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DUAL METHOD FOR SOLVING A SPECIAL PROBLEM OF QUADRATIC PROGRAMMING AS A SUBPROBLEM AT LINEARLY CONSTRAINED NONLINEAR MINIMAX APPROXIMATION

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The paper describes the dual method for solving a special problem of quadratic programming as a subproblem at linearly constrained nonlinear minimax approximation. Complete algorithm of the dual method is presented and its convergence after a finite number of steps is proved.

1. INTRODUCTION

This paper refers to a special problem of quadratic programming which occurs as a subproblem at linearly constrained nonlinear minimax approximation where a point $x^* \in R_n$ is sought such that

$$(1.1) \qquad \begin{cases} F(x^*) = \min\limits_{x \in L_n} \left(\max\limits_{i \in M_1} f_i(x) \right) \\ \text{where} \\ L_n = \{x \in R_n : a_i^T x \leqq b_i,\ i \in M_2\}. \end{cases}$$

In (1.1), $f_i(x)$, $i \in M_1$ are real-valued functions defined in the n-dimensional vector space R_n with continuous second-order derivatives and $M_1 \cup M_2 = \{1, \ldots, m\}$, $M_1 \cap M_2 = \emptyset$. Recently the problem (1.1) has been attracting a considerable attention. To solve this problem, several approaches have been developed, especially the product form of variable metric methods with generalized reduced gradients [4], the methods of recursive linear programming [6] and the methods of recursive quadratic programming [3].

The methods of recursive quadratic programming for solving the problems of minimax approximation were developed in analogy with their original application in the field of nonlinear programming, where we seek a pair $(x^*, z^*) \in N_{n+1}$ such that

$$(1.2) \qquad \begin{cases} z^* = \min\limits_{(x,z) \in N_{n+1}} z \\ \text{where} \\ N_{n+1} = \{(x, z) \in R_{n+1} : f_i(x) \leqq z,\ i \in M_1;\ a_i^T x \leqq b_i,\ i \in M_2\} \end{cases}$$

Applying the method of recursive quadratic programming [7] to the problem (1.2) we obtain one of the methods described in [3]. This method can be described roughly in the following way:

Step 1: We choose an initial feasible point $x \in L_n$ and an initial symmetric positive definite matrix G. We compute $f_i = f_i(x)$, $i \in M_1$, $a_i = g_i(x)$, $i \in M_1$ and $F = F(x) = \max_{i \in M_1} f_i(x)$ ($g_i(x)$ is the gradient of $f_i(x)$ at the point $x \in R_n$).

Step 2: We find the pair $(s, z) \in R_{n+1}$ which is the solution of the quadratic programming problem (1.3).

Step 3: If $\|s\| \leqq \varepsilon$ where ε is a small enough positive number, the computation is terminated else we find a steplength α, $0 < \alpha \leqq 1$, for which

$$F(x + \alpha s) \leqq F(x) - \eta \alpha s^{\mathrm{T}} G s$$

where $0 < 2\eta < 1$. Taking $x^+ = x + \alpha s$, we compute $f_i^+ = f_i(x^+)$, $i \in M_1$, $a_i^+ = g_i(x^+)$, $i \in M_1$ and $F^+ = F(x^+) = \max_{i \in M_1} f_i(x^+)$.

Step 4: We update the matrix G to make it positive definite and as good approximate as possible to the Hessian matrix of Lagrangian function for (1.2). Usually this purpose is attained by means of quasi-Newton corrections determined by using the differences $x^+ - x$, $a_i^+ - a_i$, $i \in M_1$ and Lagrange multipliers for quadratic programming problem (1.3).

Step 5: We set $x = x^+, f_i = f_i^+, i \in M_1, a_i = a_i^+, i \in M_1, F = F^+$ and go to Step 2.

The most important step of this method is the solution of quadratic programming subproblem in which we seek a pair $(s^*, z^*) \in R_{n+1}$ such that

$$(1.3) \quad \begin{cases} \varphi(s^*, z^*) = \min\limits_{(s,z) \in L_{n+1}} \varphi(s, z) \\ \text{where} \\ \varphi(s, z) = \frac{1}{2} s^{\mathrm{T}} G s + z \\ \text{and} \\ L_{n+1} = \{(s, z) \in R_{n+1} : f_i + a_i^{\mathrm{T}} s \leqq e_i z, \quad i \in M_1 \cup M_2\} \end{cases}$$

and where $f_i = f_i(x)$, $e_i = 1$ for $i \in M_1$ and $f_i = a_i^{\mathrm{T}} x - b_i$, $e_i = 0$ for $i \in M_2$.

It is apparent that (1.3) is a quadratic programming problem with a singular positive semidefinite matrix. Effective methods solving this problem require non-singularity, however, owing to very specific form of the matrix in our case, it is possible to develop methods overcoming the singularity drawback. In this paper, we are going to present one of these methods, which is based on the solution of dual quadratic programming problem.

The fact that the matrix G is positive definite implies that the problem (1.3) is convex and therefore we may apply the duality theory to it [1]. Thus we obtain a dual quadratic programming problem, where we seek a vector $u^* \in R_m$ such that

$$
(1.4)\qquad \begin{cases} \psi(u^*) = \min\limits_{u \in L_m} \psi(u) \\ \text{where} \\ \psi(u) = \frac{1}{2} u^{\mathrm{T}} A^{\mathrm{T}} H A u - f^{\mathrm{T}} u \\ \text{and} \\ L_m = \{u \in R_m : e^{\mathrm{T}} u = 1,\ u \geqq 0\} \end{cases}
$$

Here $H = G^{-1}$, $A = [a_1, \ldots, a_m]$ is a matrix the columns of which are vectors a_i, $i \in M_1 \cup M_2$ and furthermore

$$
f = \begin{bmatrix} f_1 \\ \hline f_m \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ \hline e_m \end{bmatrix}
$$

where $f_i = f_i(x)$, $e_i = 1$ for $i \in M_1$ and $f_i = a_i^{\mathrm{T}} x - b_i$, $e_i = 0$ for $i \in M_2$. The solution of the problem (1.3) can be obtained from the solution of the problem (1.4) by means of

$$
(1.5)\qquad \begin{cases} s^* = -HAu^* \\ \text{and} \\ z^* = f^{\mathrm{T}} u^* - (u^*)^{\mathrm{T}} A^{\mathrm{T}} H A u^* \end{cases}
$$

as it follows from the theory of duality. The vector u^* which is the solution of (1.4) is also the optimal vector of Lagrange multipliers for (1.3).

The problem (1.4) is a convex one. Hence the vector $u^* \in R_m$ is the solution of (1.4) if and only if Kuhn-Tucker conditions are valid (see [1]), i.e. if and only if

$$
(1.6)\qquad \begin{cases} e^{\mathrm{T}} u^* = 1 \\ u^* \geqq 0 \end{cases}
$$

and there exists a number z^* such that

$$
(1.7)\qquad \begin{cases} v^* = A^{\mathrm{T}} H A u^* - f + z^* e \geqq 0 \\ (v^*)^{\mathrm{T}} u^* = 0 \end{cases}
$$

The vector v^* is the vector of Lagrange multipliers for (1.4). Conditions (1.6) and (1.7) together imply that z^* in (1.7) is identical with the z^* in (1.5). This in turn implies that v^* is, at the same time, the vector of values of constraints of (1.3).

The dual method for solving (1.3) that is under examination in this paper is essentially the method of active constraints applied to (1.4) and is similar to the dual method described in [5]. Main difference between the method described in this paper and the method [5], which is conditioned by the linear constraints, consists in the fact, that (1.3) need not have any feasible solution. Further advantage of the new method as compared with the method [5] consists in a combination of non-singular and singular cases by introducing an artificial parameter $\mu > 0$. This improvement considerable simplifies the algorithm, which becomes more compact.

The algorithm described in this paper was verified in conjunction with the method for linearly constrained nonlinear minimax approximation. The results obtained will be presented in a separate paper [9].

2. ANALYSIS OF THE DUAL METHOD

Let $M = M_1 \cup M_2$ and $I \subset M$. Let $D(I)$ denotes the problem which results if we substitute I for M in the problem (1.4). Let $P(I)$ has the similar meaning with respect to the problem (1.3). The problem $D(I)$ is dual to the problem $P(I)$.

Let the Lagrange multipliers u_i, $i \in I$ be the solution of the problem $D(I)$ and the pair $(s, z) \in R_{n+1}$ be the solution of the problem $P(I)$. In order to simplify the notation we can introduce symbols u denoting the vector containing all u_i, $i \in I$, f denoting the vector of f_i, $i \in I$, e denoting the vector of e_i, $i \in I$ and A denoting the matrix containing a_i, $i \in I$ as its columns. Obviously $e_i = 1$ for $i \in M_1$ and $e_i = 0$ for $i \in M_2$. We assume troughout this section that $e \neq 0$, i.e. $I \cap M_1 \neq \emptyset$. Denote

$$\tilde{A} = \begin{bmatrix} A \\ -e^{\mathrm{T}} \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} H, & 0 \\ 0, & \mu \end{bmatrix}$$

where H is the symmetric positive definite matrix as in (1.4) and where $\mu > 0$ is some positive number. We assume troughout this section that $\tilde{A}$ has linearly independent columns. In this case $\tilde{A}^{\mathrm{T}}\tilde{H}\tilde{A}$ is positive definite and we can define the matrices

$$\left\{ \begin{array}{l} C = (\tilde{A}^{\mathrm{T}}\tilde{H}\tilde{A})^{-1} = (A^{\mathrm{T}}HA + \mu ee^{\mathrm{T}})^{-1} \\ \tilde{Q} = \tilde{H} - \tilde{H}\tilde{A}C\tilde{A}^{\mathrm{T}}\tilde{H} \\ \text{and the vector} \\ p = Ce \end{array} \right. \tag{2.1}$$

Clearly $\tilde{Q}\tilde{A} = 0$ and $\tilde{Q}\tilde{G}\tilde{Q} = \tilde{Q}$, where $\tilde{G} = \tilde{H}^{-1}$.

Definition 2.1. We say that Lagrange multipliers u_i, $i \in I$ are a basic solution of the problem $D(I)$ and that the pair $(s, z) \in R_{n+1}$ is a basic solution of the problem $P(I)$ if $v_i = a_i^{\mathrm{T}}HAu - f_i + ze_i = 0$ for all indices $i \in I$.

Lemma 2.1. Let Lagrange multipliers u_i, $i \in I$ be a basic solution of the problem $D(I)$ and let the pair $(s, z) \in R_{n+1}$ be a basic solution of the problem $P(I)$. Then

$$\left\{ \begin{array}{l} z = \mu + \dfrac{p^{\mathrm{T}}f - 1}{p^{\mathrm{T}}e} \\ u = Cf - (z - \mu)\, p \\ s = -HAu \end{array} \right. \tag{2.2}$$

Proof. Since $v_i = 0$ for all indices $i \in I$, we have $A^{\mathrm{T}}HAu - f + ze = 0$ by (1.7). Applying (1.6) we obtain

$$\tilde{A}^{\mathrm{T}}\tilde{H}\tilde{A}u - f + (z - \mu)\, e = 0$$

so that $u = Cf - (z - \mu)\, p$. Furthermore $1 = e^{\mathrm{T}}u = p^{\mathrm{T}}f - (z - \mu)\, p^{\mathrm{T}}e$ by (1.6), which means $z = \mu + (p^{\mathrm{T}}f - 1)/p^{\mathrm{T}}e$. Mutual duality of $D(I)$ and $P(I)$ brings $s = -HAu$ (see (1.5)). □

The formulae (2.2) may be formally applied to an arbitrary subset $I \subset M = M_1 \cup M_2$. However, a situation can arise in which $u_i \geqq 0$ does not hold for all indices $i \in I$. In this case there is no basic solution of the problem $D(I)$. Each problem $D(I)$, where $I \cap M_1 = \{l\}$ is a single-element subset of M_1, has a basic solution, for $u_l = 1$ holds of necessity by (1.6).

Suppose Lagrange multipliers u_i, $i \in I$ are a basic solution of the problem $D(I)$ and the pair $(s, z) \in R_{n+1}$ is a basic solution of the problem $P(I)$. If $v_i = a_i^{\mathrm{T}} HAu - f_i + ze_i \geqq 0$ for all indices $i \in M \setminus I$, then the vector $u^* = [u^{\mathrm{T}}, 0]^{\mathrm{T}}$ is the solution of the problem (1.4) (assuming a suitable ordering of indices) and the pair $(s^*, z^*) = (s, z)$ is the solution of the problem (1.3). In the other case there exists an index $k \in M \setminus I$ such that $v_k = a_k^{\mathrm{T}} HAu - f_k + ze_k < 0$, which suggests that the index k has to be added to I.

Let us set $I^+ = I \cup \{k\}$. Since the problem $D(I^+)$ need not have any basic solution, we want to find a subset $\bar{I} \subset I^+$, $k \in \bar{I}$, such that the problem $D(\bar{I})$ may have a basic solution and, at the same time, $\varphi(\bar{s}, \bar{z}) < \varphi(s, z)$, where $(\bar{s}, \bar{z}) \in R_{n+1}$ is a basic solution of the problem $P(\bar{I})$ and $\varphi(s, z)$ is defined by (1.3).

Let $D_\lambda(I^+)$ be the problem we obtain from $D(I^+)$ after substituting $f_i(\lambda) = f_i + (1 - \lambda) v_i$, $i \in I^+$ for f_i, $i \in I^+$ and let $P_\lambda(I^+)$ have an analogical meaning with respect to $P(I^+)$. Let us suppose $0 \leqq \lambda \leqq 1$. Let Lagrange multipliers $u_i(\lambda)$, $i \in I^+$ be the solution of the problem $D_\lambda(I^+)$ and let $(s(\lambda), z(\lambda)) \in R_{n+1}$ be the solution of the problem $P_\lambda(I^+)$. Let $u(\lambda)$ denote the vector containing $u_i(\lambda)$, $i \in I$.

Lagrange multipliers $u_i(0) = u_i$, $i \in I$ and $u_k(0) = 0$ are a basic solution of the problem $D_0(I^+)$. We want to find the maximum value of the parameter λ such that Lagrange multipliers $u_i(\lambda)$, $i \in I^+$ are a basic solution of the problem $D_\lambda(I^+)$.

Lemma 2.2. Suppose Lagrange multipliers $u_i(0)$, $i \in I^+$, $u_i(\lambda)$, $i \in I^+$ and the pairs $(s(0), z(0)) \in R_{n+1}$, $(s(\lambda), z(\lambda)) \in R_{n+1}$ are basic solutions of the problems $D_0(I^+)$, $D_\lambda(I^+)$ and $P_0(I^+)$, $P_\lambda(I^+)$ respectively. Let us denote $\tilde{a}_k = [a_k^{\mathrm{T}}, -e_k]^{\mathrm{T}}$ and introduce

$$
\text{(2.3)} \qquad \begin{cases} q_k = C\tilde{A}^{\mathrm{T}}\tilde{H}\tilde{a}_k = CA^{\mathrm{T}}Ha_k + \mu e_k p \\ \beta_k = e_k - e^{\mathrm{T}} q_k \\ \gamma_k = \beta_k / p^{\mathrm{T}} e \\ \delta_k = \tilde{a}_k^{\mathrm{T}}\tilde{Q}\tilde{a}_k = a_k^{\mathrm{T}} H(a_k - Aq_k) + \mu e_k \beta_k \end{cases}
$$

assuming $\beta_k \gamma_k + \delta_k \neq 0$. Then

$$
\text{(2.4)} \qquad \begin{cases} u(\lambda) = u(0) - \alpha(q_k + \gamma_k p) \\ u_k(\lambda) = u_k(0) + \alpha \\ z(\lambda) = z(0) + \alpha\gamma_k \end{cases}
$$

where

$$
\text{(2.5)} \qquad \alpha = -\lambda \frac{v_k}{\beta_k \gamma_k + \delta_k}
$$

Proof. Using the definition of a basic solution we can write

$$\begin{bmatrix} \tilde{A}^T\tilde{H}\tilde{A}, & \tilde{A}^T\tilde{H}\tilde{a}_k \\ \tilde{a}_k^T\tilde{H}\tilde{A}, & \tilde{a}_k^T\tilde{H}\tilde{a}_k \end{bmatrix} \begin{bmatrix} u(\lambda) \\ u_k(\lambda) \end{bmatrix} = \begin{bmatrix} f - (z(\lambda) - \mu)\, e \\ f_k + (1 - \lambda)\, v_k - (z(\lambda) - \mu)\, e_k \end{bmatrix}$$

(see also the proof of Lemma 2.1). Subtracting these equations for various values of the parameter λ, we get the equation

$$\begin{bmatrix} \tilde{A}^T\tilde{H}\tilde{A}, & \tilde{A}^T\tilde{H}\tilde{a}_k \\ \tilde{a}_k^T\tilde{H}\tilde{A}, & \tilde{a}_k^T\tilde{H}\tilde{a}_k \end{bmatrix} \begin{bmatrix} u(\lambda) - u(0) \\ u_k(\lambda) - u_k(0) \end{bmatrix} = - \begin{bmatrix} (z(\lambda) - z(0))\, e \\ \lambda v_k + (z(\lambda) - z(0))\, e_k \end{bmatrix}$$

which has the solution (2.4) with α given by (2.5) (see also [5], proof of Lemma 2.2). □

Lemma 2.3. Let the assumptions of Lemma 2.2 hold. Then

$$\varphi(s(\lambda), z(\lambda)) = \varphi(s(0), z(0)) + \tfrac{1}{2}\alpha(\beta_k\gamma_k + \delta_k)\,(u_k(\lambda) + u_k(0)) \tag{2.6}$$

Proof. Using (1.5) and (2.4) we get

$$\begin{aligned} s(\lambda) - s(0) &= -HA(u(\lambda) - u(0)) - Ha_k(u_k(\lambda) - u_k(0)) = \\ &= \alpha HA(q_k + \gamma_k p) - \alpha Ha_k \end{aligned} \tag{2.7}$$

and

$$s(0) = -HA\,u(0) - Ha_k\,u_k(0) \tag{2.8}$$

Considering (2.1) we obtain

$$A^THAq_k = \tilde{A}^T\tilde{H}\tilde{A}q_k - \mu ee^Tq_k = A^THa_k + \mu\beta_k e$$

and

$$A^THAp = \tilde{A}^T\tilde{H}\tilde{A}p - \mu ee^Tp = (1 - \mu e^Tp)\, e$$

which, applying (2.3), yields

$$A^THA(q_k + \gamma_k p) = A^THa_k + \gamma_k e \tag{2.9}$$

Now, using (2.3) and (2.7)–(2.9), we can write

$$\begin{aligned} (s(\lambda) - s(0))^T\, G\, s(0) &= -\alpha((q_k + \gamma_k p)^T A^T - a_k^T)\, HA\, u(0) - \\ - \alpha((q_k + \gamma_k p)^T A^T - a_k^T)\, Ha_k\, u_k(0) &= -\alpha(a_k^THA + \gamma_k e^T - a_k^THA)\, u(0) - \\ - \alpha(\mu e_k\beta_k - \delta_k + \gamma_k e^Tq_k - \mu e_k\gamma_k e^Tp)\, u_k(0) &= \\ = -\alpha\gamma_k + \alpha(e_k\gamma_k - e^Tq_k\gamma_k + \delta_k)\, u_k(0) &= \\ = -\alpha\gamma_k + \alpha(\beta_k\gamma_k + \delta_k)\, u_k(0) & \end{aligned}$$

and

$$\begin{aligned} (s(\lambda) - s(0))^T\, G(s(\lambda) - s(0)) &= \alpha^2((q_k + \gamma_k p)^T A^T - a_k^T)\, H(A(q_k + \gamma_k p) + a_k) = \\ = \alpha^2(a_k^THA + \gamma_k e^T)\,(q_k + \gamma_k p) + \alpha^2 a_k^THa_k - 2\alpha^2 a_k^THA(q_k + \gamma_k p) &= \\ = \alpha^2(\gamma_k e^Tq_k + \beta_k\gamma_k) + \alpha^2(\delta_k - \mu e_k\beta_k - \gamma_k e^Tq_k + \mu e_k\gamma_k e^Tp) = \alpha^2(\beta_k\gamma_k + \delta_k) & \end{aligned}$$

which, applying (1.3), yields

$$\varphi(s(\lambda), z(\lambda)) = \varphi(s(0), z(0)) + (s(\lambda) - s(0))^{\mathrm{T}} G\, s(0) +$$
$$+ \tfrac{1}{2}(s(\lambda) - s(0))^{\mathrm{T}} G(s(\lambda) - s(0)) + (z(\lambda) - z(0)) =$$
$$= \varphi(s(0), z(0)) - \alpha\gamma_k + \alpha(\beta_k\gamma_k + \delta_k)\, u_k(0) + \tfrac{1}{2}\alpha(\beta_k\gamma_k + \delta_k)\,(u_k(\lambda) -$$
$$- u_k(0)) + \alpha\gamma_k = \varphi(s(0), z(0)) + \tfrac{1}{2}\alpha(\beta_k\gamma_k + \delta_k)\,(u_k(\lambda) + u_k(0))$$

and the lemma is proved. □

The maximum value of the parameter λ, for which the problem $D_\lambda(I^+)$ has a basic solution, is determined by the condition $u(\lambda) \geqq 0$. Let us write

$$\text{(2.10)} \qquad \begin{cases} \alpha_1 = -\dfrac{v_k}{\beta_k\gamma_k + \delta_k} \\ \alpha_2 = \dfrac{u_j(0)}{q_{kj} + \gamma_k p_j} = \min\limits_{i \in \bar{I}} \left(\dfrac{u_i(0)}{q_{ki} + \gamma_k p_i} \right) \end{cases}$$

where $\bar{I} = \{i \in I,\ q_{ki} + \gamma_k p_i > 0\}$, q_{ki} is the ith component of the vector q_k and p_i is the ith component of the vector p. Let us set $\alpha = \min(\alpha_1, \alpha_2)$. Then the maximum value λ_0 of the parameter λ is defined as $\lambda_0 = \alpha/\alpha_1$.

When $\alpha = \alpha_1$ (i.e. $\lambda_0 = 1$), Lagrange multipliers $u_i(\lambda_0)$, $i \in I^+$ are a basic solution of the problem $D(I^+)$ and we can set $\bar{I} = I^+$. In this case we can construct matrices $\bar{A} = [A, a_k]$ and

$$\text{(2.11)} \qquad \bar{C} = \begin{bmatrix} C + \dfrac{q_k q_k^{\mathrm{T}}}{\delta_k}, & -\dfrac{q_k}{\delta_k} \\ -\dfrac{q_k^{\mathrm{T}}}{\delta_k}, & \dfrac{1}{\delta_k} \end{bmatrix}$$

(for the derivation of this formula see for instance [2]). We show that the assumption $\beta_k\gamma_k + \delta_k \neq 0$ implies the inequality $\delta_k \neq 0$, so that the formula (2.11) is valid. Assume on the contrary that $\delta_k = 0$. Then there exists a nonzero vector w such that $\tilde{a}_k = \tilde{A}w$ (see (2.3)), i.e.

$$a_k = Aw$$
$$e_k = e^{\mathrm{T}}w$$

Hence we obtain by (2.1) and (2.3)

$$\beta_k = e_k - e^{\mathrm{T}}q_k = e^{\mathrm{T}}w - e^{\mathrm{T}}C\tilde{A}^{\mathrm{T}}\tilde{H}\tilde{A}w = e^{\mathrm{T}}w - e^{\mathrm{T}}w = 0$$

so that $\delta_k + \beta_k\gamma_k = 0$, which is in contradiction with the assumption $\delta_k + \beta_k\gamma_k \neq 0$.

When $\alpha \neq \alpha_1$ (i.e. $\lambda_0 < 1$), we have $u_j(\lambda_0) = 0$ by (2.10). Let us set $I_1 = I \setminus \{j\}$, $I_1^+ = I^+ \setminus \{j\}$, and $v_i^{(1)} = (1 - \lambda_0)\, v_i$, $i \in I_1^+$. Let $D_\lambda(I_1^+)$ denotes the problem resulting from the problem $D(I_1^+)$ after substituting $f_i(\lambda) = f_i + (1 - \lambda)\, v_i^{(1)}$, $i \in I_1^+$ (where $0 \leqq \lambda \leqq 1$) for f_i, $i \in I_1^+$. Lagrange multipliers $u_i^{(1)}(0) = u_i(\lambda_0)$, $i \in I_1^+$ are a basic

solution of the problem $D_0(I_1^+)$. Again we want to find the maximum value of the parameter λ, for which Lagrange multipliers $u_i^{(1)}(\lambda)$, $i \in I_1^+$ are a basic solution of the problem $D_\lambda(I_1^+)$. On this purpose we can apply the preceding process (Lemma 2.2 and Lemma 2.3), with the exception that instead of values referring to the problem $D(I^+)$ we use values referring to the problem $D(I_1^+)$. Especially, the matrices A, C are to be replaced with the matrices A_1, C_1 respectively such that $A_1 = A^{(j)}$, which is the matrix A with the jth column removed, and

$$C_1 = C^{(jj)} - \frac{C_j^{(j)}(C_j^{(j)})^{\mathrm{T}}}{C_{jj}} \tag{2.12}$$

where $C^{(jj)}$ results from C by removing the jth row and the jth column, $C_j^{(j)}$ results from the jth column of C by removing the element C_{jj}. (For the derivation of this formula see for instance [2]).

Suppose $\lambda_0^{(1)} = \alpha^{(1)}/\alpha_1^{(1)}$ is the maximum value of the parameter λ, for which the problem $D_\lambda(I_1^+)$ has a basic solution. If $\lambda_0^{(1)} = 1$, we set $\bar{I} = I_1^+$, else we repeat all the process. In this manner we obtain a sequence $I_1^+, \ldots, I_p^+$ of subsets of the set I^+. The cardinality of each of these subsets is by one element less than the cardinality of its predecessor. But the set I^+ is finite, $I^+ \cap M_1 \neq 0$ and the problem $D(I_p^+)$ has a basic solution provided $I_p^+ \cap M_1$ is a single-element subset of M_1, therefore we attain, after a finite number of steps, a subset $I_p^+ \subset I^+$, $k \in I_p^+$ such that the problem $D(I_p^+)$ has a basic solution. Thus we can set $\bar{I} = I_p^+$.

So far we have been treating the case $\beta_k\gamma_k + \delta_k \neq 0$. Now let suppose $\beta_k\gamma_k + \delta_k = 0$ In this case, there exists no nonzero value of the parameter λ such that the problem $D_\lambda(I^+)$ has a basic solution. On the other side, the problem $D_0(I^+)$ has several basic solutions that are defined by the equations

$$\begin{cases} u(\alpha) = u(0) - \alpha(q_k + \gamma_k p) \\ u_k(\alpha) = u_k(0) + \alpha \end{cases} \tag{2.13}$$

The condition $\beta_k\gamma_k + \delta_k = 0$ is valid only if $\beta_k = 0$, $\gamma_k = 0$ and $\delta_k = 0$ (this is implied by the fact that C is positive definite, Q is positive semidefinite, and by (2.3)). Therefore $z(\alpha) = z(0)$ (see (2.4)) and $s(\alpha) = s(0)$ (see (2.7)) since $\delta_k = 0$ implies $\tilde{Q}\tilde{a}_k = \tilde{H}\tilde{a}_k - \tilde{H}\tilde{A}q_k = 0$ which gives $Ha_k - HAq_k = 0$. The problem $P_0(I^+)$ has a unique solution $(s(\alpha), z(\alpha)) = (s(0), z(0))$ so that $\varphi(s(\alpha), z(\alpha)) = \varphi(s(0), z(0))$ for an arbitrary value of the parameter α.

Lagrange multipliers $u_i(\alpha)$, $i \in I^+$ are a basic solution of the problem $D_0(I^+)$ only if $u(\alpha) \geqq 0$. When $\tilde{I} = \{i \in I, q_{ki} + \gamma_k p_i > 0\} \neq \emptyset$, there exists a finite value α_2 in (2.10). When we choose $\alpha = \alpha_2$, then $u_j(\alpha) = 0$ holds for some index $j \in I$. Let us set $I_1 = I \setminus \{j\}$, $I_1^+ = I^+ \setminus \{j\}$, and $v_i^{(1)} = v_i$, $i \in I_1^+$. Let $D_\lambda(I_1^+)$ denotes the problem $D(I_1^+)$ after substituting $f_i(\lambda) = f_i + (1 - \lambda)\, v_i^{(1)}$, $i \in I_1^+$ (where $0 \leqq \lambda \leqq 1$) for f_i, $i \in I_1^+$. Then Lagrange multipliers $u_i^{(1)}(0) = u_i(\alpha)$, $i \in I_1^+$ are a basic solution of the problem $D_0(I_1^+)$. Thus we can proceed in the same way as we did in the case where $\beta_k\gamma_k + \delta_k \neq 0$ and $\alpha \neq \alpha_1$, only for $\beta_k\gamma_k + \delta_k = 0$ we set formally $\alpha_1 = \infty$ in (2.10).

When $k \in M_1$, we have $\tilde{I} \neq \emptyset$ since (2.3) implies

$$e^{\mathrm{T}}(q_k + \gamma_k p) = e^{\mathrm{T}} q_k + \frac{e_k - e^{\mathrm{T}} q_k}{e^{\mathrm{T}} p} e^{\mathrm{T}} p = e_k = 1$$

Therefore there exists at least one index $i \in I$ such that $q_{ki} + \gamma_k p_i > 0$. When $k \in M_2$ and $\tilde{I} = \emptyset$, the problem $P(I^+)$ has no solution.

Lemma 2.4. Let the pair $(s(0), z(0)) \in R_{n+1}$ be a basic solution of the problem $P_0(I^+)$. Let $\beta_k \gamma_k + \delta_k = 0$ and $a_k^{\mathrm{T}} s(0) + f_k - z(0)\, e_k > 0$. If $k \in M_2$ and $\tilde{I} = \emptyset$, where $\tilde{I}$ is the same set as in (2.10), the problem $P(I^+)$ has no solution.

Proof. We have $a_k^{\mathrm{T}} s(0) + f_k - z(0)\, e_k > 0$ and $A^{\mathrm{T}} s(0) + f - z(0)\, e = 0$ by the assumption. Let the pair $(s^+, z^+) \in R_{n+1}$ be a solution of the problem $P(I^+)$ (it need not be a basic solution). Then $a_k^{\mathrm{T}} s^+ + f_k - z^+ e_k \leqq 0$ and $A^{\mathrm{T}} s^+ + f - z^+ e \leqq 0$ must hold. Since $k \in M_2$, we have $e_k = 0$, so that

$$a_k^{\mathrm{T}}(s^+ - s(0)) < 0$$

$$A^{\mathrm{T}}(s^+ - s(0)) - (z^+ - z(0))\, e \leqq 0$$

Since $\beta_k \gamma_k + \delta_k = 0$, we have $\beta_k = 0$, $\gamma_k = 0$ and $\delta_k = 0$, which by (2.3) implies $\tilde{Q} \tilde{a}_k = \tilde{H} \tilde{a}_k - \tilde{H} \tilde{A} q_k = 0$ and therefore $H a_k - H A q_k = 0$. Since the matrix H is positive definite, we get $a_k = A q_k$, so that

$$q_k^{\mathrm{T}} A^{\mathrm{T}}(s^+ - s(0)) < 0 \tag{2.14a}$$

$$A^{\mathrm{T}}(s^+ - s(0)) \leqq (z^+ - z(0))\, e \tag{2.14b}$$

Since $\tilde{I} = \emptyset$ and $\gamma_k = 0$, the inequalities $q_{ki} \leqq 0$ must hold for all indices $i \in I$. Furthermore $\beta_k = e_k - e^{\mathrm{T}} q_k = 0$, so that $e^{\mathrm{T}} q_k = e_k = 0$. Thus we get

$$q_k^{\mathrm{T}} A^{\mathrm{T}}(s^+ - s(0)) \geqq (z^+ - z(0))\, q_k^{\mathrm{T}} e = 0$$

from (2.14b), which is in contradiction with (2.14a). Therefore the problem $P(I^+)$ has no solution. □

We have shown that either the problem $P(I^+)$ has no solution or we can construct a subset $\bar{I} \subset I^+$, $k \in \bar{I}$ such that the problem $(P(\bar{I})$ has a basic solution. It remains to prove that $\varphi(\bar{s}, \bar{z}) > \varphi(s, z)$.

Theorem 2.1. Suppose $(s, z) \in R_{n+1}$ is the solution of the problem $P(I)$ and $(\bar{s}, \bar{z}) \in \in R_{n+1}$ is the solution of the problem $P(\bar{I})$. Then $\varphi(\bar{s}, \bar{z}) > \varphi(s, z)$.

Proof. The set $\bar{I}$ is obtained after a finite number of steps, in which we construct subsets $\bar{I} = I_p^+ \subset \ldots \subset I_1^+ \subset I^+$. Since all steps are formally equivalent, it suffices to analyze the first step. Let $(s(0), z(0)) \in R_{n+1}$ be the solution of the problem $P_0(I^+)$ and $(s(\alpha), z(\alpha)) \in R_{n+1}$ be the solution of the problem $P_0(I_1^+)$. Two cases are possible. If $\beta_k \gamma_k + \delta_k = 0$, then $s(\alpha) = s(0)$ and $z(\alpha) = z(0)$ so that $\varphi(s(\alpha), z(\alpha)) = \varphi(s(0), z(0))$. If $\beta_k \gamma_k + \delta_k \neq 0$, we get, by (2.4) and (2.6),

$$\varphi(s(\alpha), z(\alpha)) = \varphi(s(0), z(0)) + \tfrac{1}{2}\alpha(\beta_k\gamma_k + \delta_k)\,(2\,u_k(0) + \alpha)$$

But $\beta_k\gamma_k + \delta_k > 0$ (because $\beta_k\gamma_k + \delta_k \geqq 0$ and $\beta_k\gamma_k + \delta_k \neq 0$), $u_k(0) \geqq 0$, and $\alpha \geqq 0$, so that $\varphi(s(\alpha), z(\alpha)) \geqq \varphi(s(0), z(0))$ and $\varphi(s(\alpha), z(\alpha)) = \varphi(s(0), z(0))$ holds if and only if $\alpha = 0$. Combining both cases into one, we obtain

$$\varphi(\alpha) \geqq \varphi(0) \tag{2.15a}$$

$$\varphi(\alpha) = \varphi(0) \langle = \rangle s(\alpha) = s(0),\ z(\alpha) = z(0) \tag{2.15b}$$

where $\varphi(\alpha) = \varphi(s(\alpha), z(\alpha))$ and $\varphi(0) = \varphi(s(0), z(0))$. Now we are proving that $\varphi(\bar{s}, \bar{z}) > \varphi(s, z)$. The validity of (2.15a) in each step yields $\varphi(\bar{s}, \bar{z}) \geqq \varphi(s, z)$. Now let us suppose $\varphi(\bar{s}, \bar{z}) = \varphi(s, z)$. Since (2.15b) is valid in each step, we have $\bar{s} = s$ and $\bar{z} = z$. Therefore

$$\bar{v}_k = \bar{z}e_k - a_k^{\mathrm{T}}\bar{s} - f_k = ze_k - a_k^{\mathrm{T}}s - f_k = v_k < 0$$

which is a contradiction, for $k \in \bar{I}$ and $(\bar{s}, \bar{z}) \in R_{n+1}$ is a basic solution of the problem $P(\bar{I})$, and consequently, $\bar{v}_k = 0$. □

3. ALGORITHM OF THE DUAL METHOD

In Section 2, we have described the construction of the major step of the dual method for solving the problem (1.3). Now we are describing the algorithm that contains these major steps.

For improving the stability of our algorithm, we use the triangular decomposition $R^{\mathrm{T}}R = \tilde{A}^{\mathrm{T}}\tilde{H}\tilde{A}$ instead of the inversion $C = (\tilde{A}^{\mathrm{T}}\tilde{H}\tilde{A})^{-1}$. The upper triangular matrix R is computed recursively. Let $(R^{+})^{\mathrm{T}}R^{+} = (\tilde{A}^{+})^{\mathrm{T}}\tilde{H}\tilde{A}^{+}$, where $\tilde{A}^{+} = [\tilde{A}, \tilde{a}_k]$. Then

$$\left\{\begin{array}{ll} & R^{+} = \begin{bmatrix} R, & r_1 \\ 0, & r_2 \end{bmatrix} \\ \text{where} & \\ & R^{\mathrm{T}}r_1 = A^{\mathrm{T}}Ha_k + \mu e_k e \\ & r_2^2 = a_k^{\mathrm{T}}Ha_k + \mu e_k^2 - r_1^{\mathrm{T}}r_1 \end{array}\right. \tag{3.1}$$

(see for instance [8]). After deleting the index j from the set I, we need find the decomposition $(R^{-})^{\mathrm{T}}R^{-} = (\tilde{A}^{-})^{\mathrm{T}}\tilde{H}\tilde{A}^{-}$, where $\tilde{A}^{-}$ results from the matrix $\tilde{A}$ after deleting the column $\tilde{a}_j$. Let P be a permutation matrix which transfers the column $\tilde{a}_j$ of the matrix $\tilde{A}$ to the last position so that RP is an upper Hessenberg matrix. Let Q be an orthogonal matrix such that $QRP = \tilde{R}$, where $\tilde{R}$ is an upper triangular matrix. Then $\tilde{R}^{\mathrm{T}}\tilde{R} = P^{\mathrm{T}}R^{\mathrm{T}}Q^{\mathrm{T}}QRP = P^{\mathrm{T}}R^{\mathrm{T}}RP$, so that

$$\left\{\begin{array}{ll} & \tilde{R} = \begin{bmatrix} R^{-}, & \tilde{r}_1 \\ 0, & \tilde{r}_2 \end{bmatrix} \\ \text{and} & \\ & (R^{-})^{\mathrm{T}}R^{-} = (\tilde{A}^{-})^{\mathrm{T}}\tilde{H}\tilde{A}^{-} \end{array}\right. \tag{3.2}$$

The number $\mu > 0$ in (2.1) must be taken to guarantee the positive definiteness of the matrix $\tilde{A}^T\tilde{H}\tilde{A}$. Most advantageous choice of the number $\mu > 0$ is that, which makes the matrix $\tilde{A}^T\tilde{H}\tilde{A}$ optimally conditioned. However, it is computationally time-consuming. Therefore we use simple choice $\mu = 1$ in the algorithm.

Algorithm 3.1.

Step 1: Choose arbitrarily an index $k \in M_1$ and set $\mu = 1$.

Step 2: Set $I := \{k\}$, $u := [1]$. $e := [1]$, $A := [a_k]$, $R := [\sqrt{(a_k^T H a_k + \mu)}]$ and calculate $z := f_k - a_k^T H a_k$.

Step 3: Calculate $s := -HAu$ and

$$v_k := ze_k - f_k - a_k^T s = \min_{i \in M \setminus I} (ze_i - f_i - a_i^T s)$$

where $f_i = f_i(x)$, $e_i = 1$ for $i \in M_1$ and $f_i = a_i^T x - b_i$, $e_i = 0$ for $i \in M_2$. If $v_k \geqq 0$, terminate computation ($(s, z) \in R_{n+1}$ is the solution of the problem (1.3)). If $v_k < 0$, set $u_k := 0$ and go to Step 4.

Step 4: Determine the vector p as a solution of the equation $R^T R p = e$ and the vector q_k as a solution of the equation $R^T R q_k = A^T H a_k + \mu e_k e$. Calculate $\beta_k := e_k - e^T q_k$, $\gamma_k := \beta_k / p^T e$ and $\delta_k := r_2^2$ where r_2^2 is given by (3.1). If $\beta_k \gamma_k + \delta_k = 0$, set $\alpha_1 := \infty$ else set

$$\alpha_1 := -\frac{v_k}{\beta_k \gamma_k + \delta_k}$$

Determine the set $\tilde{I} := \{i \in I : q_{ki} + \gamma_k p_i > 0\}$. If $\tilde{I} = \emptyset$, set $\alpha_2 := \infty$ else set

$$\alpha_2 := \frac{u_j}{q_{kj} + \gamma_k p_j} = \min_{i \in \tilde{I}} \left(\frac{u_i}{q_{ki} + \gamma_k p_i} \right)$$

Set $\alpha := \min(\alpha_1, \alpha_2)$. If $\alpha = \infty$, terminate computation (the problem (1.3) has no solution). In the opposite case calculate $u := u - \alpha(q_k + \gamma_k p)$, $u_k := u_k + \alpha$, $z := z + \alpha\gamma_k$ and $v_k := (1 - \alpha/\alpha_1) v_k$. If $\alpha = \alpha_1$ go to Step 5. If $\alpha \neq \alpha_1$ go to Step 6.

Step 5: Set $I := I \cup \{k\}$, $u := [u^T, u_k]^T$, $e := [e^T, e_k]^T$, $A := [A, a_k]$ and $R := R^+$ where R^+ is the upper triangular matrix determined from (3.1). Go to Step 3.

Step 6: Set $I := I \setminus \{j\}$, $u := u^{(j)}$, $e := e^{(j)}$ and $A := A^{(j)}$, where $u^{(j)}$ and $e^{(j)}$ result from u and e after deleting the elements u_j and e_j respectively and where $A^{(j)}$ results from A after deleting the column a_j. Set $R := R^-$ where R^- is the upper triangular matrix determined from (3.2). If $I \cap M_1 \neq \emptyset$ go to Step 4. If $I \cap M_1 = \emptyset$ then set $z := z - v_k$ and go to Step 5.

Step 6 of Algorithm 3.1 contains a test on $I \cap M_1 = \emptyset$. We show that this case arises only if $k \in M_1$, so that, after applying Step 5, we get $I \cap M_1 \neq \emptyset$.

Lemma 3.1. Let the assumptions of Lemma 2.2 hold. Let $k \in M_2$ and let $I \cap M_1 = \{j\}$ be a single-element subset of M_1. Then $u_j(\lambda) = u_j(0)$.

Proof. The vector e contains only one nonzero element $e_j = 1$. Thus, applying (2.3), we get

$$q_{kj} + \gamma_k p_j = e^T q_k + \gamma_k e^T p = e^T q_k + ((e_k - e^T q_k)/e^T p)\, e^T p = e_k = 0$$

since $k \in M_2$ implies $e_k = 0$. Using (2.4) we obtain $u_j(\lambda) = u_j(0)$. □

The case $I \cap M_1 = \emptyset$ and $I \cap M_2 = \emptyset$ can also occurs in Step 6 of Algorithm 3.1. If it is the case, we have $I = \emptyset$ and we must work formally with empty vectors and empty matrices.

Now we are proving the convergence of the dual method for solving the problem (1.3).

Theorem 3.1. Algorithm 3.1 finds the solution of the problem (1.3) after a finite number of steps or it shows, after a finite number of steps, that the problem (1.3) has no solution.

Proof. During the execution of Algorithm 3.1 we construct a sequence of subsets $\bar{I}_j \subset M$, $j \geqq 0$. Theorem 2.1 assures the validity of $\varphi(\bar{s}_j, \bar{z}_j) > \varphi(\bar{s}_{j-1}, \bar{z}_{j-1})$, $j \geqq 1$ (the pair $(\bar{s}_j, \bar{z}_j) \in R_{n+1}$ is a basic solution of the problem $P(\bar{I}_j)$). Hence the sets $\bar{I}_j \subset M$, $j \geqq 0$ must be distinct. Since M is finite, the sequence of mutually distinct subsets $\bar{I}_j \subset M$, $j \geqq 0$ is also finite. Therefore, Algorithm 3.1 must terminate after a finite number of steps either in Step 3 (the solution of (1.3) is found) or in Step 4 (the problem (1.3) has no solution). □

4. CONCLUSION

Algorithm 3.1 was implemented and tested in the connection with methods for linearly constrained nonlinear minimax approximation. Successful results obtained will be presented in the forthcoming paper [9]. Main advantage of Algorithm 3.1 consists in little demans on computer storage. It uses only the matrices A, H needed for minimax approximation, the triangular matrix R of order $n + 1$ and several $n + 1$ dimensional vectors. For $n = 30$ and $m = 300$, we have to store about 10 000 numbers. Simplex based methods, which can be also used for problems with singular matrices, require the tableau with $n + m + 1$ rows and $2(n + 1) + m + 1$ columns. Since the matrices A, H have to be stored for minimax approximation, these methods require 120 000 numbers for $n = 30$ and $m = 300$. This comparison prefers Algorithm 3.1 because m is usually large for typical problems of minimax approximation.

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