KYBERNETIKA — VOLUME 20 (1984), NUMBER 3

# TIME OPTIMAL CONTROL OF A SECOND ORDER NONLINEAR PLANT

K. BALACHANDRAN, D. SOMASUNDARAM

In this paper a second order non linear relay controlled plant is considered. The equation of the switching curve and the controllable region are determined. It is established that the error and error derivative can be reduced to zero simultaneously and in the shortest possible time with at most one switching reversal of the control. The same result is obtained for the unstable case provided that the initial values of error and error derivative fall in a controllable region.

## 1. INTRODUCTION

Time optimal control of a number of second order linear plants has been thoroughly studied by Athans and Falb [1]. Lee and Markus [2] discussed the time optimal control problem for a class of second order nonlinear systems. It has been shown in [2] that the time optimal controllers for these non linear systems are relay controllers. A number of examples are given but no analytical treatment for obtaining the equation of the switching curve is presented. Almuzara and Flugge-Lotz [3] treated a non linear system in which the nonlinear function is periodic. Boettiger and Haas [4] and James [5], respectively, analysed the time optimal control of a soft spring and Van der Pol Oscillator. In [9] Boltyanskii gave an example of a nonlinear system where the region of controllability is not completely filled with optimal trajectories. Vakilzadeh and Keshavarz [7, 8] considered the analytical treatment of second order nonlinear plants with different types of nonlinearity. In this paper we shall consider a special type of second order nonlinear plant.

## 2. ANALYSIS

Consider a plant whose dynamics are described by the second order nonlinear equation

$$\ddot{c} + a\dot{c}|\dot{c}| + bc = Ku\,, \quad a > 0 \quad u = \pm 1\,. \tag{1}$$

If $b > 0$, the plant is a stable one. On the other hand if $b < 0$, the plant is an unstable one. The plant represents a forced vibrating system whose damping force is proportional to the square of the velocity.

Here our object is to design a controller for simultaneous reduction $c(t)$ and $\dot{c}(t)$ to zero when the input $r(t) = 0$. Therefore, the systems error differential equation is given by

$$\ddot{e} + a\dot{e}|\dot{e}| + be = -Ku . \tag{2}$$

The reduction of error and error derivative to zero will mean the reduction of output $c(t)$ and output derivative $\dot{c}(t)$ to zero.

Now $\ddot{e} = (\mathrm{d}\dot{e}/\mathrm{d}t) = \dot{e}(\mathrm{d}\dot{e}/\mathrm{d}e)$, the equation (2) can be written as

$$\dot{e}\frac{\mathrm{d}\dot{e}}{\mathrm{d}e} + a\dot{e}|\dot{e}| + be = -Ku . \tag{3}$$

The solution of the nonlinear differential equation (3) for $\dot{e} > 0$, when $y_1 = ae$, $y_2 = \sqrt{(a/(aK))}\dot{e}$ and $\alpha = b/(aK)$ is

$$y_2 = +[(\tfrac{1}{2}\alpha - u - \alpha y_1) + (\beta_2^2 - \tfrac{1}{2}\alpha + u + \alpha\beta_1)\exp(2\beta_1 - 2y_1)]^{1/2} \quad u = \pm 1 \tag{4}$$

and for $\dot{e} < 0$

$$y_2 = -[(\tfrac{1}{2}\alpha + u + \alpha y_1) + (\beta_2^2 - \tfrac{1}{2}\alpha - u - \alpha\beta_1)\exp(2y_1 - 2\beta_1)]^{1/2} \quad u = \pm 1 \tag{5}$$

where $\beta_1 = y_1(0)$ and $\beta_2 = y_2(0)$.

## 3. SWITCHING CURVE AND TIME OPTIMAL CONTROL LAW

Figure 1 shows the phase plane trajectories for the stable case ($\alpha > 0$). The solid curves are for control $u = +1$ and the dashed ones for $u = -1$, the arrows show the direction of increasing time. When $\alpha = 1$, these trajectories originate from the initial state $(\beta_1, \beta_2)$ and approach to the point $(-1, 0)$ for $u = +1$ and $(1, 0)$ for $u = -1$. In general if $\alpha > 0$, the trajectories approach to the point $(-\alpha^{-1}, 0)$ for $u = +1$ and $(\alpha^{-1}, 0)$ for $u = -1$.

The two forced trajectories which pass through the origin are denoted by $\gamma_+$ for control $u = +1$ and $\gamma_-$ for control $u = -1$. The switching curve is the union of segments AO and BO. From equations (4) and (5), the equation of switching curve is obtained and is given by

$$y_2 = -\frac{y_1}{|y_1|}[\tfrac{1}{2}\alpha - 1 + \alpha|y_1| + (1 - \tfrac{1}{2}\alpha)\exp(2|y_1|)]^{1/2}. \tag{6}$$

The switching curve as well as the variation of $\alpha$ is shown in Fig. 2.

As seen from Fig. 1 for all states initially above the switching curve, we must first apply the control $u = +1$ and when the state reaches the switching curve then the control $u = -1$. Conversely for all states initially below the switching curve we must first apply the control $u = -1$ and when the state reaches the switching curve the control $u = +1$. Hence all the initial states $(\beta_1, \beta_2)$ can be brought to the origin $(0, 0)$ simultaneously and in the shortest possible time by the application of one of the four possible control sequences

$$[+1], [-1], [+1, -1], [-1, +1] \tag{7}$$

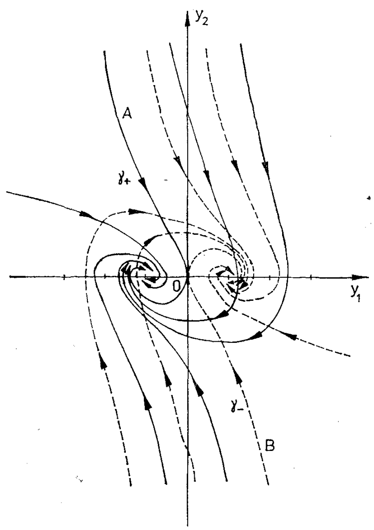


**Fig. 1.** The $u = +1$ and $u = -1$ forced trajectories in the $y_1y_2$-plane when $\alpha = 1$.

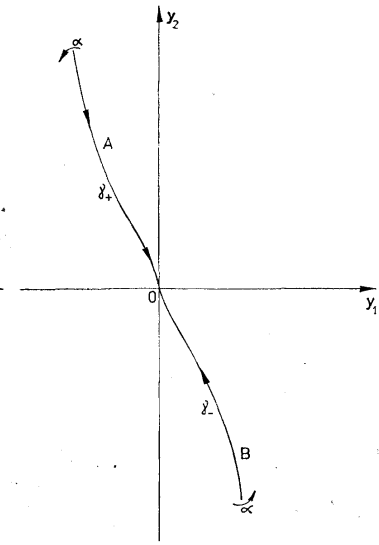


**Fig. 2.** The switching curve in the $y_1y_2$-plane when $\alpha > 0$.

## 4. UNSTABLE CASE AND CONTROLLABLE REGION

**Case (i) $a > 0$ and $b < 0$**

Figure 3 shows the phase plane trajectories for $\alpha = -1$. A separatrix is a trajectory which tends toward a saddle point as $t \to \pm\infty$. From Fig. 3 it is observed that the separatrices tend to $(+1, 0)$ for control $u = +1$ and $(-1, 0)$ for control $u = -1$. It is easy to see that the domain of controllability is the interior of the region bounded by the two separatrices. Fig. 4 shows the controllable region and the switching curve. The equation of the switching curve is given by

$$y_2 = -\frac{y_1}{|y_1|}\left[\tfrac{3}{2} - |y_1| + \tfrac{3}{2}\exp(2|y_1|)\right]^{1/2} \tag{8}$$

Using the shape of the forced trajectories shown in Fig. 3, we can prove that by similar argument as above the control sequences (7) are the only candidates for the time optimal control of this case. Hence, if the initial values of $y_1$ and $y_2$ fall in the interior of the region bounded by the two separatrices then $(y_1, y_2)$ can be brought to the origin $(0, 0)$ by the application of one of the four possible control sequences (7). The variation of $\alpha$ is also shown in Fig. 4.

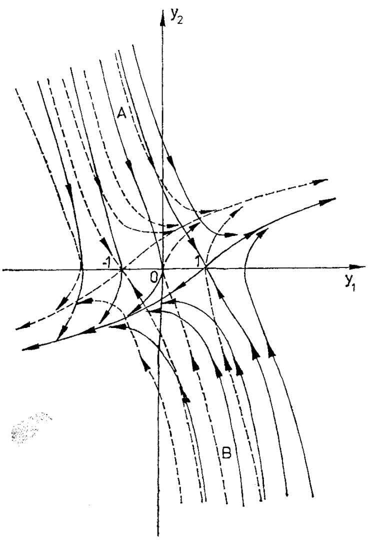


**Fig. 3.** The $u = +1$ and $u = -1$ forced trajectories in the $y_1y_2$-plane when $\alpha = -1$.

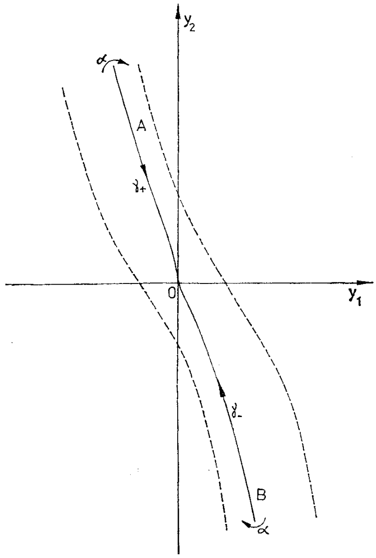


**Fig 4.** The switching curve and the controllable region in the $y_1y_2$-plane when $\alpha < 0$.

**Case (ii)** $b = 0$ **and** $a < 0$.

In this case, the system (1) is unstable which is equivalent to the following system with $a > 0$

$$\ddot{c} - a\dot{c}|\dot{c}| = Ku . \tag{9}$$

The forced trajectories are shown in Fig. 5 and the equations are given as follows: for $\dot{e} > 0$

$$y_2 = +[(\beta_2^2 - u)\exp(2y_1 - 2\beta_1) + u]^{1/2} \tag{10}$$

and for $\dot{e} < 0$

$$y_2 = -[(\beta_2^2 + u)\exp(2\beta_1 - 2y_1) - u]^{1/2} \tag{11}$$

From (10) and (11), the equation of the switching curve is

$$y_2 = -\frac{y_1}{|y_1|}\left[1 - \exp\left(-2|y_1|\right)\right]^{1/2} \tag{12}$$

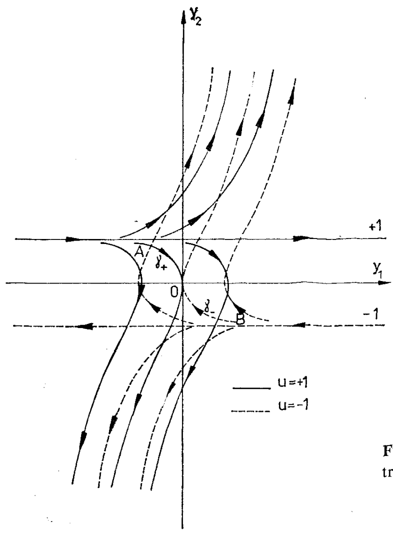


**Fig. 5.** The $u = +1$ and $u = -1$ forced trajectories in the $y_1y_2$-plane when $\alpha = 0$.

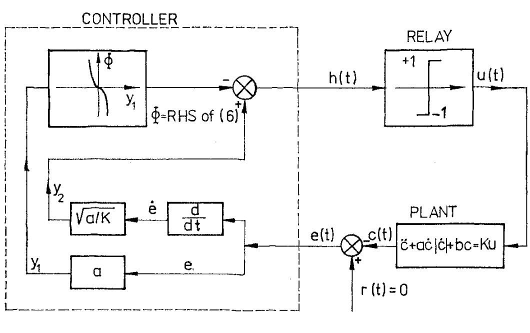


**Fig. 6.** Block diagram of relay control of the nonlinear plant with designed controller.

By the process of elimination, we can also see that the control sequences (7) are the only candidates for the time optimal control of this case.

When $a = 0$, the system (1) becomes linear. The cases $b > 0$ and $b < 0$ have been respectively treated in [1] and [6]. Figure 6 shows the relay controlled plant with designed controller.

## 5. CONCLUSION

In this paper the time optimal control of a second order nonlinear plant was discussed. The equation of the switching curve was derived. If $r(t) = 0$, the designed controller will bring any output $c(t)$ and output derivative $\dot{c}(t)$ to zero simultaneously and in the shortest possible time with atmost one switching reversal of the relay. The same conclusion was obtained for unstable case provided that the error and its derivative fall in the controllable region. It is interesting to note that as $\alpha \to 0$, the plant is stable if $a > 0$ and unstable if $a < 0$; in this case, the equation of the switching curve can be made independent of the values of constant gain $(K)$ and the coefficient of nonlinearity $(a)$ of the plant. The stable case has been fully treated in[7].

(Received May 16, 1983.)

## REFERENCES

[1] M. Athans and P. L. Falb: Optimal Control. McGraw Hill, New York 1966.

[2] E. B. Lee and L. Markus: Foundations of Optimal Control Theory. John Wiley, New York 1967.

[3] J. L. G. Almuzara and I. Flugge-Lotz: Minimum time control of a nonlinear system. J. Differential Equations *4* (1968), 1, 12—39.

[4] A. Boettiger and V. B. Haas: Synthesis of time-optimal control of a second order nonlinear process. J. Optim. Theory Appl. *4* (1969), 1, 22—39.

[5] E. M. James: Time optimal control and the Van der Pol oscillator. J. Inst. Math. Appl. *13* (1974), 1, 67—81.

[6] I. Vakilzadeh: Bang-bang control of a plant with one positive and one negative real pole. J. Optim. Theory Appl. *24* (1978), 2, 315—324.

[7] I. Vakilzadeh and A. A. Keshavarz: Bang-bang control of a second order nonlinear stable plant with second order nonlinearity. Kybernetika *18* (1982), 1, 66—71.

[8] I. Vakilzadeh and A. A. Keshavarz: Bang-bang control of a second order nonlinear unstable plant with third order nonlinearity. Internat. J. Control *34* (1981), 3, 457—463.

[9] V. G. Boltyanskii: An example of nonlinear synthesis (in Russian). Differencial'nye Uravneniya 6 (1970), 4, 644—649; (in English) Differential Equations 6 (1970), 4, 495—498.

*K. Balachandran, D. Somasudaram, Department of Mathematics, Bharathiar University, Coimbatore-641 046. Tamil Nadu, India.*