{*(0)} = \sqrt{(\mu)} \, U_R(k),$$

where

$$U_R(k) = k \sqrt{\frac{2 \lg (1/k)}{1 - k^2}}, \quad 0 < k < 1.$$

But for $U_R(k)$ we have $0 < U_R(k) < 1$ and in addition to it $U_R(k)$ is an increasing function of k. Therefore it holds

$$y_{*(0)} < y_{(0)}$$

and for decreasing k the modus $y_{*(0)}$ shifts from $y_{(0)}$ to the left.

Further we shall take into account the knowledge of μ and k. First we shall consider the combination: μ unknown, k known. For the moments of Rayleigh distribution it holds

$$(4.4) \qquad\qquad \alpha_i = (2\mu)^{i/2} \, \Gamma\left(\frac{i+2}{2}\right), \quad i = 1, 2, ..., p,$$

$\Gamma(n)$ being the gamma function of argument n. Putting $i = 1$ we gain from (4.4)

$$(4.5) \qquad\qquad\qquad \mu = \frac{2\alpha_1^2}{\pi}$$

and from (3.4)

$$(4.6) \qquad\qquad\qquad \alpha_1 = \frac{2}{1+k} \, {}_*\alpha_1,$$

so that

$$(4.7) \qquad\qquad\qquad \mu = \frac{8}{\pi(1+k)^2} \, {}_*\alpha_1^2,$$

where ${}_*\alpha_1$ can be estimated from measurement results after processing the photography.

The second couple will have the form: both μ and k are unknown. The loss of information on k will be replaced by increasing number of hitherto used moments

$_*\alpha_i$. Putting $i = 2$ in (4.4) we get making use of (4.5)

$$(4.8) \qquad\qquad \alpha_2 = \frac{4}{\pi}\,\alpha_1^2\,.$$

But from (4.6) it follows

$$(4.9) \qquad\qquad k = \frac{2\,_*\alpha_1}{\alpha_1} - 1\,.$$

Inserting these two last equations into

$$(4.10) \qquad\qquad \alpha_2 = \frac{3}{1 + k + k^2}\,_*\alpha_2\,,$$

following from (3.4) for $i = 2$, we obtain after some algebra a quadratic equation in α_1. Its solution has the form

$$(4.11) \qquad\qquad _{1,2}\alpha_1 = {}_*\alpha_1 \pm \tfrac{1}{2}\sqrt{\left[3\left(\pi\,_*\alpha_2 - 4\,_*\alpha_1^2\right)\right]}\,.$$

We now proceed to concentrate on the discriminant D. Obviously it can be expressed in the following form

$$D = 3\left(\pi\,_*\alpha_2 - 4\,_*\alpha_1^2\right) = 4\,_*\alpha_1^2\left[4\,\frac{1 + k + k^2}{(1 + k)^2} - 3\right],$$

so that for $0 < k < 1$ it is $D > 0$. On the other hand it follows from (4.6) that $\alpha_1 > {} > {}_*\alpha_1$ and so the only acceptable root of (4.11) is the positively signed root.

Applying this root in (4.5) and (4.9) we see that both α_1 and k are functions of moments $_*\alpha_i\ (i = 1, 2)$ only.

4.2. Normal distribution

If Y follows the Gaussian distribution with p.d.f.

$$(4.12) \qquad\qquad g(y) = \frac{1}{\sqrt{(2\pi)}\,\sigma}\,e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,, \quad 0 < y < \infty\,,$$

(3.9) attains the form

(4.13)
$$g_*(y_*) = \frac{1}{(1 - k)\sqrt{(2\pi)}\,\sigma}\int_{y_*}^{y^*/k}\frac{1}{y}\,e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy = \frac{\mu}{(1 - k)\sqrt{(2\pi)}}\int_{U_1}^{U_2}\frac{1}{Vw + 1}\,e^{-\frac{1}{2}w^2}\,dw\,,$$

where

$$U_1 = \frac{y_* - \mu}{\sigma}\quad\text{and}\quad U_2 = \frac{y_* - k\mu}{k\sigma}\,;$$

V is the variation coefficient and k the reduction coefficient. Unfortunately the integral in (4.13) cannot be transformed into some known integral the values of which are

tabulated. For practical need we could use only the series expansion or numerical integration. The variation coefficient V is assumed to be known, since the parameters μ and σ can be estimated using the aforementioned procedure. This shows that the verification of the hypothesis "$g(y)$ follows the normal distribution" is not so easy in practice.

Now by means of (3.10) we can determine the modus $y_{*(0)}$ of $g_*(y_*)$. For normally distributed Y we get after some algebra

$$y_{*(0)} = 2\mu\, U_N(k)\,,$$

where

$$U_N(k) = \frac{k}{1+k}\,, \quad 0 < k < 1\,.$$

But for $U_N(k)$ we have $0 < U_N(k) < 0.5$ and in addition to it $U_N(k)$ is again an increasing function of k. Hence we deduce that our conclusion on the relationship between the modus $y_{*(0)}$ and the modus $y_{(0)}$ are identical with that outlined in Section 4.1.

Further we shall express the parameters μ and σ^2 as functions of the moments $_*\alpha_i$ only.

The first case will be again: k is known. Using the same procedure as in Section 4.1. we obtain

$$(4.14) \qquad\qquad \mu = \alpha_1 = \frac{2}{1+k}\,\alpha_1^*$$

and

$$(4.15) \qquad\qquad \sigma^2 = \frac{3}{1+k+k^2}\,_*\alpha_2 - \frac{4}{(1+k)^2}\,_*\alpha_1^2\,.$$

The solution of the second case (k is unknown) will be based on the same idea as in Section 4.1. Obviously the corresponding system of equations is given by

$$(4.16) \qquad\qquad _*\alpha_1 = \frac{1+k}{2}\,\alpha_1\,,$$

$$(4.17) \qquad\qquad _*\alpha_2 = \frac{1+k+k^2}{3}\,(\alpha_1^2 + \sigma^2)\,,$$

$$(4.18) \qquad\qquad _*\alpha_3 = \tfrac{1}{4}(1 + k + k^2 + k^3)(\alpha_1^3 + 3\alpha_1\sigma^2)\,.$$

In the next step we express α_1 and σ^2 as functions of only one unknown constant k as we carried it out in (4.14) and (4.15). The application of these two equations in (4.18) leads after some complex algebra to the following equation of the fifth order in k

$$(4.19) \qquad\qquad Z_5 k^5 + Z_4 k^4 + Z_3 k^3 + Z_2 k^2 + Z_1 k + Z_0 = 0\,,$$

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where

$$(4.20) \quad \begin{cases} Z_5 = Z_0 = 8\ _*\alpha_1^3 + 9\ _*\alpha_1\ _*\alpha_2 - 2\ _*\alpha_3\,, \\ Z_4 = Z_1 = 16\ _*\alpha_1^3 + 27\ _*\alpha_1\ _*\alpha_2 - 8\ _*\alpha_3\,, \\ Z_3 = Z_2 = 24\ _*\alpha_1^3 + 36\ _*\alpha_1\ _*\alpha_2 - 14\ _*\alpha_3\,. \end{cases}$$

We see that (4.19) is a positive reciprocal equation which can be rewritten in form

$$(4.21) \qquad A(k^5 + 1) + Bk(k^3 + 1) + Ck^2(k + 1) = 0$$

where

$$(4.22) \qquad A = Z_5 = Z_0\,; \quad B = Z_4 = Z_1\,; \quad C = Z_3 = Z_2\,.$$

It is known that any positive reciprocal equation of odd order has a root $k_1 = -1$. The knowledge of this root permits to gain a positive reciprocal equation of the fourth order that may be transformed into the form

$$(4.23) \qquad A\left(k^2 + \frac{1}{k^2}\right) + (B - A)\left(k + \frac{1}{k}\right) + (A - B + C) = 0\,.$$

Making here the substitution

$$(4.24) \qquad\qquad\qquad k + \frac{1}{k} = u\,,$$

we arrive at a quadratic equation for unknown u

$$(4.25) \qquad\qquad Au^2 + (B - A)u + (C - B - A) = 0\,,$$

Its solution gives two quadratic equations for unknown k

$$(4.26) \qquad k + \frac{1}{k} = \frac{1}{2A}\left[A - B + \sqrt{(B^2 + 5A^2 + 2AB - 4AC)}\right],$$

$$(4.27) \qquad k + \frac{1}{k} = \frac{1}{2A}\left[A - B - \sqrt{(B^2 + 5A^2 + 2AB - 4AC)}\right]$$

and from here we find the other roots k_2 to k_5. Of course, the coefficients A, B and C are functions of the known moments $_*\alpha_i$ $(i = 1, 2, 3)$ only.

Investigate now the discriminant D_1. It can be rewritten into the form

$$D_1 = B^2 + 5A^2 + 2AB - 4AC = 4\big\{_*\alpha_3\big(_*\alpha_3 + 8\ _*\alpha_1^3 - 18\ _*\alpha_1\ _*\alpha_2\big) +$$
$$+ \; 16\ _*\alpha_1^6 + 81\ _*\alpha_1^2\ _*\alpha_2^2 + 72\ _*\alpha_1^4\ _*\alpha_2\big\}\,.$$

Applying the Schwartz inequality we easily prove that D_1 takes finite and positive value.

Comprehensively from five roots k_1 to k_5 of (4.19) we may consider only those

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fulfilling the condition $0 < k < 1$. We find that only the root

$$(4.28) \qquad k_5 = \frac{1}{4A}\left[A - B - \sqrt{(D_1)} + \sqrt{((A - B - \sqrt{(D_1)})^2 - 16A^2)}\right]$$

possesses the required property.

The knowledge of the value k enables to determine the parameters μ and σ^2 given by (4.14) and (4.15), respectively, and using them the p.d.fs $g(y)$ and $g_*(y_*)$, respectively, given by (4.12) and (4.13).

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REFERENCES

[1] R. Bexon, C. D. Bishop and J. Gibbs: Aerosol sizing by holography using the QUANTIMET Cambridge Instruments. News of Interest on Image Analysis Techniques, Equipment and Applications, 1975, No. 3.
[2] V. Horálek: Statistical Model for the Analysis of Aerosol Size Particle Distribution (in Czech). Res. Report SVÚSS, 1979.
[3] J. Likeš and J. Laga: Basic Statistical Tables (in Czech). SNTL, Praha 1978.
[4] M. Abramowitz and I. A. Stegun: Handbook of Mathematical Functions. Dover Publication, Inc., New York 1970.
[5] Z. Horák, F. Krupka and V. Šindelář: Technical Physics (in Czech). SNTL, Praha 1961.
[6] M. Miler: Holography (in Czech). SNTL, Praha 1974.

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