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RELATIVE CONTROLLABILITY OF NONLINEAR SYSTEMS WITH TIME VARYING DELAYS IN CONTROL

K. BALACHANDRAN, D. SOMASUNDARAM

Using the measure of noncompactness of a set and Darbo's fixed point theorem, sufficient conditions for relative controllability of nonlinear systems with time varying multiple delays in control and implicit derivative are established. The delays in [11] are of distributed in nature while here are time varying.

1. INTRODUCTION

The problem of controllability of dynamical systems described by nonlinear ordinary differential equations has been investigated by many authors [1, 2, 7, 9, 10], with the help of Schauder's fixed point theorem. In particular Klamka [5, 6] has studied the controllability of nonlinear systems with different types of delay in control. In [3] Dacka introduced a new method of analysis based on the notion of measure of noncompactness of a set and Darbo's fixed point theorem for the study of the controllability of nonlinear systems with implicit derivative. This method is extended by Dacka [4] to perturbed nonlinear system with time varying delay in control and implicit derivative. In [11], sufficient conditions for controllability of nonlinear systems having implicit derivative with distributed delays in control have been derived. The purpose of this paper is to study the controllability of nonlinear systems with time varying multiple delays in control and implicit derivative by suitably adopting the technique of Dacka. The results generalise the results of Klamka [5] and Dacka [4].

2. BASIC ASSUMPTIONS AND DEFINITIONS

Consider the following nonlinear time varying systems with time varying multiple delays in the control, represented by the equation

$$\dot{x}(t) = A(t, x(t))\, x(t) + \sum_{i=0}^{M} B_i(t, x(t))\, u(h_i(t)) + f(t, x(t), \dot{x}(t), u(t)) \tag{1}$$

where the state $x(t)$ is an n-vector and the control $u(t)$ is a p-vector, $A(t, x)$ is an $n \times n$ matrix, $B_i(t, x)$ for $i = 0, 1, \ldots, M$ are $n \times p$ matrices and $f(t, x, \dot{x}, u)$ is an n-vector function.

Assume that the elements a_{jk} of A $(j, k = 1, 2, \ldots, n)$ and b_{ijk} of B_i $(j = 1, 2, \ldots$ $\ldots, n,\ k = 1, 2, \ldots, p)$ for $i = 0, 1, \ldots, M$ are continuous functions and fulfill the following conditions

(2) $$|a_{jk}(t, x)| \leqq N \quad \text{for} \quad t \in [t_0, t_1] \quad \text{and} \quad x \in \mathbb{R}^n$$

(3) $$|b_{ijk}(t, x)| \leqq L_i \quad \text{for} \quad t \in [t_0, t_1] \quad \text{and} \quad x \in \mathbb{R}^n$$

where N and L_i $(i = 0, 1, \ldots, M)$ are positive real constants. Furthermore, assume that the function $f(t, x, y, u)$ is continuous and satisfies the conditions

(4) $$|f(t, x, y, u)| \leqq K \quad \text{for} \quad t \in [t_0, t_1],\ x, y \in \mathbb{R}^n \quad \text{and} \quad u \in \mathbb{R}^p$$

and for every $y, \bar{y} \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $t \in [t_0, t_1]$

(5) $$|f(t, x, y, u) - f(t, x, \bar{y}, u)| \leqq k|y - \bar{y}|$$

where K and k are positive real constants and $0 \leqq k < 1$.

Assume that the functions h_i: $[t_0, t_1] \to \mathbb{R}$, $i = 0, 1, \ldots, M$ are twice continuously differentiable and strictly increasing in $[t_0. t_1]$. Moreover

(6) $$h_i(t) \leqq t \quad \text{for} \quad t \in [t_0, t_1]\,, \quad i = 0, 1, \ldots, M\,.$$

Let us introduce the time-lead functions

$$r_i(t)\colon [h_i(t_0), h_i(t_1)] \to [t_0, t_1]\,, \quad i = 0, 1, \ldots, M$$

such that $r_i(h_i(t)) = t$ for $t \in [t_0, t_1]$. Further assume that $h_0(t) = t$ and for $t = t_1$, the function $h_i(t)$ satisfy the inequalities

(7) $$h_M(t_1) \leqq h_{M-1}(t_1) \leqq \ldots \leqq h_{m+1}(t_1) \leqq t_0 = h_m(t_1) < h_{m-1}(t_1) = \\ = h_1(t_1) = h_0(t_1) = t_1$$

Define the norm of a continuous $n \times p$ matrix valued function $S(t)$ by

$$\|S(t)\| = \max_i \sum_{j=1}^{p} \max_{t_0 \leqq t \leqq t_1} |s_{ij}(t)|$$

where s_{ij} are elements of S. Let us define the Cartesian product as $C^1_{n+p}[t_0, t_1] = C^1_n[t_0, t_1] \times C_p[t_0, t_1]$. For the measure μ of noncompactness of a set and Darbo's fixed point theorem and the common modulus of continuity $\dot{w}$, w_0 see [3, 8, 11]. The following definitions of complete state of the system (1) at time t and relative controllability are assumed [5].

Definition 1. The set $y(t) = \{x(t), \beta(t, s)\}$ where $\beta(t, s) = u(s)$ for $s \in [\min_i h_i(t), t)$ is said to be the complete state of the system (1) at time t.

Definition 2. The system (1) is said to be globally relatively controllable on $[t_0, t_1]$ if for every complete state $y(t_0)$ and every vector $x_1 \in \mathbb{R}^n$, there exists a control $u(t)$ defined on $[t_0, t_1]$ such that the corresponding trajectory of the system (1) satisfies $x(t_1) = x_1$.

Definition 3. The system (1) is said to be locally relatively controllable on $[t_0, t_1]$ in the domain $D \subset \mathbb{R}^n$, if for every complete state $y(t_0)$, such that $x(t_0) \in D$ and every vector $x_1 \in D \subset \mathbb{R}^n$, there exists a control $u(t)$ defined on $[t_0, t_1]$ such that the corresponding trajectory of the system (1) satisfies $x(t_1) = x_1$.

For each $[z, v] \in C^1_{n+p}[t_0, t_1]$, we consider the following system

$$\dot{x}(t) = A(t, z(t))\, x(t) + \sum_{i=0}^{M} B_i(t, z(t))\, u(h_i(t)) + f(t, z(t), \dot{z}(t), v(t)) \,. \tag{8}$$

The solution of the differential equation (8) with condition $x(t_0) = x_0$ can be expressed in the form

$$x(t) = F(t, t_0; z)\, x_0 + \int_{t_0}^{t} F(t, s; z) \sum_{i=0}^{M} B_i(s, z)\, u(h_i(s))\, \mathrm{d}s + \\ + \int_{t_0}^{t} F(t, s; z) f(s, z, \dot{z}, v)\, \mathrm{d}s. \tag{9}$$

where $F(t, t_0; z)$ is the transition matrix of the linear system

$$\dot{x}(t) = A(t, z(t))\, x(t) \tag{10}$$

with initial condition $F(t_0, t_0; z) = I$, the identity matrix. Using the time lead function $r_i(t)$, the formula (9) can be written as

$$x(t) = F(t, t_0; z) \left[x_0 + \sum_{i=0}^{M} \int_{h_i(t_0)}^{h_i(t)} F(t_0, r_i(s); z)\, B_i(r_i(s), z)\, \dot{r}_i(s)\, u(s)\, \mathrm{d}s + \right. \\ \left. + \int_{t_0}^{t} F(t_0, s; z) f(s, z, \dot{z}, v)\, \mathrm{d}s \right]. \tag{11}$$

By (7), the equality (11) for $t = t_1$ can be expressed in the following form

$$x(t_1) = F(t_1, t_0; z) \left[x_0 + \sum_{i=0}^{m} \int_{h_i(t_0)}^{t_0} F(t_0, r_i(s); z)\, B_i(r_i(s), z)\, \dot{r}_i(s)\, \beta(t_0, s)\, \mathrm{d}s + \right. \\ + \sum_{i=m+1}^{M} \int_{h_i(t_0)}^{h_i(t_1)} F(t_0, r_i(s); z)\, B_i(\dot{r}_i(s), z)\, \dot{r}_i(s)\, \beta(t_0, s)\, \mathrm{d}s + \\ + \sum_{i=0}^{m} \int_{t_0}^{t_1} F(t_0, r_i(s); z)\, B_i(r_i(s), z)\, \dot{r}_i(s)\, u(s)\, \mathrm{d}s + \\ \left. + \int_{t_0}^{t_1} F(t_0, s; z) f(s, z, \dot{z}, v)\, \mathrm{d}s \right]. \tag{12}$$

For brevity, let us introduce the following notations

(13) $$G_i(t_0, s; z) = \sum_{j=0}^{i} F(t_0, r_j(s); z)\, B_j(r_j(s), z)\, \dot{r}_j(s) \quad \text{for} \quad i = 0, 1, \ldots, M$$

(14) $$q(y(t_0), x_1; z, v) = F(t_0, t_1; z)\, x_1 - x_0 - \int_{t_0}^{t_1} F(t_0, s; z) f(s, z, \dot{z}, v)\, ds -$$
$$- \sum_{i=0}^{m} \int_{h_i(t_0)}^{t_0} F(t_0, r_i(s); z)\, B_i(r_i(s), z)\, \dot{r}_i(s)\, \beta(t_0, s)\, ds -$$
$$- \sum_{i=m+1}^{M} \int_{h_i(t_0)}^{h_i(t_1)} F(t_0, r_i(s); z)\, B_i(r_i(s), z)\, \dot{r}_i(s)\, \beta(t_0, s)\, ds.$$

Define the controllability matrix $W(t_0, t_1; z)$ by

(15) $$W(t_0, t_1; z) = \int_{t_0}^{t_1} G_m(t_0, s; z)\, G'_m(t_0, s; z)\, ds$$

where the prime indicates the matrix transpose.

3. MAIN RESULTS

Theorem 1. Given the system (1) with conditions (2) to (7) and

(16) $$\inf_{z \in C^1_n[t_0, t_1]} \det W(t_0, t_1; z) > 0$$

then the system (1) is globally relatively controllable on $[t_0, t_1]$.

Proof. For each fixed element $[z, v] \in C^1_{n+p}[t_0, t_1]$, define the control $u(t)$ for $t \in [t_0, t_1]$ as follows

(17) $$u(t) = G'_m(t_0, t; z)\, W^{-1}(t_0, t_1; z)\, q(y(t_0), x_1; z, v)$$

where $y(t_0)$ and x_1 are chosen arbitrarily. Inserting (17) into (11) we obtain the following equality

$$x(t) = F(t, t_0; z)\Big[x_0 + \sum_{i=0}^{m} \int_{h_i(t_0)}^{t_0} F(t_0, r_i(s); z)\, B_i(r_i(s), z)\, \dot{r}_i(s)\, \beta(t_0, s)\, ds +$$

(18) $$+ \sum_{i=m+1}^{M} \int_{h_i(t_0)}^{h_i(t)} F(t_0, r_i(s); z)\, B_i(r_i(s), z)\, \dot{r}_i(s)\, \beta(t_0, s)\, ds +$$

$$+ \sum_{i=0}^{m} \int_{t_0}^{t} F(t_0, r_i(s); z)\, B_i(r_i(s), z)\, \dot{r}_i(s)\, G'_m(t_0, s; z)\, W^{-1}(t_0, t_1; z)\, q(y(t_0), x_1; z, v)\, ds +$$
$$+ \int_{t_0}^{t} F(t_0, s; z)\, f(s, z, \dot{z}, v)\, ds \Big].$$

By using (13) to (15) and (17), it is easy to see that $x(t)$ in (18) satisfies the condition $x(t_1) = x_1$. Let us consider the right-hand sides of (17) and (18) as a pair of operators $T_2([z, v])(t)$ and $T_1([z, v])(t)$, respectively. Define the nonlinear operator T by

(19) $$T([z, v])(t) = [T_1([z, v])(t), T_2([z, v])(t)]$$

It is easy to see that T is continuous and maps the space $C^1_{n+p}[t_0, t_1]$ into itself. Consider the closed convex subset of $C^1_{n+p}[t_0, t_1]$

(20) $$H = \{[z, v]: \|v\| \leqq N_1, \|z\| \leqq N_2, \|Dz\| \leqq N_3\}$$

where the positive constants N_1, N_2 and N_3 are defined by

(21) $$N_1 = [|x_1| \exp(nN(t_1 - t_0)) + C_1] C_2$$

(22) $$C_1 = |x_0| + \sum_{i=0}^{m} (t_0 - h_i(t_0)) npL_i a_i b \exp(nN(r_i(t_0) - t_0)) +$$

$$+ \sum_{i=m+1}^{M} (h_i(t_1) - h_i(t_0)) npL_i a_i b \exp(nN(t_1 - t_0)) + K(t_1 - t_0) \exp(nN(t_1 - t_0))$$

(23) $$a_i = \|\dot{r}_i(s)\|, \quad b = \|\beta(t_0, s)\|$$

(24) $$C_2 = \sup_{z \in C^1_n[t_0, t_1]} \|W^{-1}(t_0, t_1; z)\| \sum_{i=0}^{m} npL_i a_i \exp(nN(t_1 - t_0))$$

(25) $$N_2 = \exp(nN(t_1 - t_0)) [C_1 + (t_1 - t_0) C_3(|x_1| \exp(nN(t_1 - t_0) + C_1)$$

$$. \sum_{i=0}^{m} npL_i a_i \exp(nN(r_i(t_1) - r_i(t_0))]$$

(26) $$C_3 = \sup_{z \in C^1_n[t_0, t_1]} \|W^{-1}(t_0, t_1; z)\| \sum_{i=0}^{m} npL_i a_i \exp(nN(r_i(t_1) - r_i(t_0)))$$

(27) $$N_3 = n^2 N N_2 + \sum_{i=0}^{M} L_i npN_1 + K$$

The mapping T transforms the set H defined by (20) into H. Let us note that all the functions of the form $G'_m(t_0, t; z)$ are equicontinuous if the functions $z \in C^1_n[t_0, t_1]$ are arbitrarily taken, but such that they satisfy the inequalities $\|z\| \leqq N_2$, $\|Dz\| \leqq N_3$. By using the differentiability of $r_i(t)$, the equicontinuity of the functions $B_i(r_i(t), z)$ follows from the equicontinuity of $z \in H$ and the equicontinuity of $F(t_0, r_i(t); z)$ follows from the fact that all the functions $z \in H$ are uniformly bounded. Denote $w(G'_m, h)$ the common modulus of continuity of all functions $G'_m(t_0, t; z)$. Then we have

(28) $$w(T_2([z, v]), h) \leqq w(G'_m, h)\, a$$

where $a = \sup_{[z,v] \in H} [\|W^{-1}(t_0, t_1; z)\|\, q(y(t_0), x_1; z, v)]$.

From the relation (28) it follows that all functions $T_2([z, v])(t)$ have a uniformly bounded modulus of continuity, hence they are equicontinuous.

Further the functions $T_1([z, v])(t)$ are also equicontinuous when $[z, v] \in H$, since they have uniformly bounded derivatives. As in [11] the modulus of continuity of the function for $t, s \in [t_0, t_1]$ can be made as

(29) $$|DT_1([z, v])(t) - DT_1([z, v])(s)| \leqq k|\dot{z}(t) - \dot{z}(s)| + \beta(|t - s|)$$

and so $w(DT_1([z, v]), h) \leqq kw(Dz, h) + \beta(h)$, where β is a nonnegative function such that $\lim_{h\to 0^+} \beta(h) = 0$. Thus we have for any set $E \subset H$,

$$w_0(DT_1E) \leqq kw_0(DE_1) \quad \text{and} \quad w_0(T_2E) = 0$$

where E_1 is the natural projection of the set E on the space $C_n^1[t_0, t_1]$. Hence it follows that

$$\mu(TE) \leqq k\,\mu(E)\,.$$

By Darbo's fixed point theorem the operator T has at least one fixed point, hence there exists functions $z^+ \in C_n^1[t_0, t_1]$ and $v^+ \in C_p[t_0, t_1]$ such that

$$z^+(t) = T_1([z^+, v^+])(t) \tag{30}$$

$$v^+(t) = T_2([z^+, v^+])(t) \tag{31}$$

Differentiating with respect to t, we easily verify that $x(t)$ given by (30) is a solution to the system (1) for the control $u(t)$ given by (31). The control $u(t) = v^+(t)$ steers the system (1) the from initial complete state $y(t_0)$ to $x_1 \in \mathbb{R}^n$, on $[t_0, t_1]$, since $y(t_0)$ and x_1 have been chosen arbitrarily, then by Definition 2, the system (1) is globally relatively controllable on $[t_0, t_1]$.

Remark 1. If we assume that the function f appearing in equation (1) satisfies also Lipschitz condition with respect to the variable x, then we can obtain the unique response determined by any control.

Remark 2. The case for $M = 1$ and f is independent of u has been studied in [4]. The trivial case $M = 0$, A and B are independent of x has been considered in the paper [3].

Corollary 1. Given the system (1) with conditions as in Theorem 1, then the perturbed system

$$\dot{x}(t) = [A(t, x) + \varepsilon\bar{A}(t, x)]\, x + \sum_{i=0}^{M} [B_i(t, x) + \varepsilon\bar{B}_i(t, x)]\, u(h_i(t)) + f(t, x, \dot{x}, u) \tag{32}$$

with $\bar{A}(t, x)$ and $\bar{B}_i(t, x)$, $i = 0, 1, \ldots, M$ satisfying the same type of conditions as imposed on $A(t, x)$ and $B_i(t, x)$ is globally relatively controllable on $[t_0, t_1]$, provided ε is sufficiently small.

Proof. Here we have to show that the determinant of the modified controllability matrix $W(t_0, t_1; z, \varepsilon)$ for the system (32) has a positive infimum. By similar argument as in [9], it is easy to show that by taking ε small enough, there exists a constant $c > 0$ such that

$$\inf_{z\in C^1{}_n[t_0, t_1]} \det W(t_0, t_1; z, \varepsilon) \geqq c$$

Then by Theorem 1, the system is globally relatively controllable on $[t_0, t_1]$. □

Example.

$$\dot{x}_1(t) = g(t, x_1, x_2)\, x_2(t) + u_1(t) + u_1(t - h) + \frac{x_1(t) + x_2(t)}{1 + x_1^2(t) + x_2^2(t)} + \sin \tfrac{1}{2}\, \dot{x}_1(t)$$

$$\dot{x}_2(t) = -g(t, x_1, x_2)\, x_1(t) + t\, u_2(t) + t\, u_2(t - h) + \frac{x_1(t)}{1 + x_2^2(t)}$$

where $g(t, x_1, x_2)$ is a continuous function in its arguments.

Observe that all the function are continuous and bounded. Moreover f satisfies Lipschitz condition with respect to the variable $\dot{x}$ with the constant $k = \frac{1}{2}$. For any fixed $z \in C_2^1[t_0, t_1]$ the state transition matrix $F(t, t_0; z)$ has the following form

$$F(t, t_0; z) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

where $a = \cos \int_{t_0}^{t} g(\tau, z_1, z_2)\, d\tau$, $b = \sin \int_{t_0}^{t} g(\tau, z_1, z_2)\, d\tau$
and the controllability matrix $W(t_0, t_1; z)$ is

$$W(t_0, t_1; z) = \int_{t_0}^{t_1 - h} \begin{bmatrix} a_1^2 + b_2^2 & a_1 b_1 - a_2 b_2 \\ a_1 b_1 - a_2 b_2 & b_1^2 + a_2^2 \end{bmatrix} ds \, +$$

$$+ \int_{t_1 - h}^{t_1} \begin{bmatrix} a^2 + s^2 b^2 & (1 - s^2)\, ab \\ (1 - s^2)\, ab & b^2 + s^2 a^2 \end{bmatrix} ds$$

where

$$a_2 = s a_1 + \bar{a} h\,, \quad b_2 = s b_1 + \bar{b} h$$

$$a_1 = a + \bar{a}\,, \qquad b_1 = b + \bar{b}$$

and

$$\bar{a} = \cos \int_{t_0}^{s+h} g(\tau, z_1, z_2)\, d\tau\,, \quad \bar{b} = \sin \int_{t_0}^{s+h} g(\tau, z_1, z_2)\, d\tau\,.$$

If $t_1 > t_0 + h$, then the infimum of det $W(t_0, t_1; z)$ is greater then zero. Thus from Theorem 1, the dynamical system in question is globally relatively controllable.

4. LOCAL CONTROLLABILITY RESULTS

The technique used to prove Theorem 1 can also be applied to derive sufficient conditions for the local relative controllability on $[t_0, t_1]$ of the dynamical system (1). Consider the following subset of $[t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p$.

(33) $$D = [(t, x, \dot{x}, u): t \in [t_0, t_1], |x| \leqq \alpha, |\dot{x}| \leqq \beta, |u| \leqq \gamma, \alpha, \beta, \gamma \text{ being some positive constants}]$$

Theorem 2. Given the system (1) with conditions (2) to (7) in D, we assume that the constants N, L_i and K being bounds for the elements of the matrices A, B_i and for the function f, and that the states $y(t_0)$ and x_1 are such that the constants N_1, N_2

and N_3 from Theorem 1 satisfy the inequalities

$$N_1 \leqq \gamma\,, \quad N_2 \leqq \alpha\,, \quad N_3 \leqq \beta \tag{34}$$

and further,

$$\inf_{|z| \leqq \alpha} \det W(t_0, t_1; z) > 0\,. \tag{35}$$

Then the system (1) is locally relatively controllable on $[t_0, t_1]$.

Proof. The proof of Theorem 2 is similar to that of Theorem 1. It is enough to note that if the domain of the matrices $A(t, x)$, $B_i(t, x)$, $i = 0, 1, \ldots, M$ and the function $f(t, x, \dot{x}, u)$ is restricted to D, then the inequalities (34) imply that

$$|T_1([z, v])(t)| \leqq \alpha\,, \quad |T_2([z, v])(t)| \leqq \gamma$$
$$|DT_1([z, v])(t)| \leqq \beta$$

Hence the theorem. □

Corollary 2. Given the system (1) with conditions as in Theorem 1, then the perturbed system (32) with $\bar{A}(t, x)$ and $\bar{B}_i(t, x)$, $i = 0, 1, \ldots, M$ satisfying the same type of conditions as imposed on $A(t, x)$ and $B_i(t, x)$, $i = 0, 1, \ldots, M$ in D is locally relatively controllable on $[t_0, t_1]$ provided ε is sufficiently small.

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K. Balachandran, D. Somasundaram, Department of Mathematics, Bharathiar University, Coimbatore 641 046, Tamil Nadu. India.