A PROCEDURE FOR DESIGNING STABILIZING OUTPUT FEEDBACK CONTROLLERS

JAN LUNZE

This new approach to designing stabilizing constant-gain output feedback controllers is based on quantitative robustness properties of feedback control. A sufficient stabilizability condition is derived for an important class of unstable linear systems, which can be directly interpreted in terms of the system modes. Then, an instructive design procedure is presented to calculate appropriate feedback matrices. It can be used to find a solution of the general servomechanism problem with least dynamical order. For PI-control the design algorithm yields an explicit expression for the controller matrices.

0. INTRODUCTION

The problems of stability and stabilization of unstable linear systems

$$\dot{x} = Ax + Bu\,, \quad x(0) = x_0$$
$$y = Cx + Du$$

by means of constant output feedback

$$u = Ky$$

are of fundamental importance in systems theory and have attracted considerable attention. However, up to now these problems have not been completely solved. The existence of stabilizing output feedback matrices is established only in form of necessary *or* sufficient conditions [1], [17], [20]–[22] or for special classes of unstable systems [2], [4], [10], [12], [13]. The design procedures presented in [2], [3], [8], [18] can be used only if the number of outputs and inputs is rather high compared with the order of the system.

Most of these results have been obtained by a discussion of the characteristic polynomial of the closed-loop system in relation to the elements of the matrix K and are expressed in algebraic relations. Although these results essentially originate from structural properties of the system [14], [15], [19], they cannot be satisfactorily interpreted in terms of the technological and physical characteristics of the system to be controlled.

In this paper a completely new approach will be presented to solving the stabilizability and stabilization problems. Based on the robustness of multivariable feedback controllers, the stabilization of an important class of unstable linear systems is considered. The results summarized in Theorem 1 lead to an instructive design algorithm. Theorem 2 states a sufficient condition for the stabilizability. This condition can be easily interpreted in terms of those properties, which are commonly used in control theory and by control engineers to describe the nature of the control system.

1. THE PROBLEM OF STABILIZATION BY OUTPUT FEEDBACK

Consider a linear dynamical system

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\,\bar{x}(t) + \bar{B}\,u(t)\,, \quad \bar{x}(0) = x_0 \\ y(t) &= \bar{C}\,\bar{x}(t) + \bar{D}\,u(t) \end{aligned} \tag{1}$$

where $\bar{x}(t) \in \mathbb{R}^n$ is the vector of system states, $u(t) \in \mathbb{R}^m$ the vector of inputs, and $y(t) \in \in \mathbb{R}^r$ the vector of outputs. $\bar{A}, \bar{B}, \bar{C}$ and $\bar{D}$ are constant matrices with appropriate dimensions.

Let the system (1) be unstable, i.e. the matrix $\bar{A}$ has some eigenvalues with non-negative real parts. Then the system is to be stabilized by means of a constant output feedback

$$u(t) = K\,y(t)\,. \tag{2}$$

If the matrix $I - K\bar{D}$ is invertible, the closed-loop system (1), (2) is described by

$$\dot{\bar{x}} = [\bar{A} + \bar{B}(I - K\bar{D})^{-1}\,K\bar{C}]\,\bar{x}\,, \quad \bar{x}(0) = \bar{x}\,.$$

It is stable, if all the eigenvalues of

$$\tilde{A} = \bar{A} + \bar{B}(I - K\bar{D})^{-1}\,K\bar{C} \tag{3}$$

have negative real parts.

Problem 1. (Problem of stabilizability.) Under what conditions does a feedback gain matrix $K \in \mathbb{R}^{m\times r}$ exist such that the matrix $(I - K\bar{D})$ is nonsingular and the closed-loop system matrix $\tilde{A}$ is stable?

Problem 2. (Problem of stabilization.) Assume that the system (1) is stabilizable by output feedback (2). Which feedback matrix $K \in \mathbb{R}^{m\times r}$ does the system (1) stabilize?

Both problems are solved in this paper for system (1), which satisfies the following

Assumption 1. It is assumed that the system (1) can be decomposed into two subsystems

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{x}\,, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0\,, \quad \boldsymbol{x}(t) \in \mathbb{R}^{n_1} \tag{4}$$

$$\boldsymbol{s} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \qquad\qquad \boldsymbol{s}(t) \in \mathbb{R}^{p}$$

$$\dot{\boldsymbol{z}} = \boldsymbol{F}\boldsymbol{z} + \boldsymbol{G}\boldsymbol{s}\,, \quad \boldsymbol{z}(0) = \boldsymbol{z}_0\,, \quad \boldsymbol{z}(t) \in \mathbb{R}^{n_2} \tag{5}$$

$$\boldsymbol{y} = \boldsymbol{z} \tag{6}$$

where subsystem (4) is stable and subsystem (5) includes all the unstable modes of (1).

As shown in Section 6, this assumption is satisfied by an important class of unstable control systems.

2. A SIMPLE EXAMPLE

At the first sight, all systems of the form (4)–(6) seem to be stabilizable, because all the unstable state variables can be directly measured. However, the special structure of the system explained in Assumption 1 does not ensure the stabilizability by output feedback. To demonstrate this and to motivate our further investigations let us consider the single-input single-output (SISO) system with $n_1 = n_2 = 1$

$$\begin{aligned} \dot{x} &= ax + bu\,, \quad x(0) = x_0 \\ s &= x \\ \dot{z} &= fz + gs\,, \quad z(0) = z_0 \\ y &= z\,. \end{aligned} \tag{7}$$

Obviously, this system satisfies Assumption 1, if $a < 0$ and $f > 0$. Without loss of generality it is assumed that $b > 0$ and $g > 0$.

The closed-loop system (2), (7) is described by

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} a & bk \\ g & f \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}.$$

It is stable if and only if

$$f < -a \tag{8}$$

$$k < \frac{fa}{bg}\,. \tag{9}$$

Thus, the system (7) is stabilizable if and only if eqn. (8) holds, i.e. if the stability degree of the subsystem (5) is greater than the instability of the subsystem (5). Then the system can be stabilized by output feedback (2) with gains satisfying eqn. (9). Note that the absolute value of the gain must be greater than $|fa/bg|$.

This simple example points out that a system (1) of the form (4)–(6) can be stabilized by output feedback only under some restrictions concerning the eigenvalues or time constants, respectively, of both subsystems. These restrictions occur although the unstable modes of the system can be directly measured! The following investigations yield a generalization of the restriction (8) and a way to calculate stabilizing output feedback matrices.

3. NECESSARY CONDITION FOR STABILIZABILITY BY OUTPUT FEEDBACK

A necessary condition for the stabilizability of the system (4)–(6) is that all the unstable modes of (5) must be controllable by input $\boldsymbol{u}$ and observable by output $\boldsymbol{y}$. While the observability is ensured by eqn. (6), the controllability depends on the properties of both subsystems. It can be checked by means of the necessary and sufficient condition given in [4]. Here, the following assumptions are made, which are sufficient and "nearly" necessary for the controllability of the modes of (5) (cf. [4], Section 3.2).

Assumption 2. It is assumed that the static transition matrix $\boldsymbol{K}_s$ of subsystem (4)

$$\boldsymbol{K}_s = \boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \tag{10}$$

satisfies the condition

$$\operatorname{rg} \boldsymbol{K}_s = r \tag{11}$$

and that the pair $(\boldsymbol{F}, \boldsymbol{G})$ of subsystem (5) is controllable.

4. DESIGN OF A STABILIZING OUTPUT FEEDBACK

4.1. The robustness approach

The design should be carried out by means of the following philosophy, which is motivated by the example of Section 2. To stabilize the system (4)–(6) the unstable eigenvalues of the matrix $\boldsymbol{F}$ must be shifted into the left half complex plane. This problem would be trivial, if subsystem (4) has very small eigenvalues in relation to the eigenvalues of $\boldsymbol{F}$ and can be approximated by a static system

$$\boldsymbol{s} = \boldsymbol{K}_s\boldsymbol{u} . \tag{12}$$

Then the output feedback (2) would represent a state feedback to the system (5), (6), (12) and Assumption 2 would ensure arbitrary pole assignability. The controller could be chosen so as to give all the eigenvalues of the closed-loop system (2), (5), (6), (12) some real value $-\beta < 0$, i.e. there exists some transformation matrix $\boldsymbol{T}$ such that

$$\boldsymbol{T}^{-1}(\boldsymbol{F} + \boldsymbol{G}\boldsymbol{K}_s\boldsymbol{K})\,\boldsymbol{T} = \operatorname{diag} -\beta . \tag{13}$$

To stabilize the original system (4)–(6) rather than the modified system (5), (6), (12), this feedback must be robust enough to tolerate the dynamical effects of subsystem (4). A sufficient condition for this robustness leads to the following theorem, which describes the main result of this paper.

Theorem 1. Consider the system (1) satisfying Assumption 1 and Assumption 2. Choose the matrix $\tilde{K}$ such that all the eigenvalues of

$$\bar{F} = F + G\tilde{K}$$

have the same positive real value $-\beta < 0$. If the inequality

$$\lambda_p \left[\int_0^\infty \left| CA^{-1} \exp(At) B \right| \mathrm{d}t \left| K_s^+ \tilde{K} G \right| \right] < 1 \tag{14}$$

is satisfied, then the close-loop system (1), (2) is stable with

$$K = K_s^+ \tilde{K} . \tag{15}$$

Here, λ_p denotes the maximum eigenvalue (Perron root) of the indicated non-negative matrix. The symbol $|.|$ signifies that all elements of the matrix are replaced by their absolute values. K_s^+ is the pseudoinverse of K_s

$$K_s^+ = K_s'(K_s K_s')^{-1} . \tag{16}$$

This result will be proved in Section 4.3.

4.2. Design algorithm

Theorem 1 leads to an instructive design algorithm, which proceeds in the following steps:

Step 1. Decompose the overall system (1) into the stable subsystem (4) and the unstable subsystem (5). Check Assumption 1 and 2.

Step 2. Calculate a feedback matrix K so as to satisfy eqn. (13) for some $\beta > 0$ by means of design procedures for state feedback (cf. [9], [16]).

Step 3. Check the stability condition (14), (15). If eqn. (14) is not satisfied proceed with Step 2 choosing a smaller value of β.

This design procedure is very simple indeed. It can be used to solve different problems of feedback control (for examples see Section 5).

4.3. Proof of Theorem 1

To prove Theorem 1 the following lemma will be used.

Lemma 1. Consider a stable linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{17}$$

with unity feedback

$$u = y + v\,. \tag{18}$$

Then the closed-loop system (17), (18) is stable if the inequality

$$\lambda_p(|G(s)|) < 1 \tag{19}$$

holds for all s on the Nyquist contour $\mathfrak{D}$, where $G(s)$ is the transfer function matrix of system (17)

$$G(s) = C(sI - A)^{-1} B\,.$$

Proof. In the frequency domain the closed-loop system (17), (18) is described by

$$y(s) = G(s)\, u(s) \tag{17'}$$

and

$$u(s) = y(s) + v(s)\,. \tag{18'}$$

First, it will be proved that this system is input-output (I/0) stable if the inequality (19) holds.

According to the generalized Nyquist stability criterion used for stable system (17), the closed-loop system (17'), (18') is stable if and only if the graph of $\det(F(s))$ for $s \in \mathfrak{D}$ does not encircle the origin of the complex plane (cf. [9], [16]), where

$$F(s) = I - G(s)\,.$$

This criterion is satisfied if eqn. (19) holds, because

$$\operatorname{Re}[\det(F(s))] = \operatorname{Re}[\prod_i \lambda_i(F(s))] = \operatorname{Re}[\prod_i (1 - \lambda_i(G(s)))] \geqq$$
$$\geqq \prod_i (1 - |\lambda_i(G(s))|) \geqq \prod_i (1 - \lambda_p(|G(s)|)) \geqq 0$$

follows from (19) with $\lambda_i(.)$ denoting the eigenvalues of the given matrices.

Now, it will be proved that I/0-stability of the system (17'), (18') in the frequency domain implies Lyapunov-stability of the system (17), (18) in the time domain, which is characterized by the fact that all eigenvalues of the closed-loop system matrix

$$\bar{A} = A + BC$$

have negative real parts. To do this realize that if (17'), (18') is I/0-stable then all modes of (17), (18) that are controllable via $v(t)$ and observable via $y(t)$ have negative real parts. In order to prove Lemma 1 it must be shown that all other eigenvalues of $\bar{A}$ are stable.

Using Hautus' controllability criterion [7], the eigenvalues $\lambda_i(\bar{A})$ of the closed-loop system (17), (18) which are not controllable via $v(t)$ or not observable via $y(t)$ are given by

$$\operatorname{rg}(\bar{A} - \lambda_i I, B) < n \tag{20}$$
$$\operatorname{rg}(\bar{A}' - \lambda_i I, C') < n$$

respectively, where n is the order of the matrices A and $\bar{A}$. For these eigenvalues

$$\operatorname{rg}(A - \lambda_i I, B) = \operatorname{rg}(A - \lambda_i I + BC, B) = \operatorname{rg}(A - \lambda_i I, B) < n$$
$$\operatorname{rg}(\bar{A}' - \lambda_i I, C') = \operatorname{rg}(A' - \lambda_i I + C'B', C') = \operatorname{rg}(A' - \lambda_i I, C') < n$$

hold. Hence, these eigenvalues of $\bar{A}$ are also eigenvalues of A. Thus they are stable. This completes the proof. □

Proof of Theorem 1. The closed-loop system described by eqns. (2), (4)–(6) and (15) can be written as

$$(21)\qquad \begin{aligned} \dot{x} &= Ax + Bu \\ s_1 &= Cx + CA^{-1}Bu \end{aligned}$$

$$(22)\qquad \begin{aligned} \dot{z} &= (F + G\tilde{K})\, z + Gs_1 \\ u &= Kz\,, \end{aligned}$$

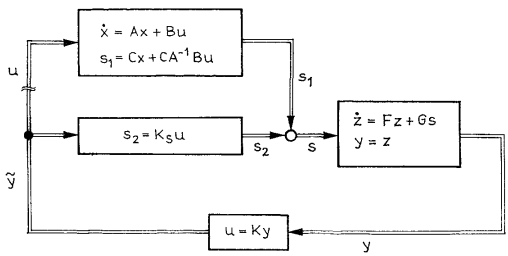

Fig. 1. Decomposition of the closed-loop system (2), (4)–(6) with $s = s_1 + s_2$.

where eqns. (15), (16) are used (Fig. 1). According to the location indicated in Figure 1, the system (21), (22) can be interpreted as open-loop system

$$(23)\qquad \begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A & 0 \\ GC & F + G\tilde{K} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B \\ GCA^{-1}B \end{pmatrix} u \\ \tilde{y} &= (0 \;\; K) \begin{pmatrix} x \\ z \end{pmatrix} \end{aligned}$$

and unity feedback

$$u = \tilde{y}\,.$$

The stability of this system can be proved by means of Lemma 1. In order to get a simple expression for the transfer function matrix $G(s)$ of the system (23), eqn. (23) is transformed by

$$\begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -T^{-1}GCA^{-1} & T^{-1} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

with $\boldsymbol{T}$ from (13) into

$$\begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \mathbf{0} \\ -\beta \boldsymbol{T}^{-1}\boldsymbol{G}\boldsymbol{C}\boldsymbol{A}^{-1} & -\beta \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} + \begin{pmatrix} \boldsymbol{B} \\ \mathbf{0} \end{pmatrix} u \tag{24}$$

$$\tilde{y} = (\boldsymbol{K}\boldsymbol{G}\boldsymbol{C}\boldsymbol{A}^{-1} \ \ \boldsymbol{K}\boldsymbol{T}) \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix}.$$

This system represents a chain connection of the subsystems

$$\dot{\bar{x}} = A\bar{x} + \boldsymbol{B}u \tag{25}$$

$$\bar{s} = \boldsymbol{C}A^{-1}\bar{x}$$

and

$$\dot{\bar{z}} = -\beta\bar{z} - \beta \boldsymbol{T}^{-1}\boldsymbol{G}\bar{s} \tag{26}$$

$$\tilde{y} = \boldsymbol{K}\boldsymbol{T}\bar{z} + \boldsymbol{K}\boldsymbol{G}\bar{s}\,.$$

Hence the transfer function matrix $\boldsymbol{G}(s)$ of the system (23) is given by

$$\boldsymbol{G}(s) = \boldsymbol{G}_2(s)\,\boldsymbol{G}_1(s) \tag{27}$$

with

$$\boldsymbol{G}_1(s) = \int_0^\infty \boldsymbol{C}A^{-1} \exp(At)\,\boldsymbol{B} \exp(-st)\,\mathrm{d}t \tag{28}$$

$$\boldsymbol{G}_2(s) = \boldsymbol{K}\boldsymbol{G} - \beta \boldsymbol{K}\boldsymbol{T}(s\boldsymbol{I} + \beta \boldsymbol{I})^{-1}\,\boldsymbol{T}^{-1}\boldsymbol{G}\,. \tag{29}$$

From (28)

$$|\boldsymbol{G}_1(s)| \leqq \int_0^\infty |\boldsymbol{C}A^{-1} \exp(At)\,\boldsymbol{B}|\,\mathrm{d}t \tag{30}$$

follows. Eqn. (29) yields

$$|\boldsymbol{G}_2(s)| = |\beta/(s + \beta)|\,|\boldsymbol{K}\boldsymbol{G}| \leqq |\boldsymbol{K}\boldsymbol{G}| \quad \text{for} \quad s \in \mathfrak{D}\,, \quad \beta > 0\,. \tag{31}$$

Now, the stability condition (14) of Theorem 1 follows directly from Lemma 1. □

5. A SUFFICIENT CONDITIONS FOR STABILIZABILITY BY OUTPUT FEEDBACK

Theorem 1 can be used to derive a sufficient condition for the stabilizability by output feedback. Consider the system

$$\dot{x} = \operatorname{diag}(-\lambda_i)\,x + \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{n_1} \end{pmatrix} u \tag{32}$$

$$s = (c_1 \dots c_{n_1})\,x$$

$$
(33) \qquad \dot{z} = \begin{pmatrix} 0 & 1 & 0 \ldots & 0 \\ & & \ddots & \\ -a_1 & -a_2 & \ldots & -a_{n_2} \end{pmatrix} z + \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} s
$$

$$
y = z
$$

with $\dim u = \dim s = 1$, which satisfies the Assumptions 1 and 2. The design procedure of Section 4.2 shall be used to find a vector k' such that the feedback

$$
u = k'y
$$

stabilizes the given system (32), (33). From (10) and (15)

$$
(34) \qquad \begin{aligned} k' &= \frac{1}{k_s} \tilde{k}' \\ k_s &= \sum_{i=1}^{n_1} c_i \end{aligned}
$$

follow. $\tilde{k}$ has to be chosen such that the matrix

$$
\tilde{F} = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & \cdot 1 \\ -a_1 & -a_2 & \ldots & -a_{n_2} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} (\tilde{k}_1 \ldots \tilde{k}_{n_2}) = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & \cdot 1 \\ -\tilde{a}_1 & -\tilde{a}_2 & \ldots & -\tilde{a}_{n_2} \end{pmatrix}
$$

has an eigenvalue $-\beta$ with multiplicity n_2. As the scalars a_i and $\tilde{a}_i$ in F or $\tilde{F}$, respectively, are the coefficients of the characteristic polynomial of F or $\tilde{F}$, respectively, $\tilde{k}$ can be calculated easily. In particular,

$$
(35) \qquad \tilde{k}_{n_2} = -n\beta_2 - \sum_{i=1}^{n_2} \omega_i = -(n_2\beta + \sum_{i=1}^{n_2} \operatorname{Re}(\omega_i))
$$

holds, where ω_i are the eigenvalues of F.

To check the stability condition (14), use the relation

$$
(36) \qquad \int_0^\infty \left| (c_1 \ldots c_{n_1}) \operatorname{diag} \frac{1}{-\lambda_i} \operatorname{diag} \exp(-\lambda_i t) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{n_1} \end{pmatrix} \right| \mathrm{d}t \leqq \sum_{i=1}^{n_1} \left| \frac{c_i}{\lambda_i} \right| .
$$

The closed-loop system is stable, if for some $\beta > 0$ the inequality

$$
(37) \qquad \frac{\sum_{i=1}^{n_1} \left| \frac{c_i}{\lambda_i} \right|}{\left| \sum_{i=1}^{n_1} c_i \right|} (n_2\beta + \sum_{i=1}^{n_2} \operatorname{Re}(\omega_i)) < 1
$$

holds. Hence, a suffcient condition for stabilizability is given by

Theorem 2. A sufficient condition for the stabilizability of systems (4)–(6) with dim $\boldsymbol{u}$ = dim $\boldsymbol{s}$ = 1 satisfying Assumptions 1 and 2 is given by

$$(38) \qquad \sum_{i=1}^{n_2} \operatorname{Re}(\omega_i) < \frac{\left|\sum_{i=1}^{n_1} c_i\right|}{\sum_{i=1}^{n_1} \left|\frac{c_i}{\lambda_i}\right|},$$

where c_i and λ_i are defined by the canonical form (32) of subsystem (4).

This condition is a generalization of the inequality (8). As eqn. (8) states a necessary and sufficient condition for stabilizability in case of $n_1 = n_2 = 1$, eqn. (38) can be considered as a sufficient and "nearly" necessary condition for the stabilizability of higher order systems. Therefore, the stability condition of Theorem 1 is not very conservative.

The result of Theorem 2 can be interpreted in the sense of the design philosophy described in Section 4.1. The overall system is stabilizable by constant output feedback if the subsystem (4) is fast enough in relation to subsystem (5), so that it can be approximated by a static system (12). Inequality (38) describes, what "fast enough" means. It provides a bound on the dynamics of subsystem (4) that can be tolerated by the controller. In contrast to the singular perturbation approach, which proceeds with a similar design philosophy, the robustness approach, as adopted here, leads to quantitative bounds rather than qualitative statements.

Note that eqn. (38) can be satisfied if the subsystem (5) has not only unstable eigenvalues but stable ones too. Then $\sum \operatorname{Re}(\omega_i)$ may be very small even if the overall system has severely unstable modes. Therefore, the system (1) can be made to satisfy condition (38) by transforming some stable eigenvalues with small real part from subsystem (4) into subsystem (5). The only price for it is an additional measurement of these states.

The result of Theorem 2 can be extended to multi-input, multi-output systems using similar canonical forms of both subsystems as in eqns. (32) and (33) [11].

6. APPLICATIONS

To demonstrate the usefulness of the proposed design procedure and the sufficient stabilizability condition several applications are considered in the following section.

The influence of the actuator dynamics in feedback systems

Identify subsystem (5) with the real plant and subsystem (4) with the actuator of a feedback system. If the actuator dynamics are neglected the controller (2) represents a state feedback to the resulting plant (4), (12). Then Theorems 1 and 2 can be used to investigate the stabilization problem in the presence of actuator dynamics. Accordingly, the plant remains stabilizable, if the actuator is fast enough so as to satisfy the quantitative bound on its dynamical behaviour given by eqn. (38). Then a stabilizing output feedback of the overall system can be found by the algorithm described in Section 4.2.

The general servomechanism problem

A problem that often occurs in linear multivariable control is to find a feedback such that the output $\boldsymbol{y}(t)$ tracks asymptotically a given command input $\boldsymbol{v}(t)$ independently of unmeasurable disturbances. If the command and disturbance signals are solutions of homogenenous linear differential equations

$$\dot{\boldsymbol{r}} = \boldsymbol{R}\boldsymbol{r}\,, \quad \boldsymbol{r}(\boldsymbol{0}) = \boldsymbol{r}_0 \tag{39}$$

$$\boldsymbol{v} = \boldsymbol{V}\boldsymbol{r}$$

where all the eigenvalues of $\boldsymbol{R}$ have non-negative real parts, then the feedback must include an internal model (servocompensator) of the external signals [5]

$$\dot{\boldsymbol{z}} = \operatorname{diag} \boldsymbol{R}\boldsymbol{z} + \boldsymbol{G}(\boldsymbol{y} - \boldsymbol{v})\,. \tag{40}$$

The block diagonal matrix $\operatorname{diag} \boldsymbol{R}$ consists of r blocks, and $\boldsymbol{G}$ is chosen such that the pair $(\operatorname{diag} \boldsymbol{R}, \boldsymbol{G})$ is controllable. If a compensator with input $\boldsymbol{y}$ and $\boldsymbol{z}$ and output $\boldsymbol{u}$ is used that ensures closed-loop stability for $\boldsymbol{v} \equiv \boldsymbol{0}$, then asymptotic regulation occurs for $\boldsymbol{v}(t)$ from (39)

$$\boldsymbol{y}(t) - \boldsymbol{v}(t) \to \boldsymbol{0} \quad \text{for} \quad t \to \infty$$

for all initial states $\boldsymbol{x}_0$ and $\boldsymbol{r}_0$.

The overall feedback controller consists of the servocompensator (40) as well as the stabilizing compensator. Whereas the former is given by the properties of the external signals, there is some freedom in the design of the latter. In order to get a low dynamical order of the controller the question arises, whether a static feedback

$$\boldsymbol{u} = \boldsymbol{K}_1\boldsymbol{y} + \boldsymbol{K}_2\boldsymbol{z} \tag{41}$$

can be used rather than a dynamic compensator to stabilize the extended plant (1), (40). It can be answered by means of Theorems 1 and 2.

Interpreting the real plant (1) as subsystem (4) and the servocompensator (40) as subsystem (5) the results of the Sections 4 and 5 lead to the following

Corollary 1. Consider a SISO plant (1). Assume that the plant (1) is stable and satisfies eqn. (11). Then there exists a static feedback

$$\boldsymbol{u} = \boldsymbol{K}_2\boldsymbol{z} \tag{42}$$

stabilizing the extended plant (1), (40), if the inequality (38) is satisfied, where ω_i are the eigenvalues of the servocompensator (40) and (λ_i, c_i) are given by the canonical form (32) of the plant (1). A stabilizing feedback (42) can be found by means of the procedure of Section 4.2.

If the output y is fed back as well (cf. eqn. (40)) then the freedom in choosing the matrix K_1 can be utilized to make the plant as fast as possible in order to satisfy the inequality (38).

Multivariable I-controller

If the servomechanism problem is considered for step signals $v(t) = \bar{v}\,\sigma(t)$, which are commonly used in process control, the servocompensator (40) has the form

$$\dot{z} = y - v\,. \tag{43}$$

Then condition (11) must *necessarily* be satisfied to ensure the stabilizability of the extended plant (1), (43) by static output feedback (41) [12]. Using Theorems 1 and 2 with $F = 0$, $G = I$ and $K_2 = -\beta K_s^+$ eqn. (38) is satisfied, and the following corollary can be derived.

Corollary 2. For every stable system (1) satisfying eqn. (11) there exists a multivariable I-controller (42), (43) such that the closed-loop system (1), (42), (43) is stable. Corresponding controller matrices K_2 are given by

$$K_2 = -\beta K_s^+ \tag{44}$$

with $0 < \beta < \bar{\beta}$

$$\bar{\beta} = 1 \Big/ \lambda_p \left[\int_0^\infty |CA^{-1} \exp(At)\, B|\, \mathrm{d}t\, |K_s^+| \right]. \tag{45}$$

While the existence of such an I-controller is already known [6], [12], the eqns. (44) and (45) give a broader class of controller matrices than the results of [12], [13].

This example gives a further argument that the stability condition (14) given in Theorem 1 is not very conservative, although it is sufficient but in general not necessary for the stability of the closed-loop system.

CONCLUSIONS

Using the methods of robust control a design procedure and a sufficient condition for the stabilizability of a class of unstable linear systems by means of constant-gain output feedback have been given. As demonstrated by several examples the class of systems considered here is important e.g. in process control. The main advantage of the obtained results is the possibility to interprete the stabilizability condition directly in terms of the system properties. The design procedure is instructive and very simple.

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Dr. sc. techn. Jan Lunze, Zentralinstitut für Kybernetik und Informationsprozesse der Akademie der Wissenschaften der DDR, Kurstraße 33, DDR-1086 Berlin. German Democratic Republic.