A NEW CRITERION FOR STRONG OBSERVABILITY

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Given the transfer-function matrix $T(s) = P(s) q^{-1}(s)$ of a linear time-invariant system where $P(s)$ is a $p \times m$ polynomial matrix and $q(s)$ is a polynomial, a new criterion for strong observability of state space realizations of $T(s)$ is expressed in terms of the parameters in $P(s)$ and $q(s)$.

1. INTRODUCTION

Apart from controllability and observability an important concept in linear system theory is strong observability [1]–[4]. This latter concept appears naturally in the study of singular quadratic optimal control problems [4]. A strongly (alternatively perfectly) observable is that system for which an observer can determine the state vector $x(0) = x_0$ with the knowledge of output $y(t)$ only, information regarding input $u(t)$ not required. The criteria of these concepts in terms of system matrices (A, B, C, D) are well known [1]. In this correspondence we shall formulate these criteria in terms of parameters contained in transfer-function matrix representation of the system in the form $P(s) q^{-1}(s)$.

Given a transfer-function matrix description $T(s)$ of a system it may be of interest to decide whether any minimal realization of this system is strongly observable or not. So one has to find out at first the minimal realization by applying the existing methods and then to apply the available criteria given in terms of state space description to check whether this realization of the system is strongly observable or not. If instead one could check the strong observability of state realization with the help of parameters of $T(s)$ the first step could be avoided to get the desired information. The aim of this short note is to formulate a criterion of strong observability of the state-space realization of a given system in terms of the parameters of its transfer-function matrix.

The linear time-invariant systems under our investigation are represented by the $p \times m$ transfer-function matrix

(1.1) $$T(s) = P(s) q^{-1}(s)$$

where $P(s)$ is a $p \times m$ polynomial matrix having the form

(1.2) $$P(s) = P_0 s^d + P_1 s^{d-1} + \ldots + P_d,$$

in which the P_i's $(i = 0, 1, \ldots, d)$ are $p \times m$ real constant matrices, and

(1.3) $$q(s) = s^d + q_1 s^{d-1} + \ldots + q_{d-1} s + q_d.$$

Consequently (1.1) can as well be represented as $P(s) \, Q^{-1}(s)$ where $P(s)$ is given as before by (1.2) and

(1.4) $$Q(s) = Q_0 s^d + Q_1 s^{d-1} + \ldots + Q_d$$

in which $Q_i = q_i I_m$ $(i = 0, 1, \ldots, d)$ and $q_0 = 1$, and so Q_0 is nonsingular. Let $\Sigma = (A, B, C, D)$ represent the state space realization of (1.1) having dimension n. Then Σ is known (cf. [1]) to be controllable, observable and strongly observable iff

(1.4abc) $$\text{rank } L = n; \quad \text{rank } M = n; \quad \mathscr{R}(M) \cap \mathscr{R}(N) = \{0\}$$

respectively where

$$L = \begin{bmatrix} A^{n-1}B & A^{n-2}B \ldots B \end{bmatrix}$$

$$M = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} D & & \\ CB & D & \\ \vdots & \vdots & \\ CA^{n-2}B & CA^{n-3}B & D \end{bmatrix}$$

and $\mathscr{R}(M)$ and $\mathscr{R}(N)$ are range space of M and N respectively.

In the next section we shall express the criteria of strong observability of a state space realization of (1.1) in terms of P_i's and Q_i's $(i = 0, 1, \ldots, d)$.

2. THE NEW CRITERION

Substituting (1.2) and (1.3) in

(2.1) $$C(sI - A)^{-1} B + D = P(s) \, Q^{-1}(s)$$

and equating the coefficients of like powers of s and s^{-1} from both sides of the resultant expression we get

(2.2) $$\begin{bmatrix} N & 0 \\ H & N \end{bmatrix} \begin{bmatrix} \hat{Q} & 0 \\ \bar{Q} & \hat{Q} \end{bmatrix} = \begin{bmatrix} \hat{P} & 0 \\ \bar{P} & \hat{P} \end{bmatrix}$$

where $H, \hat{Q}, \bar{Q}, \hat{P}$ and $\bar{P}$ are $n \times n$ block matrices having the form

(2.3) $$H = \begin{bmatrix} CA^{n-1}B & \ldots & CB \\ \vdots & \vdots & \vdots \\ CA^{2n-2}B & \ldots & CA^{n-1}B \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} Q_0 & & & & 0 \\ \vdots & \ddots & & & \\ Q_d & \ldots & Q_0 & & \\ \vdots & & \ddots & \ddots & \\ 0 & \ldots & Q_d & \ldots & Q_0 \end{bmatrix}$$

$$\bar{Q} = \begin{bmatrix} 0 & \dots & Q_d & \dots & Q_1 \\ & & & & \ddots \\ & 0 & & & Q_d \\ & & & & \vdots \\ & & & & 0 \end{bmatrix}, \quad \hat{P} = \begin{bmatrix} P_0 & & & & \\ \vdots & \ddots & & & \\ P_d & \dots & P_0 & & \\ \vdots & & \ddots & & \\ 0 & \dots & P_d & \dots & P_0 \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} 0 & \dots & P_d & \dots & P_1 \\ & & & & \ddots & \vdots \\ & 0 & & & & P_d \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

The matrix $\hat{P}$ is the same for d equal to n and $n - 1$. So is also the matrix $\bar{Q}$. Consequently $\hat{P}$ and $\bar{Q}$ do not contain P_d and Q_d respectively when $d = n$.

Given the transfer-function matrix $T(s)$ one can determine the order 'n' of the system as the degree of the least common denominator of all minors of $T(s)$. Since $H = ML$ and assuming a controllable realization i.e., rank $L = n$, we have

(2.4) $$\mathscr{R}(H) = \mathscr{R}(M).$$

Again $\hat{Q}$ is nonsingular as Q_0 is so. From (2.2) solving for N and H we get

(2.5) $$N = \hat{P}\hat{Q}^{-1} \quad \text{and} \quad H = \left(\bar{P} - \hat{P}\hat{Q}^{-1}\bar{Q}\right)\hat{Q}^{-1}.$$

Therefore we get the following criterion for strong observability of any minimal state space realization of the given transfer function matrix $T(s)$.

2.6. Theorem. Any minimal realization of the system described by the transfer function matrix $T(s)$ in (1.1) is strongly observable iff

(2.7) $$\mathscr{R}(\hat{P}) \cap \mathscr{R}(\bar{P} - \hat{P}\hat{Q}^{-1}\bar{Q}) = \{0\}.$$

Proof. From (1.4c) we know that the system (1.1) having the minimal realization (A, B, C, D) is strongly observable iff $\mathscr{R}(M) \cap \mathscr{R}(N) = \{0\}$. A minimal realization is controllable, hence $\mathscr{R}(H) = \mathscr{R}(M)$ and we have from (2.5) that the system (1.1) is strongly observable iff (2.7) holds. □

In the above theorem $\hat{Q}$ is a triangular matrix with unit diagonal elements and therefore has a triangular inverse:

$$\begin{bmatrix} I & & & & \\ Q_1 & I & & & \\ Q_2 & Q_1 & I & & \\ \vdots & \vdots & \vdots & & \\ Q_{n-1} & Q_{n-2} & Q_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & & & & \\ R_1 & I & & & \\ R_2 & R_1 & I & & \\ \vdots & \vdots & & \ddots & \\ R_{n-1} & R_{n-2} & \dots & R_1 & I \end{bmatrix}$$

in which

$$R_1 = -Q_1, \quad -R_i = Q_i + Q_{i-1}R_1 + \dots + Q_1R_{i-1}, \quad i = 2, \dots, n - 1,$$

can be computed recursively without any major problem.

3. CONCLUSION

Different properties of multivariable control systems can be characterized in terms of state variable as well as transfer-function description. So the criteria to ensure that a particular system possesses such a property must be available to the advantage of the designer in both the descriptions. Although criteria for strong observability are available in state-space description, it is unfortunately not available in terms of transferfunction description of the system. The criterion (2.6) given in this paper is likely to fill up this gap.

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