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DUALITY AND OPTIMALITY CONDITIONS IN ABSTRACT CONCAVE MAXIMIZATION

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This paper is devoted to abstract concave maximization problems. On the basis of separation theorems which are given in Section 1 a unified duality theory associated with Lagrange multipliers is developed in Section 2. In Section 3 the well-known Kuhn-Tucker-Lagrange conditions are generalized and proved.

0. INTRODUCTION

There is no doubt that one of the most important domains of optimization theory is concave maximization, the most delicate problems of which are the duality theory and the necessary and sufficient optimality conditions.

Duality theory for one-objective concave maximization is, how it is generally known, already completely developed. For vector optimization, however, the duality question is mote complicated, since the dual gap in scalar optimization cannot be easily transformed into vector optimization. The literature devoted to duality in vector optimization can be divided into two groups. In the first group duality is developed on the basis of the concept of conjugate maps (see Tanino, Sawaragi [12], Zowe [13], Gros [14]) and the results present a generalization of Fenchel's duality theorem.

The second group is more or less associated with so called Lagrange multipliers. Results of this group are however, ununified and most of them are proved only in the space of finite dimension (see Tran Quoc Chien [4], Podinovskij, Nogin [2], Minami [7], Neuwenhuis [8]).

In Section 2 of this paper using separation theorems a duality theory is developed for vector concave maximization. As to necessary and sufficient optimality conditions in Section 3 the Kuhn-Tucker-Lagrange conditions are generalized for vector optimization.

Before giving main results we present some basic notions and preliminaries.

1. NOTATIONS AND PRELIMINARIES

Let E be a real vector space with its algebraical dual E^*. For an arbitrary subset A of E, $(L_A)_0$ denotes the vector subspace parallel to the affine hull L of A. If V is a vector subspace of E, the *core of A relative to V* is defined by

$$\mathrm{cor}_V(A) = \{x \in A \mid \forall v \in V \exists \eta > 0 : x + \alpha v \in A \text{ if } |\alpha| < \eta\},$$

in particular, $\mathrm{cor}_{(L_A)_0}(A)$ is the usual *intrinsic core of A*, labelled by $\mathrm{icr}(A)$, and $\mathrm{cor}_E(A)$ is the usual *core* of A, labelled by $\mathrm{cor}(A)$. The *linear access of A*, denoted by $\mathrm{lin}(A)$, is by definition the set

$$A \cup \{x \in E \mid \exists y \in A \setminus \{x\} : [y, x) \subset A\}.$$

Finally, the *admissible cone* $K(A, x)$ of A from the point x is the set of points u such that there exists a real $\varepsilon > 0$ with $x + \alpha u \in A$ for all $\alpha \in (0, \varepsilon)$.

Now let E_+ be a convex cone in E such that

$$(-E_+) \cap (E_+) = \{0\}$$
$$\mathrm{lin}(E_+) = E_+$$
$$\mathrm{cor}(E_+) \neq \emptyset.$$

We have then an ordering generated by the cone E_+ as follows for any $a, b \in E$

$$a > b \quad \text{iff} \quad a - b \in \mathrm{cor}(E_+)$$
$$a \geq b \quad \text{iff} \quad a - b \in E_+ \setminus \{0\}$$
$$a \geqq b \quad \text{iff} \quad a - b \in E_+$$
$$a \mathrel{\overline{\gtrless}} b \quad \text{iff} \quad b - a \notin E_+ \setminus \{0\}.$$

A linear functional $f \in E^*$ is called *nonnegative* if

$$\langle f, a\rangle \geqq 0 \quad \forall a \in E_+.$$

The set of all nonnegative linear functionals on E is denoted by E_+^*. $f \in E_+^*$ is called *positive* if

$$\langle f, a\rangle > 0 \quad \forall a \in E_+ \setminus \{0\}.$$

Given a subset $A \subset E$, a point $a \in A$, we say that a is a *weak maximal* of A iff

$$b > a \Rightarrow b \notin A$$

a is a *Pareto maximal* of A iff

$$b \geqq a \Rightarrow b \notin A,$$

a is a *Borwein maximal* of A iff

$$\mathrm{lin}(K(A, a)) \cap E_+ = \{0\},$$

and a is a *proper maximal* of A iff there exists a positive linear functional f on E with

$$\langle f, b\rangle \leqq \langle f, a\rangle \quad \forall b \in A .$$

Analogously are defined a *weak, Pareto, Borwein* and *proper minimal* of A.

Now let $a \in E$. We say that a is a *supremal (infimal)* of A if $(A - a) \cap \operatorname{cor}(E_+) = \emptyset$, $((a - A) \cap \operatorname{cor}(E_+) = \emptyset)$ and for any $b < a$ $(b > a)$, $(A - b) \cap \operatorname{cor}(E_+) \neq \emptyset$, $((b - A) \cap \operatorname{cor}(E_+) \neq \emptyset)$.

Remark 1.1. The following implications are evident: proper maximal minimal $\Rightarrow$ $\Rightarrow$ Borwein maximal minimal $\Rightarrow$ Pareto maximal minimal $\Rightarrow$ weak maximal minimal.

Remark 1.2. Neuwenhuis [8] has developed a duality theory just for the supremal points.

We recall now some separation theorems which play a crucial role in convex optimization.

Theorem 1.1. (Bair [5].) Let E be a real vector space and F a real ordered vector space with $F_+ \neq \{0\}$. For each finite family $\{A_j \mid j \in J\}$ of convex subsets of E such that A_j, $j \in J$, has nonempty intrinsic core there exist a family $\{y_j \mid j \in J\}$ of points in $F_+ \cup (-E_+)$ and a family $\{L_j \mid j \in J\}$ of linear mappings of E to F with the following properties:

$$\bigcap_{j\in J} \operatorname{icr}(A_j) = \emptyset \quad \text{if and only if}$$

$$(1) \qquad A_j \subset \{x \in E \mid L_j(x) \leqq y_j\} \quad \forall j \in J$$

$$(2) \qquad \sum_{j\in J} L_j = 0 \quad \text{and} \quad \sum_{j\in J} y_j \leqq 0$$

$$(3) \qquad \exists j_0 \in J : L_{j_0} \neq 0 \quad \text{and} \quad A_{j_0} \not\subset \{x \in E \mid L_{j_0}(x) = y_{j_0}\}$$

Theorem 1.2. (Bair [5].) Let E be a real vector space and F a real ordered vector space with $F_+ \neq \{0\}$. For each finite family $\{A_j \mid j \in J\}$ of $n + 1$ nonempty convex subsets of E such that n sets A_j, $j \in J \setminus \{j_1\}$ have a nonempty core, there exist a family $\{y_j \mid j \in J\}$ of points in $F_+U(-F_+)$ and a family $\{L_j \mid j \in J\}$ of linear mappings of E to F for which properties (1), (2) and (3) of Theorem 1.1 are satisfied if and only if

$$A_{j_1} \cap \Big(\bigcap_{j\in J\setminus\{j_1\}} \operatorname{cor} A_j\Big) = \emptyset .$$

Theorem 1.3. (Dubovitskij-Milyutin [9].) Let E be a real linear topological space with its topological dual E^*. For each finite family $\{A_j \mid j \in J\}$ of $n + 1$ nonempty convex subsets of E with $\operatorname{int}(A_j) \neq \emptyset$ $\forall j \in J \setminus \{j_1\}$ there exist a family $\{y_j \mid j \in J\}$ of real numbers and a family $\{L_j \mid j \in J\} \subset E^*$ such that

$$A_{j_1} \cap \Big(\bigcap_{j\in J\setminus\{j_1\}} \operatorname{int} A_j\Big) = \emptyset$$

if and only if properties (1), (2) and (3) of Theorem 1.1 are satisfied.

Theorem 1.4. If E is a real ordered locally convex space with nonempty weak interior of E_+^* $\operatorname{int}^w E_+^*$ and A is a nonempty convex subset of E then the Borwein maximal and the proper maximal concepts are equivalent.

Proof. It remains to prove that a Borwein maximal of A, say a_0, is also a proper maximal. Put

$$K = \operatorname{lin}(K(A, a_0)) = \overline{K(A, a_0)}.$$

We know that (see [1]

$$K^{00} = K \tag{1.1}$$

where

$$K^0 = \{f \in E^* \mid \langle f, x \rangle \leqq 1 \ \forall x \in K\}$$

is the polar of K and

$$K^{00} = \{x \in E \mid \langle f, x \rangle \leqq 1 \ \forall f \in K^0\}$$

is the bipolar of K.

If $K^0 \cap \operatorname{int}^w E_+^* = \emptyset$ there exists, by Theorem 1.3, a point $a \in E$, $a \neq 0$ such that

$$\langle f, a \rangle \leqq 0 \leqq \langle g, a \rangle \quad \forall f \in K^0, \quad \forall g \in \operatorname{int}^w E_+^* .$$

The first inequality implies, in view of (1.1), that $a \in K$ and from the second inequality it follows that $a \in E_+$ which contradicts the fact that a_0 is a Borwein maximal. Hence there exists a functional $f \in K^0 \cap \operatorname{int}^w E_+^*$ which implies that a_0 is a proper maximal. □

Theorem 1.5. (Holmes [1].) Let E be a linear space and $\Phi_1, \ldots, \Phi_n, \Psi$ linear functionals on E. Then $\Psi \in \operatorname{span}\{\Phi_1, \ldots, \Phi_n\}$ (the linear hull of $\Phi_1, \ldots, \Phi_n$) if and only if

$$\bigcap_{i=1}^{n} \ker \Phi_i \subset \ker \Psi .$$

Corollary. Let E be a real vector space and μ, η linear functionals on E such that

$$\ker \mu = \{x \in E \mid \langle \mu, x \rangle = 0\} \subset \{x \in E \mid \langle \eta, x \rangle \geqq 0\} . \tag{1.2}$$

Then there exists a real k such that

$$\eta = k \,.\, \mu .$$

Proof. It suffices, in view of Theorem 1.5, to prove that

$$\ker \mu \subset \ker \eta .$$

Indeed, if, on the contrary, there exists an $x \in \ker \mu$, $x \neq 0$ with $\langle \eta, x \rangle > 0$ then $-x \in \ker \mu$ while

$$\langle \eta, -x \rangle = -\langle \eta, x \rangle < 0$$

which contradicts (1.2). □

In what follows X, Y and Z are supposed to be real vector spaces, Y and Z are ordered by positive cones Y_+ and Z_+ with $\operatorname{cor}(Y_+) \neq \emptyset$ and $\operatorname{cor}(Z_+) \neq \emptyset$. $D \subset X$

is a nonempty convex set, $D' \subset X$ is an intrinsicly open set (i.e. $D' = \operatorname{inc}(D')$) with $D \subset D'$. F resp. G are concave operators from D' to Y resp. to Z:

$$F : D' \to Y$$

$$G : D' \to Z$$

The program

$$\text{(P)} \qquad F(x) \to \max$$

$$x \in D \,\&\, G(x) \geqq 0$$

is called a *concave maximization program.*

A point $x^* \in X$ is called a *feasible solution* of program (P) if $x^* \in D$ and $G(x^*) \geqq 0$, a *weak maximal*, a *Pareto maximal* a *Borwein maximal* and a *proper maximal solution* of program (P) if it is a feasible solution of (P) and $F(x^*)$ is, at the same time, a weak maximal, Pareto maximal, Borwein maximal and proper maximal of the set

$$Q = \bigcup_{\substack{x \in D \\ G(x) \geqq 0}} (F(x) - Y_+) . \tag{1.3}$$

2. DUALITY

Let $\mathscr{L}_+(Z, Y)$ be the set of all nonnegative linear operators of Z to Y and $\mathscr{L}$ a subset of $\mathscr{L}_+(Z, Y)$. For any $L \in \mathscr{L}$ we define

$$v_F(L) = \bigcap_{x \in D} \{y \in Y \mid y \not\leqq F(x) + \langle L, G(x) \rangle\} . \tag{2.1}$$

Then the program

$$(\mathrm{D}_{\mathscr{L}}) \qquad \bigcup_{L \in \mathscr{L}} v_F(L) \to \min$$

is called an *$\mathscr{L}$-dual* of program (P), where $L^* \in \mathscr{L}$ is called a *weak minimal, Pareto minimal, Borwein minimal* and *proper minimal* of program $(\mathrm{D}_{\mathscr{L}})$, respectively, if there exists a weak, Pareto, Borwein and proper minimal, respectively, of the set $H = \bigcup_{L \in \mathscr{L}} v_F(L)$ which belongs to $v_F(L^*)$.

From definition it follows immediately

Theorem 2.1. (Weak duality principle.) For any feasible solution x of program (P) and any $L \in \mathscr{L}$ we have

$$F(x) \not\geqq y \quad \forall y \in v_F(L) .$$

Lemma 2.1. Suppose that Slater's condition is satisfied (i.e. $\exists \bar{x} \in D : G(\bar{x}) > 0$). Let $y^* \in Q$ (see (1.3)) and $\eta \in Y_+^*$ $\eta \neq 0$ be such that

$$\langle \eta, y^* \rangle \geqq \langle \eta, y \rangle \quad \forall y \in Q . \tag{2.1}$$

Then there exists a $\xi \in Z_+^*$ with

(2.2) $$\langle \eta, y^* \rangle \geqq \langle \eta, y \rangle + \langle \xi, z \rangle \quad \forall (y, z) \in P$$

where

(2.3) $$P = \{(y, z) \in Y \times Z \mid \exists x \in D : y \leqq F(x) \,\&\, z \leqq G(x)\} .$$

Proof. Put

(2.4) $$M = \{(y, 0) \in Y \times Z \mid \langle \eta, y \rangle = \langle \eta, y^* \rangle\} .$$

It is evident that $M \cap \operatorname{cor}(P) = \emptyset$. Hence, in view of Theorem 1.2, there exists a pair $(\mu, v) \in Y^* \times Z^*$, $(\mu, v) \neq (0, 0)$ such that

$$\langle \mu, y \rangle + \langle v, z \rangle \leqq \langle \mu, u \rangle \quad \forall (y, z) \in P \quad \forall (u, 0) \in M .$$

It is easy to verify, with help of Slater's condition, that μ and v are nonnegative and $\mu \neq 0$. In particular we have

$$\langle \mu, y^* \rangle \leqq \langle \mu, y \rangle \quad \forall (y, 0) \in M$$

or, what is equivalent,

$$\ker \eta \subset \{y \in Y \mid \langle \mu, y \rangle \geqq 0\} .$$

By Corollary of Theorem 1.5 there exists a real k such that $\mu = k \,.\, \eta$. k is different from zero, since $\mu \neq 0$. Putting $\xi = (1/k)\, v$ we obtain the required functional. □

Theorem 2.2. (Strict Duality Principle 1.) Suppose that Slater's condition and the following condition

(C) $$\forall \eta \in Y_+^* , \quad \eta \text{ positive} , \quad \forall \xi \in Z_+^* \quad \exists L \in \mathscr{L} : \xi = \eta \circ L$$

are satisfied. Then if $x^* \in D$ is a proper maximal solution of program (P), there exists a Pareto minimal solution $L^* \in \mathscr{L}$ such that

$$F(x^*) \in v_P(L^*) .$$

Proof. Let $x^* \in D$ be a proper maximal solution of program (P), then by definition, $F(x^*)$ is a proper maximal of the set Q, i.e. there exists a positive linear functional $\eta \in Y_+^*$ such that

(2.5) $$\langle \eta, F(x^*) \rangle \geqq \langle \eta, y \rangle \quad \forall y \in Q .$$

By Lemma 2.1 there exists a functional $\xi \in Z_+^*$ with

(2.6) $$\langle \eta, F(x^*) \rangle \geqq \langle \eta, y \rangle + \langle \xi, z \rangle \quad \forall (y, z) \in P$$

In view of condition (C), there is an operator $L^* \in \mathscr{L}$ with

(2.7) $$\xi = \eta \circ L^*$$

From (2.6) and (2.7) it follows

(2.8) $$\langle \eta, F(x^*) \rangle \geqq \langle \eta, F(x) \rangle + \langle \eta \circ L^*, G(x) \rangle \quad \forall x \in D$$

which implies

$$F(x^*) \not\leq F(x) + \langle L^*, G(x) \rangle \quad \forall x \in D . \tag{2.9}$$

By definition $F(x^*) \in v_P(L^*)$ and then by the weak duality principle $F(x^*)$ is a Pareto minimal of H. Consequently L^* is our desired Pareto minimal solution of program $(D_{\mathscr{L}})$. □

Theorem 2.3. (Strict Duality Principle 2.) Suppose that Slater's condition and the following condition

$$(C') \qquad \forall \eta \in Y_+^* \setminus \{0\} \quad \forall \xi \in Z_+^* \quad \exists L \in \mathscr{L} : \xi = \eta \circ L$$

are fulfilled and operator $F(x)$ is strictly concave on D. Then if $x^* \in D$ is a Pareto maximal solution of program (P), there exists a Pareto minimal solution $L^* \in \mathscr{L}$ of program $(D_{\mathscr{L}})$ such that

$$F(x^*) \in v_F(L^*) .$$

Proof. Since the sets $\operatorname{cor}(P)$ and $M = \{(y, 0) \in Y \times Z \mid y \geqq F(x)\}$ are disjoint, there exist, by Theorem 1.2, $(0, 0) \neq (\eta, \xi) \in Y^* \times Z^*$ such that

$$\langle \eta, y' \rangle \geqq \langle \eta, y \rangle + \langle \xi, z \rangle \quad \forall (y', 0) \in M \quad \forall (y, z) \in P .$$

It is easy to verify that $\eta \in Y_+^* \setminus \{0\}$ and $\xi \in Z_+^*$. In particular we have

$$\langle \eta, y \rangle \geqq \langle \eta, F(x) \rangle + \langle \xi, G(x) \rangle \quad \forall x \in D \quad \forall (y, 0) \in M \tag{2.10}$$

Since $F(x)$ is strictly concave, in (2.10) equality is attained only at one point, and namely at x^*. One obtains then

$$\langle \eta, F(x^*) \rangle > \langle \eta, F(x) \rangle + \langle \xi, G(x) \rangle \quad \forall x \in D \setminus \{x^*\} \tag{2.11}$$

In view of condition (C′) there is an operator $L^* \in \mathscr{L}$ such that

$$\xi = \eta \circ L. \tag{2.12}$$

From (2.11) and (2.12) it follows then

$$\langle \eta, F(x^*) \rangle > \langle \eta, F(x) \rangle + \langle \eta \circ L^*, G(x) \rangle \quad \forall x \in D \setminus \{x^*\}$$

which implies

$$F(x^*) \not\leq F(x) + \langle L^*, G(x) \rangle \quad \forall x \in D . \tag{2.13}$$

Inequality (2.13) shows that $F(x^*) \in v_F(L^*)$ and by the weak duality principle $F(x^*)$ is a Pareto minimal of H. The proof is complete. □

Theorem 2.4. (Converse Duality Principle.) Suppose that Slater's condition and condition (C′) of Theorem 2.3 are satisfied and the set Q is algebraically closed (i.e. $Q = \operatorname{lin}(Q)$). Then if $L^* \in \mathscr{L}$ is a Pareto minimal solution of program $(D_{\mathscr{L}})$ there is a Pareto maximal solution x^* of program (P) with

$$F(x^*) \in v_F(L^*) .$$

Proof. Let $L^* \in \mathscr{L}$ be a Pareto minimal solution of program $(D_{\mathscr{L}})$. By definition there is a Pareto minimal y^* of H with $y^* \in v_F(L^*)$. We assert that

$$\forall y < y^* : y \in Q . \tag{2.14}$$

Suppose, on the contrary, there is an $y_0 < y^*$ with $y_0 \notin Q$. By Theorem 1.2 there exists a functional $\eta \in Y_+^* \setminus \{0\}$ with

$$\langle \eta, y_0 \rangle \geqq \langle \eta, y \rangle \quad \forall y \in Q$$

In view of Lemma 2.1 there is a functional $\xi \in Z_+^*$ such that

$$\langle \eta, y_0 \rangle \geqq \langle \eta, F(x) \rangle + \langle \xi, G(x) \rangle \quad \forall x \in D . \tag{2.15}$$

Then for any $L \in \mathscr{L}$ with $\xi = \eta \circ L$ (such an L always exists in view of condition (C)) and y with $y_0 < y < y^*$ we have

$$y \not\leq F(x) + \langle L, G(x) \rangle \quad \forall x \in D . \tag{2.16}$$

which means $y \in H$ what contradicts Pareto minimality of y^*. Property (2.14) is thus proved.

Fix now a $y_0 < y^*$. By (2.14) the segment $[y_0, y^*)$ is contained in Q. Consequently, for the algebraical closedness of Q, $y^* \in Q$. It means that there exists an $x^* \in D$, $G(x^*) \geqq 0$ with $F(x^*) \geqq y^*$. The weak duality principle guarantees that x^* is a Pareto maximal solution of program (P) and $F(x^*) = y^*$. The proof is complete. □

Remark 2.1. Our duality theory which has been presented above, contains duality concepts of Podinovskij, Nogin [2] and of Bitran [6] as particular cases. Indeed, if Y and Z are of finite dimension, say dim $Y = n$ and dim $Z = m$, then $\mathscr{L}$ will be sets of $(m \times m)$-matrices. If $\mathscr{L}$ is the set of all nonnegative $(n \times m)$-matrices one has the set of feasible matrices of Bitran [6]. If $\mathscr{L}$ is the set of all nonnegative $(n \times m)$-matrices rows of which are equal to each other, one obtains duality results of Podinovski, Nogin [2].

The question under what conditions the above conditions (C) and (C′) are fulfilled remains open. Maybe it serves a good direction for further investigation.

3. KUHN-TUCKER-LAGRANGE CONDITION

In this section one shall need the following generalization of differential for concave functionals.

Let X be a real vector space, $G : X \to R$ be a concave functional and $x^* \in X$. The set

$$\partial G(x^*) = \{l \in X^* : l(x) - l(x^*) \geqq G(x) - G(x^*) \ \forall x \in X\}$$

will be called superdifferential of functional G at x^*.

We recall, without proof, some important properties of superdifferential (for details see Akilov, Kutatelaze [3], Chapter II).

(i) If $x^* \in \operatorname{cor}(\operatorname{dom} G)$, where $\operatorname{dom} G = \{x \in X \mid G(x) > -\infty\}$ is the effective domain of functional G, then $\partial G(x^*) \neq \emptyset$.

(ii) If $G_1, \ldots, G_n$ are concave $x^* \in \operatorname{cor}\left(\bigcap_{i=1}^{n} \operatorname{dom} G_i\right)$ then

$$\partial(G_1 + \ldots + G_n)(x^*) = \partial G_1(x^*) + \ldots + \partial G_n(x^*)$$

(iii) If G is concave and $x^* \in \operatorname{cor}(\operatorname{dom} G)$ then

$$G(x^*) = \sup_{x \in X} G(x) \Leftrightarrow 0 \in \partial G(x^*) .$$

Given a convex set $U \subset X$ we define

$$\delta_U(x) = \begin{cases} 0 & \text{if } x \in U \\ -\infty & \text{if } x \notin U \end{cases}$$

(iv) For any $x^* \in U$ we have

$$\partial \delta_U(x^*) = U_* = \{l \in X^* \mid l(x^*) = \inf_{x \in U} l(x)\} .$$

Now let us consider the concave maximization program defined in Section 1

$$\text{(P)} \qquad F(x) \to \max \qquad x \in D \;\&\; G(x) \geqq 0$$

Theorem 3.1. (Kuhn-Tucker-Lagrange condition.) Assume that Slater's condition is fulfilled. Then a point $x^* \in D$ is a weak maximal solution of program (P) if and only if there exist functionals $\eta \in Y_+^* \setminus \{0\}$ and $\xi \in Z_+^*$ such that

$$\text{(a)} \qquad G(x^*) \geqq 0$$

$$\text{(b)} \qquad 0 \in \partial \eta \circ F(x^*) + \partial \xi \circ G(x^*) + D_*$$

$$\text{(c)} \qquad \langle \xi, G(x^*) \rangle = 0$$

Proof. Sufficiency of the Kuhn-Tucker-Lagrange condition is evident. It suffices therefore to prove necessity.

Let $x^* \in D$ be a weak maximal solution of program (P). By definition, $F(x^*)$ is a weak maximal of the set Q (see (1.3)). It implies that there exists a functional $\eta \in Y_+^* \setminus \{0\}$ with $\langle \eta, F(x^*) \rangle \geqq \langle \eta, y \rangle \; \forall y \in Q$. In view of Lemma 2.1 there exists a functional $\xi \in Z_+^*$ such that

$$\langle \eta, F(x^* \rangle \geqq \langle \eta, y \rangle + \langle \xi, z \rangle \quad \forall (y, z) \in P .$$

In particular we have

$$\langle \eta, F(x^*) \rangle = \sup_{x \in D} \{\langle \eta, F(x) \rangle + \langle \xi, G(x) \rangle\}$$

which is equivalent to

$$\langle \eta, F(x^*) \rangle = \sup_{x \in D'} \{\langle \eta, F(x^*) \rangle + \langle \xi, G(x) \rangle + \delta_D(x)\} .$$

Now it suffices to refer to properties (ii) and (iii) and we obtain condition (b). Conditions (a) and (c) are evident. The proof is complete. □

Remark 3.1. Slater's condition is not necessary for sufficiency of Kuhn-Tucker-Lagrange conditions.

Remark 3.2. If η is positive then Kuhn-Tucker-Lagrange conditions are sufficient for x^* to be a proper maximal solution of program (P).

Remark 3.3. Theorem 3.1 derives results of Holmes [1] and Minami [7] as particular cases.

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