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GENERALISED DIRECTED DIVERGENCE WITHOUT SYMMETRY

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The authors have characterized axiomatically the generalized directed divergence (which is a symmetric function of its variables) by considerably weakening the symmetry.

1. INTRODUCTION

Let

$$\Delta_n = \{(p_1, p_2, \ldots, p_n);\ p_k \geqq 0,\ k = 1, 2, \ldots, n,\ \sum_{k=1}^{n} p_k = 1\}, \quad n = 2, 3, \ldots,$$

and

$$\Delta_n^* = \{(p_1, p_2, \ldots, p_n);\ p_k > 0,\ k = 1, 2, \ldots, n,\ \sum_{k=1}^{n} p_k = 1\}, \quad n = 2, 3, \ldots,$$

be the sets of all finite n-component discrete probability distributions with non-negative elements and positive elements respectively. Let $P = (p_1, p_2, \ldots, p_n)$, $Q = (q_1, q_2, \ldots, q_n)$ and $R = (r_1, r_2, \ldots, r_n) \in \Delta_n$. The generalized directed divergence of three probability distributions P, Q and R is defined as

$$(1.1) \qquad F_n(p_1, p_2, \ldots, p_n;\ q_1, q_2, \ldots, q_n;\ r_1, r_2, \ldots, r_n) = \sum_{k=1}^{n} p_k \log \frac{q_k}{r_k},$$

$$b_k \geqq 0,\ q_k \geqq 0,\ r_k \geqq 0,\ k = 1, 2, \ldots, n,\ \sum_{k=1}^{n} p_k = 1 = \sum_{k=1}^{n} q_k = \sum_{k=1}^{n} r_k.$$

where $F_n : S_n \to \mathbb{R}$, $n = 2, 3, \ldots$, and S_n be a set of $3n$-tuples of the form $(p_1, p_2, \ldots$ $\ldots, p_n;\ q_1, q_2, \ldots, q_n;\ r_1, r_2, \ldots, r_n)$ such that $q_i = 0$ and $p_i = 0$ for all those indices i for which $r_i = 0$ and also $p_i = 0$ whenever $q_i = 0$, $i = 1, 2, \ldots, n$.

(Here the base of the logarithm is taken as 2).

Kannappan and Rathie [3] characterized (1.1) by assuming the following set of postulates.

Postulate I$_n$ (Recursivity). For all probability distributions P, Q and $R \in \Delta_n$, and $n \geqq 3$,

$$F_n(p_1, p_2, \dots, p_n; q_1, q_2, \dots, q_n; r_1, r_2, \dots, r_n) = \tag{1.2}$$
$$= F_{n-1}(p_1 + p_2, \dots, p_n; q_1 + q_2, \dots, q_n; r_1 + r_2, \dots, r_n +$$
$$+ (p_1 + p_2) F_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}; \frac{r_1}{r_1 + r_2}, \frac{r_2}{r_1 + r_2} \right)$$

with $p_1 + p_2 > 0$, $q_1 + q_2 > 0$ and $r_1 + r_2 > 0$.

Postulate II$_n$ ($n = 3$). $F_3(p_1, p_2, p_3; q_1, q_2, q_3; r_1, r_2, r_3)$ is a symmetric function of its variables $(p_i; q_i; r_i)$, $i = 1, 2, 3$.

Postulate III (Derivibility). The mapping $(x, y, z) \to f(x, y, z)$, $(x, y, z) \in J$ possesses continuous first order partial derivatives with respect to each variable $(x, y, z) \in \in (0, 1)$, where $f(x, y, z) = F_2(x, 1 - x; y, 1 - y; z, 1 - z)$ and $J = (0, 1) \times (0, 1) \times \times (0, 1) \cup \{(0, y, z), 0 \leqq y < 1, 0 \leqq z < 1\} \cup \{(1, y', z'), 0 < y' \leqq 1, 0 < z' \leqq 1\}$.

Postulate IV (Normalization).

$$f(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}) = \tfrac{1}{3} \quad \text{and} \quad f(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}) = 0 .$$

Postulate V (Nullity).

$$f(p, p, p) = 0 , \quad p \in (0, 1) .$$

The main object of this paper is to axiomatically characterized (1.1) by considerably weakening the symmetry Postulate II$_n$ ($n = 3$) assumed by Kannappan and Rathie [3] and by many other research workers.

Instead of Postulate II$_n$ ($n = 3$), we assume the following postulate:

Postulate VI$_n$. For all probability distributions P, Q and $R \in \Delta_n - \Delta_n^*$, and $n \geqq 3$,

$$F_n(p_1, p_2, \dots, p_j, \dots, p_n; q_1, q_2, \dots, q_j, \dots, q_n; r_1, r_2, \dots, r_j, \dots, r_n) = \tag{1.3}$$
$$= F_n(p_j, p_2, \dots, p_1, \dots, p_n; q_j, q_2, \dots, q_1, \dots, q_n; r_j, r_2, \dots, r_1, \dots, r_n) ,$$
$$2 \leqq j \leqq n , \quad \text{if} \quad r_1 > 0 \quad \text{and} \quad r_j = 0 \quad \text{or} \quad q_1 > 0 \quad \text{and} \quad q_j = 0$$
$$\text{or} \quad p_1 > 0 \quad \text{and} \quad p_j = 0 \quad \text{holds.}$$

Postulate VI$_n$ allows the simultaneous interchange of p_1 with p_j, q_1 with q_j and r_1 with r_j, $2 \leqq j \leqq n$ is such that either $p_1 > 0$ and $p_j = 0$ or $q_1 > 0$ and $q_j = 0$ or $r_1 > 0$ and $r_j = 0$ holds. It is obvious that Postulate II$_n$ ($n = 3$) implies Postulate VI$_n$ ($n = 3$). But the converse is not true. For example: Consider $F_n : S_n \to \mathbb{R}$ defined as

$$F_n(p_1, p_2, \ldots, p_n;\ q_1, q_2, \ldots, q_n;\ r_1, r_2, \ldots, r_n) = p_1 q_1 r_1 \quad \text{if} \quad P, Q \text{ and } R \in \Delta_n^*,$$
$$= 1 \text{ if } P, Q \text{ and } R \in (\Delta_n - \Delta_n^*).$$

Then it is easy to check that F_n satisfies VI_n but not II_n $(n = 3)$. Thus VI_n does not imply that F_n, $n \geqq 2$, is a symmetric function.

2. CHARACTERIZATION THEOREM

Theorem. Let $F_n : S_n \to \mathbb{R}$, $n = 2, 3, \ldots$, satisfy Postulates I_n $(n \geqq 3)$, III, IV, V and VI_n $(n \geqq 3)$. Then F_n is of the form

$$F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n;\ r_1, r_2, \ldots, r_n) = \sum_{k=1}^{n} p_k \log \frac{q_k}{r_k}, \tag{2.1}$$

$$p_k \geqq 0,\ q_k \geqq 0,\ r_k \geqq 0,\ k = 1, 2, \ldots, n;\ \sum_{k=1}^{n} p_k = 1 = \sum_{k=1}^{n} q_k = \sum_{k=1}^{n} r_k .$$

Proof. Before proving the main theorem, we shall prove the following lemmas:

Lemma 1. Postulates I_n $(n = 3)$ and VI_n $(n = 3) \Rightarrow$

$$F_2(0, 1;\ 0, 1;\ 0, 1) = 0 = F_2(1, 0;\ 1, 0;\ 1, 0). \tag{2.2}$$

Proof. From Postulate VI_n $(n = 3)$, we have

$$F_3(\tfrac{1}{2}, \tfrac{1}{2}, 0;\ \tfrac{1}{2}, \tfrac{1}{2}, 0;\ \tfrac{1}{2}, \tfrac{1}{2}, 0) = F_3(0, \tfrac{1}{2}, \tfrac{1}{2};\ \ 0, \tfrac{1}{2}, \tfrac{1}{2};\ 0, \tfrac{1}{2}, \tfrac{1}{2}) = \tag{2.3}$$
$$= F_3(\tfrac{1}{2}, 0, \tfrac{1}{2};\ \tfrac{1}{2}, 0, \tfrac{1}{2};\ \tfrac{1}{2}, 0, \tfrac{1}{2}).$$

which by Postulate I_n $(n = 3)$ in (2.3), we get (2.2). □

Lemma 2. Postulates I_n $(n \geqq 3)$ and VI_n $(n \geqq 3) \Rightarrow$

$$F_n(p_1, p_2, \ldots, p_n;\ q_1, q_2, \ldots, q_n;\ r_1, r_2, \ldots, r_n) = \tag{2.4}$$
$$= F_{n+1}(0, p_1, \ldots, p_n;\ 0, q_1, \ldots, q_n;\ 0, r_1, \ldots, r_n), \quad n \geqq 2 .$$

Proof. Let p_j be the first non-zero element in the probability distribution P such that $p_j > 0 \Rightarrow q_j > 0 \Rightarrow r_j > 0$, $1 \leqq j \leqq n$, and using Postulates VI_n $(n \geqq 3)$, I_n $(n \geqq 3)$ and (2.2), we get

$$F_n(p_1, p_2, \ldots, p_n;\ q_1, q_2, \ldots, q_n;\ r_1, r_2, \ldots, r_n) =$$
$$= F_n(p_j, \ldots, p_n;\ q_j, \ldots, q_n; r_j, \ldots, r_n) =$$
$$= F_n(0 + p_j, \ldots, p_n;\ 0 + q_j, \ldots, q_n; 0 + r_j, \ldots, r_n) + p_j F_2(1, 0;\ 1, 0;\ 1,\ 0) =$$
$$= F_{n+1}(p_j, 0, \ldots, p_n; q_j, 0, \ldots, q_n;\ r_j, 0, \ldots, r_n) =$$
$$\overset{(1.3)}{=} F_{n+1}(p_1, 0, \ldots, p_j, \ldots, p_n;\ q_1, 0, \ldots, q_j, \ldots, q_n;\ r_1, 0, \ldots, r_j, \ldots, r_n) =$$
$$\overset{(1.3)}{=} F_{n+1}(0, p_1, \ldots, p_n; 0, q_1, \ldots, q_n; 0, r_1, \ldots, r_n).$$
□

Lemma 3. Postulates I_n $(n \geqq 3)$ and VI_n $(n \geqq 3) \Rightarrow F_n$ has $n!$, $n = 2, 3, \ldots$, permutations $\Rightarrow F_n$, $n \geqq 2$, is a symmetric function.

Proof. Here we prove the symmetry of F_n, $n \geqq 2$, by the method of induction on n.

When $n = 2$. We have the following cases:

Case 1. When $0 < r_1 < 1$ holds in F_2:

Then, $0 < r_2 < 1$ also holds in F_2 and it implies that either

(i) $q_1 = 0 \Rightarrow p_1 = 0$, $p_2 = q_2 = 1$ in F_2; or (ii) $0 \leqq p_1 < 1$, $0 < p_2 \leqq 1$, $0 < q_1 < 1$, $0 < q_2 < 1$ in F_2.

The proof of (i) is as follows:

(2.5) $F_2(0, 1;\ 0, 1;\ r_1, r_2) \overset{(2.4)}{=} F_3(0, 0, 1;\ 0, 0, 1;\ 0, r_1, r_2) \overset{(1.3)}{=} F_3(1, 0, 0;\ 1, 0, 0;$ $r_2, r_1, 0) \overset{(1.2)}{=} F_2(1, 0;\ 1, 0;\ 1, 0) + F_2(1, 0;\ 1, 0;\ r_2, r_1) \overset{(2.2)}{=} F_2(1, 0;\ 1, 0;\ r_2, r_1).$

Similarly, the proof of (ii) follows.

Case 2. When either $r_1 = 0$ and $r_2 = 1$ or $r_1 = 1$ and $r_2 = 0$ holds in F_2:

Then, it implies either $p_1 = 0 = q_1$ and $p_2 = q_2 = 1$ or $p_1 = q_1 = 1$ and $p_2 = q_2 = 0$ in F_2.

This case is obviously true from (2.2).

Thus we have proved the symmetry of F_2 over S_2.

When $n = 3$. We have the following cases:

Case 1. When $0 < p_i < 1$, $0 < q_i < 1$, and $0 < r_i < 1$, $i = 1, 2, 3$ holds in F_3:

Then by Postulate I_n $(n = 3)$ and (2.5), we have

(2.6) $$F_3(p_1, p_2, p_3;\ q_1, q_2, q_3;\ r_1, r_2, r_3) = F_3(p_2, p_1, p_3;\ q_2, q_1, q_3;\ r_2, r_1, r_3).$$

and

(2.7) $$F_3(p_1, p_2, p_3;\ q_1, q_2, q_3;\ r_1, r_2, r_3) \overset{(2.4)}{=} F_4(0, p_1, p_2, p_3;\ 0, q_1, q_2, q_3;\ 0, r_1, r_2, r_3) \overset{(1.3)}{=} F_4(p_3, p_1, p_2, 0;\ q_3, q_1, q_2, 0;\ r_3, r_1, r_2, 0) \overset{(2.5)}{=} F_4(p_1, p_3, p_2, 0;\ q_1, q_3, q_2, 0;\ r_1, r_3, r_2, 0) \overset{(1.3)}{=} F_4(0, p_3, p_2, p_1;\ 0, q_3, q_2, q_1;\ 0, r_3, r_2, r_1) \overset{(2.4)}{=} F_3(p_3, p_2, p_1;\ q_3, q_2, q_1;\ r_3, r_2, r_1).$$

Therefore,

(2.8) $$F_3(p_1, p_2, p_3; q_1, q_2, q_3;\ r_1, r_2, r_3) \overset{(2.6)}{=} F_3(p_2, p_1, p_3;\ q_2, q_1, q_3;\ r_2, r_1, r_3) \overset{(2.7)}{=} F_3(p_3, p_1, p_2;\ q_3, q_1, q_2;\ r_3, r_1, r_2) \overset{(2.6)}{=} F_3(p_1, p_3, p_2;\ q_1, q_3, q_2;\ r_1, r_3, r_2) \overset{(2.7)}{=} F_3(p_2, p_3, p_1;\ q_2, q_3, q_1;\ r_2, r_3, r_1) \overset{(2.6)}{=} F_3(p_3, p_2, p_1;\ q_3, q_2, q_1;\ r_3, r_2, r_1).$$

From (2.8), we get the symmetry of F_3 over S_3.

Case 2. When

(i) $p_i = 0,\ i = 1, 2, 3,\ 0 < p_j < 1,\ j \neq i = 1, 2, 3,\ 0 < q_j < 1,\ 0 < r_j < 1,$ $j = 1, 2, 3$ holds in F_3:

or

(ii) $q_i = 0 \Rightarrow p_i = 0,\ \ i = 1, 2, 3,\ \ 0 < p_j < 1,\ \ 0 < q_j < 1,\ \ j \neq i = 1, 2, 3,$ $0 < r_j < 1, j = 1, 2, 3$ holds in F_3:

or

(iii) $r_i = 0 \Rightarrow q_i = 0 \Rightarrow p_i = 0, i = 1, 2, 3, 0 < p_j < 1, 0 < q_j < 1, 0 < r_j < 1,$ $j \neq i = 1, 2, 3$ holds in F_3.

In these subcases, the proof is similar to case 1.

Case 3. When

(i) $p_i = 0,\ p_j = 0,\ i \neq j = 1, 2, 3,\ p_k = 1,\ k \neq i \neq j = 1, 2, 3,\ 0 < q_k < 1,$ $0 < r_k < 1,\ k = 1, 2, 3$ holds in F_3:

or

(ii) $p_i = 0,\ q_j = 0 \Rightarrow p_j = 0,\ j \neq i = 1, 2, 3,\ p_k = 1,\ k \neq i \neq j = 1, 2, 3,$ $0 < q_k < 1,\ k \neq j = 1, 2, 3,\ 0 < r_k < 1,\ k = 1, 2, 3$ holds in F_3:

or

(iii) $p_i = 0,\ r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0,\ i \neq j = 1, 2, 3,\ p_k = 1,\ k \neq i \neq j =$ $= 1, 2, 3,\ 0 < q_k < 1,\ 0 < r_k < 1,\ k \neq j = 1, 2, 3$ holds in F_3:

or

(iv) $q_i = 0 \Rightarrow p_i = 0,\ q_j = 0 \Rightarrow p_j = 0,\ i \neq j = 1, 2, 3,\ p_k = q_k = 1,\ k \neq i \neq$ $\neq j = 1, 2, 3,\ 0 < r_k < 1,\ k = 1, 2, 3$ holds in F_3:

or

(v) $q_i = 0 \Rightarrow p_i = 0,\ r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0,\ i \neq j = 1, 2, 3,\ p_k = q_k = 1,$ $k \neq i \neq j = 1, 2, 3,\ 0 < r_k < 1,\ k \neq j = 1, 2, 3$ holds in F_3:

In case (i), we have

$$(2.9)\quad F_3(0, 0, 1;\ q_1, q_2, q_3;\ r_1, r_2, r_3) \overset{(1.3)}{=} F_3(1, 0, 0;\ q_3, q_2, q_1;\ r_3, r_2, r_1)$$
$$\overset{(2.6)}{=} F_3(0, 1, 0;\ q_2, q_3, q_1;\ r_2, r_3, r_1) \overset{(2.7)}{=} F_3(0, 1, 0;\ q_1, q_3, q_2;\ r_1, r_3, r_2)$$
$$\overset{(2.6)}{=} F_3(1, 0, 0;\ q_3, q_1, q_2;\ r_3, r_1, r_2) \overset{(2.7)}{=} F_3(0, 0, 1;\ q_2, q_3, q_1;\ r_2, r_3, r_1)\,.$$

Thus (2.9) shows that F_3 is a symmetric function. Similarly, the proof of other sub-cases follows from sub case (i).

Case 4. When $r_i = 0 \Rightarrow q_i = 0 \Rightarrow p_i = 0,\ \ r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0,\ \ i \neq j =$ $= 1, 2, 3,\ p_k = q_k = r_k = 1,\ k \neq i \neq j = 1, 2, 3$ holds in F_3:

Then, by Postulate VI_n $(n = 3)$, we have

$$F_3(0, 0, 1;\ 0, 0, 1;\ 0, 0, 1) = F_3(1, 0, 0;\ 1, 0, 0;\ 1, 0, 0) =$$
$$= F_3(0, 1, 0;\ 0, 1, 0;\ 0, 1, 0)$$

Hence we have proved the symmetry of F_3 completely.

When $n = 4$. We have the following cases:

Case 1. When $0 < p_i < 1$, $0 < q_i < 1$ and $0 < r_i < 1$, $i = 1, 2, 3, 4$ holds in F_4:
Then, we have

$$(2.10)\qquad F_4(p_1, p_2, p_3, p_4;\ q_1, q_2, q_3, q_4;\ r_1, r_2, r_3, r_4) = \\ = F_4(p_2, p_1, p_3, p_4;\ q_2, q_1, q_3, q_4;\ r_2, r_1, r_3, r_4)$$

and

$$(2.11)\qquad F_4(p_1, p_2, p_3, p_4;\ q_1, q_2, q_3, q_4;\ r_1, r_2, r_3, r_4) = \\ \overset{(2.4)}{=} F_5(0, p_1, p_2, p_3, p_4;\ 0, q_1, q_2, q_3, q_4;\ 0, r_1, r_2, r_3, r_4) \\ \overset{(1.3)}{=} F_5(p_3, p_1, p_2, 0, p_4;\ q_3, q_1, q_2, 0, q_4;\ r_3, r_1, r_2, 0, r_4) \\ \overset{(2.10)}{=} F_5(p_1, p_3, p_2, 0, p_4;\ q_1, q_3, q_2, 0, q_4;\ r_1, r_3, r_2, 0, r_4) \\ \overset{(1.3)}{=} F_5(0, p_3, p_2, p_1, p_4;\ 0, q_3, q_2, q_1, q_4;\ 0, r_3, r_2, r_1, r_4) \\ \overset{(2.4)}{=} F_4(p_3, p_2, p_1, p_4;\ q_3, q_2, q_1, q_4;\ r_3, r_2, r_1, r_4)\,.$$

Similarly, we can show

$$(2.12)\qquad F_4(p_1, p_2, p_3, p_4;\ q_1, q_2, q_3, q_4;\ r_1, r_2, r_3, r_4) = \\ = F_4(p_4, p_2, p_3, p_1;\ q_4, q_2, q_3, q_1;\ r_4, r_2, r_3, r_1)\,.$$

$$(2.13)\qquad F_4(p_1, p_2, p_3, p_4;\ \underset{\text{I}}{q_1, q_2, q_3, q_4};\ r_1, r_2, r_3, r_4) = \\ \overset{(2.11)}{=} F_4(p_3, p_2, p_1, p_4;\ \underset{\text{II}}{q_3, q_2, q_1, q_4};\ r_3, r_2, r_1, r_4) \\ \overset{(2.12)}{=} F_4(p_4, p_2, p_1, p_3;\ \underset{\text{III}}{q_4, q_2, q_1, q_3};\ r_4, r_2, r_1, r_3) \\ \overset{(2.10)}{=} F_4(p_2, p_4, p_1, p_3;\ q_2, q_4, q_1, q_3;\ r_2, r_4, r_1, r_3) \\ \overset{(2.11)}{=} F_4(p_1, p_4, p_2, p_3;\ \underset{\text{IV}}{q_1, q_4, q_2, q_3};\ r_1, r_4, r_2, r_3) \\ \overset{(2.12)}{=} F_4(p_3, p_4, p_2, p_1;\ \underset{\text{V}}{q_3, q_4, q_2, q_1};\ r_3, r_4, r_2, r_1) \\ \overset{(2.10)}{=} F_4(p_4, p_3, p_2, p_1;\ q_4, q_3, q_2, q_1;\ r_4, r_3, r_2, r_1) \\ \overset{(2.12)}{=} F_4(p_1, p_3, p_2, p_4;\ \underset{\text{VI}}{q_1, q_3, q_2, q_4};\ r_1, r_3, r_2, r_4)$$

Using Postulate I_n ($n = 4$) and symmetry of F_2 and F_3 in I, II, III, IV, V and VI of (2.13), we have $4! = 24$ permutations of $F_4 \Rightarrow F_4$ is symmetric.

Case 2. When

(i) $p_i = 0$, $i = 1, 2, 3, 4$, $0 < p_j < 1$, $i \neq j = 1, 2, 3, 4$, $0 < q_j < 1$, $0 < r_j < 1$, $j = 1, 2, 3, 4$ holds in F_4:

or

(ii) $q_i = 0 \Rightarrow p_i = 0$, $i = 1, 2, 3, 4$, $0 < p_j < 1$, $0 < q_j < 1$, $i \neq j = 1, 2, 3, 4$, $0 < r_j < 1$, $j = 1, 2, 3, 4$ holds in F_4:

or

(iii) $r_i = 0 \Rightarrow q_i = 0 \Rightarrow p_i = 0$, $i = 1, 2, 3, 4$, $0 < p_j < 1$, $0 < q_j < 1$, $0 < r_j < 1$, $j \neq i = 1, 2, 3, 4$ holds in F_4:

The above sub-cases follows from case 1.

Case 3. When

(i) $p_i = 0,\ \ p_j = 0,\ \ i \neq j = 1, 2, 3, 4,\ \ 0 < p_k < 1,\ \ k \neq i \neq j = 1, 2, 3, 4,$ $0 < q_k < 1,\ 0 < r_k < 1,\ k = 1, 2, 3, 4$ holds in F_4:

or

(ii) $p_i = 0, q_j = 0 \Rightarrow p_j = 0,\ \ i \neq j = 1, 2, 3, 4,\ \ 0 < p_k < 1,\ \ k \neq i \neq j = 1, 2, 3, 4,\ \ 0 < q_k < 1,\ \ k \neq j = 1, 2, 3, 4,\ \ 0 < r_k < 1,\ \ k = 1, 2, 3, 4,$ holds in F_4:

or

(iii) $p_i = 0,\ r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0,\ i \neq j = 1, 2, 3, 4,\ 0 < p_k < 1,\ k \neq i \neq j = 1, 2, 3, 4,\ 0 < q_k < 1,\ 0 < r_k < 1,\ k \neq j = 1, 2, 3, 4$ holds in F_4:

or

(iv) $q_i = 0 \Rightarrow p_i = 0, q_j = 0 \Rightarrow p_j = 0, i \neq j = 1, 2, 3, 4,\ 0 < p_k < 1,\ 0 < q_k < 1, k \neq i \neq j = 1, 2, 3, 4,\ \ 0 < r_k < 1, k = 1, 2, 3, 4$ holds in F_4:

or

(v) $q_i = 0 \Rightarrow p_i = 0, r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0, i \neq j = 1, 2, 3, 4, 0 < p_k < 1,$ $0 < q_k < 1,\ k \neq i \neq j = 1, 2, 3, 4,\ 0 < r_k < 1,\ k \neq j = 1, 2, 3, 4$ holds in F_4:

or

(vi) $r_i = 0 \Rightarrow q_i = 0 \Rightarrow p_i = 0,\ \ r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0,\ \ i \neq j = 1, 2, 3, 4,$ $0 < p_k < 1,\ 0 < q_k < 1,\ 0 < r_k < 1,\ k \neq i \neq j = 1, 2, 3, 4$ holds in F_4:

Let us assume $p_1 = 0 = p_{10}, p_2 = 0 = p_{20}$ in (i) and using (2.10), (2.11) and (2.12) in F_4, we get

(2.14)
$$\begin{aligned}
&F_4(p_{10}, p_{20}, p_3, p_4;\ \underset{\text{I}}{q_1}, q_2, q_3, q_4;\ r_1, r_2, r_3, r_4) = \\
&\overset{(2.11)}{=} F_4(p_3, p_{20}, p_{10}, p_4;\ \underset{\text{II}}{q_3}, q_2, q_1, q_4;\ r_3, r_2, r_1, r_4) \\
&\overset{(2.12)}{=} F_4(p_4, p_{20}, p_{10}, p_3;\ \underset{\text{III}}{q_4}, q_2, q_1, q_3;\ r_4, r_2, r_1, r_3) \\
&\overset{(2.10)}{=} F_4(p_{20}, p_4, p_{10}, p_3;\ q_2, q_4, q_1, q_3;\ r_2, r_4, r_1, r_3) \\
&\overset{(2.11)}{=} F_4(p_{10}, p_4, p_{20}, p_3;\ \underset{\text{IV}}{q_1}, q_4, q_2, q_3;\ r_1, r_4, r_2, r_3) \\
&\overset{(2.12)}{=} F_4(p_3, p_4, p_{20}, p_{10};\ \underset{\text{V}}{q_3}, q_4, q_2, q_1;\ r_3, r_4, r_2, r_1) \\
&\overset{(2.10)}{=} F_4(p_4, p_3, p_{20}, p_{10};\ q_4, q_3, q_2, q_1;\ r_4, r_3, r_2, r_1) \\
&\overset{(2.12)}{=} F_4(p_{10}, p_3, p_{20}, p_4;\ \underset{\text{VI}}{q_1}, q_3, q_2, q_4;\ r_1, r_3, r_2, r_4).
\end{aligned}$$

Now we shall show below that I of (2.14) contributes 4 permutations of F_4 which are as follows:

(2.15) (a)
$$\begin{aligned}
&F_4(p_{10}, p_{20}, p_3, p_4;\ q_1, q_2, q_3, q_4;\ r_1, r_2, r_3, r_4) = \\
&\overset{(2.4)}{=} F_5(0, p_{10}, p_{20}, p_3, p_4;\ 0, q_1, q_2, q_3, q_4;\ 0, r_1, r_2, r_3, r_4) \\
&\overset{(1.3)}{=} F_5(p_{20}, p_{10}, 0, p_3, p_4;\ q_2, q_1, 0, q_3, q_4;\ r_2, r_1, 0, r_3, r_4) \\
&\overset{(2.4)}{=} F_6(0, p_{20}, p_{10}, 0, p_3, p_4;\ 0, q_2, q_1, 0, q_3, q_4;\ 0, r_2, r_1, 0, r_3, r_4) \\
&\overset{(1.3)}{=} F_6(p_{10}, p_{20}, 0, 0, p_3, p_4;\ q_1, q_2, 0, 0, q_3, q_4;\ r_1, r_2, 0, 0, r_3, r_4)
\end{aligned}$$

$$\overset{(1.3)}{=} F_6(0, p_{20}, 0, p_{10}, p_3, p_4;\ 0, q_2, 0, q_1, q_3, q_4;\ 0, r_2, 0, r_1, r_3, r_4)$$
$$\overset{(2.4)}{=} F_5(p_{20}, 0, p_{10}, p_3, p_4;\ q_2, 0, q_1, q_3, q_4;\ r_2, 0, r_1, r_3, r_4)$$
$$\overset{(1.3)}{=} F_5(0, p_{20}, p_{10}, p_3, p_4;\ 0, q_2, q_1, q_3, q_4;\ 0, r_2, r_1, r_3, r_4)$$
$$\overset{(2.4)}{=} F_4(p_{20}, p_{10}, p_3, p_4;\ q_2, q_1, q_3, q_4;\ r_2, r_1, r_3, r_4)\,.$$

(2.16) (b)
$$F_4(p_{10}, p_{20}, p_3, p_4;\ q_1, q_2, q_3, q_4;\ r_1, r_2, r_3, r_4) =$$
$$\overset{(2.4)}{=} F_5(0, p_{10}, p_{20}, p_3, p_4;\ 0, q_1, q_2, q_3, q_4;\ 0, r_1, r_2, r_3, r_4)$$
$$\overset{(1.3)}{=} F_5(p_3, p_{10}, p_{20}, 0, p_4;\ q_3, q_1, q_2, 0, q_4;\ r_3, r_1, r_2, 0, r_4)$$
$$\overset{(2.4),(1.3)}{=} F_6(p_4, p_3, p_{10}, p_{20}, 0, 0;\ q_4, q_3, q_1, q_2, 0, 0;\ r_4, r_3, r_1, r_2, 0, 0)$$
$$\overset{(1.3),(2.4)}{=} F_5(p_3, p_{10}, p_{20}, p_4, 0;\ q_3, q_1, q_2, q_4, 0;\ r_3, r_1, r_2, r_4, 0)$$
$$\overset{(1.3),(2.4)}{=} F_4(p_{10}, p_{20}, p_4, p_3;\ q_1, q_2, q_4, q_3;\ r_1, r_2, r_4, r_3)\,.$$

Similarly, we can show that

(2.17)
$$F_4(p_{10}, p_{20}, p_3, p_4;\ q_1, q_2, q_3, q_4;\ r_1, r_2, r_3, r_4) =$$
$$= F_4(p_{20}, p_{10}, p_4, p_3;\ q_2, q_1, q_4, q_3;\ r_2, r_1, r_4, r_3)\,.$$

Now using Postulate I_n $(n = 4)$ and symmetry of F_2 and F_3 in II, III, IV, V and VI of (2.14) and (2.15), (2.16) and (2.17) in I of (2.14) would yield 4! permutations of $F_4 \Rightarrow$ symmetry of F_4. Similarly, the proof of other subcases follows from sub case (i) of case 3.

Case 4. When

(i) $p_i = 0,\ p_j = 0,\ p_k = 0,\ i \neq j \neq k = 1, 2, 3, 4,\ p_l = 1,\ l \neq i \neq j \neq k = 1, 2, 3, 4,\ \ 0 < q_l < 1,\ 0 < r_l < 1,\ l = 1, 2, 3, 4$ holds in F_4:

or

(ii) $p_i = 0,\ p_j = 0,\ q_k = 0 \Rightarrow p_k = 0,\ i \neq j \neq k = 1, 2, 3, 4,\ p_l = 1,\ l \neq i \neq j \neq k = 1, 2, 3, 4,\ 0 < q_l < 1, l \neq k = 1, 2, 3, 4,\ 0 < r_l < 1, l = 1, 2, 3, 4,$ holds in F_4:

or

(iii) $p_i = 0,\ p_j = 0,\ r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0,\ i \neq j \neq k = 1, 2, 3, 4,\ p_l = 1,\ l \neq i \neq j \neq k = 1, 2, 3, 4,\ 0 < q_l < 1,\ 0 < r_l < 1,\ l \neq k = 1, 2, 3, 4,$ holds in F_4:

or

(iv) $p_i = 0,\ q_j = 0 \Rightarrow p_j = 0,\ q_k = 0 \Rightarrow p_k = 0,\ i \neq j \neq k = 1, 2, 3, 4,\ p_l = 1,\ l \neq i \neq j \neq k = 1, 2, 3, 4,\ 0 < q_l < 1,\ l \neq j \neq k = 1, 2, 3, 4,\ 0 < r_l < 1,\ l = 1, 2, 3, 4,$ holds in F_4:

or

(v) $p_i = 0, q_j = 0 \Rightarrow p_j = 0, r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0, i \neq j \neq k = 1, 2, 3, 4,\ p_l = 1,\ l \neq i \neq j \neq k = 1, 2, 3, 4,\ 0 < q_l < 1,\ l \neq j \neq k = 1, 2, 3, 4,\ 0 < r_l < 1,\ l \neq k = 1, 2, 3, 4$ holds in F_4:

or

(vi) $p_i = 0,\quad r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0,\quad r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0,\quad i \neq j \neq$ $\neq k = 1, 2, 3, 4,\ p_l = 1,\ l \neq i \neq j \neq k = 1, 2, 3, 4,\ 0 < q_l < 1,\ 0 < r_l < 1,$ $l \neq j \neq k = 1, 2, 3, 4$ holds in F_4:

or

(vii) $q_i = 0 \Rightarrow p_i = 0,\ q_j = 0 \Rightarrow p_j = 0,\ q_k = 0 \Rightarrow p_k = 0,\ i \neq j \neq k = 1, 2, 3, 4,$ $p_l = q_l = 1,\ l \neq i \neq j \neq k = 1, 2, 3, 4,\ 0 < r_l < 1,\ l = 1, 2, 3, 4$ holds in F_4:

or

(viii) $q_i = 0 \Rightarrow p_i = 0,\quad q_j = 0 \Rightarrow p_j = 0,\quad r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0,\quad i \neq j \neq$ $\neq k = 1, 2, 3, 4,\ p_l = q_l = 1,\ l \neq i \neq j \neq k = 1, 2, 3, 4,\ 0 < r_l < 1,\ l \neq$ $\neq k = 1, 2, 3, 4$ holds in F_4:

or

(ix) $q_i = 0 \Rightarrow p_i = 0,\ r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0,\ r_k = 0 \Rightarrow q_k = 0 \Rightarrow p_k = 0,\ i \neq$ $\neq j \neq k = 1, 2, 3, 4,\ p_l = q_l = 1,\ l \neq i \neq j \neq k = 1, 2, 3, 4,\ 0 < r_l < 1,$ $l \neq j \neq k = 1, 2, 3, 4$ holds in F_4:

Let us assume $p_1 = 0 = p_{10}, p_2 = 0 = p_{20},\ p_3 = 0 = p_{30}$ and $p_4 = 1$, in subcase (i) of case 4 and using (2.10), (2.11) and (2.12), we get

(2.18)
$$
\begin{aligned}
&F_4(p_{10}, p_{20}, p_{30}, p_4;\ \underset{\text{I}}{q_1}, q_2, q_3, q_4;\ r_1, r_2, r_3, r_4) = \\
&\overset{(2.11)}{=} F_4(p_{30}, p_{20}, p_{10}, p_4;\ \underset{\text{II}}{q_3}, q_2, q_1, q_4;\ r_3, r_2, r_1, r_4) \\
&\overset{(2.12)}{=} F_4(p_4, p_{20}, p_{10}, p_{30};\ \underset{\text{III}}{q_4}, q_2, q_1;\ q_3,\ r_4, r_2, r_1, r_3) \\
&\overset{(2.10)}{=} F_4(p_{20}, p_4, p_{10}, p_{30};\ q_2, q_4, q_1, q_3;\ r_2, r_4, r_1, r_3) \\
&\overset{(2.11)}{=} F_4(p_{10}, p_4, p_{20}, p_{30};\ \underset{\text{IV}}{q_1}, q_4, q_2, q_3;\ r_1, r_4, r_2, r_3) \\
&\overset{(2.12)}{=} F_4(p_{30}, p_4, p_{20}, p_{10};\ \underset{\text{V}}{q_3}, q_4, q_2, q_1;\ r_3, r_4, r_2, r_1) \\
&\overset{(2.10)}{=} F_4(p_4, p_{30}, p_{20}, p_{10};\ q_4, q_3, q_2, q_1;\ r_4, r_3, r_2, r_1) \\
&\overset{(2.12)}{=} F_4(p_{10}, p_{30}, p_{20}, p_4;\ \underset{\text{VI}}{q_1}, q_3, q_2, q_4;\ r_1, r_3, r_2, r_4)\,.
\end{aligned}
$$

Using Postulate I_n $(n = 4)$ and symmetry of F_2 and F_3 in III, IV and V of (2.18), and (2.15), (2.16) and (2.17) in I, II and VI of (2.18), we get 4! permutations of $F_4 \Rightarrow$ $\Rightarrow$ the function F_4 is a symmetric function. Similarly, the proof of other subcases of case 4 follows from subcase (i) of case 4.

Case 5. When $r_i = 0 \Rightarrow q_i = 0 \Rightarrow p_i = 0,\ r_j = 0 \Rightarrow q_j = 0 \Rightarrow p_j = 0,\ r_k = 0 \Rightarrow$ $\Rightarrow q_k = 0 \Rightarrow p_k = 0,\ i \neq j \neq k = 1, 2, 3, 4,\ p_l = q_l = r_l = 1,\ l \neq i \neq j \neq k =$ $= 1, 2, 3, 4$ holds in F_4:

Then symmetry of F_4, obviously, follows by applying Postulate VI_n $(n = 4)$ in F_4.

From case 1 to case 5, discussed above, we conclude that F_4 is a symmetric function for all set of values of p's, q's and r's.

When $n = m$

From the above results, we conclude:

(i) If F_2 has 2! permutations, then F_2 is a symmetric function;

(ii) If F_3 has 3! permutations, then F_3 is a symmetric function;

(iii) If F_4 has 4! permutations, then F_4 is a symmetric function;

Assuming that F_{m-1}, $m \geqq 5$ is a symmetric function and thus it has $(m-1)!$ permutations, we shall prove that F_m has $m!$ permutations which imply F_m is a symmetric function for $m \geqq 5$. We proceed as follows:

Case 1. When $0 < p_i < 1$, $0 < q_i < 1$, and $0 < q_i < 1$, $i = 1, 2, \ldots, m$ holds in F_m:
Then we have

$$(2.19)\qquad F_m(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_m; r_1, r_2, \ldots, r_m) = \\ = F_m(p_2, p_1, \ldots, p_m; q_2, q_1, \ldots, q_m; r_2, r_1, \ldots, r_m)$$

and by Lemma 2 and Postulate VI_n $(n \geqq 5)$ in the function F_m, $m \geqq 5$, we get

$$(2.20)\qquad F_m(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_m; r_1, r_2, \ldots, r_m) \overset{(1)}{=} \\ = F_m(p_3, p_2, p_1, p_4, \ldots, p_m; q_3, q_2, q_1, q_4, \ldots, q_m; r_3, r_2, r_1, r_4, \ldots, r_m) \overset{(2)}{=} \\ = F_m(p_4, p_2, p_3, p_1, p_5, \ldots, p_m; q_4, q_2, q_3, q_1, q_5, \ldots, q_m; r_4, r_2, r_3, r_1, r_5, \ldots, r_m) \overset{(3)}{=} \\ \overset{(4)}{=} F_m(p_5, p_2, p_3, p_4, p_1, p_6, \ldots, p_m; q_5, q_2, q_3, q_4, q_1, q_6, \ldots \\ \ldots, q_m; r_5, r_2, r_3, r_4, r_1, r_6, \ldots, r_m) = \ldots = \\ \overset{(m-2)\text{th}}{=} F_m(p_{m-1}, p_2, \ldots, p_{m-2}, p_1, p_m; q_{m-1}, q_2, \ldots, q_{m-2}, q_1, q_m; r_{m-1}, r_2, \ldots, r_1, r_m) \overset{(m-1)\text{th}}{=} \\ = F_m(p_m, p_2, \ldots, p_{m-1}, p_1; q_m, q_2, \ldots, q_{m-1}, q_1; r_m, r_2, \ldots, r_{m-1}, r_2)\,.$$

Using Postulate I_n $(n \geqq 5)$ and symmetry of F_2 in (2,), (3), ..., $(m-1)$th of (2.20), we get

$$(2.21)\qquad F_m(p_2, p_3, p_1, p_4, \ldots, p_m; q_2, q_3, q_1, q_4, \ldots, q_m; r_2, r_3, r_1, r_4, \ldots, r_m) \overset{(2)}{=} \\ \overset{(3)}{=} F(p_2, p_4, p_3, p_1, p_5, \ldots, p_m; q_2, q_4, q_3, q_1, q_5, \ldots, q_m; r_2, r_4, r_3, r_1, r_5, \ldots, r_m) = \\ = \ldots \overset{(m-1)\text{th}}{=} F_m(p_2, p_m, \ldots, p_{m-1}, p_1; q_2, q_m, \ldots, q_{m-1}, q_1; r_2, r_m, \ldots, r_{m-1}, r_1)\,.$$

Again using Lemma 2 and Postulate VI_n $(n \geqq 5)$ (as used in obtaining (2.11) and (2.12)) in (2), (3), ..., $(m-1)$th of (2.21), we get

$$(2.22)\qquad F_m(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_m; r_1, r_2, \ldots, r_m) \overset{(1)}{=} \\ \overset{(2)}{=} F_m(p_1, p_3, p_2, p_4, \ldots, p_m; q_1, q_3, q_2, q_4, \ldots, q_m; r_1, r_3, r_2, r_4, \ldots, r_m) = \\ \overset{(3)}{=} F_m(p_1, p_4, p_3, p_2, p_5, \ldots, p_m; q_1, q_4, q_3, q_2, q_5, \ldots, q_m; r_1, r_4, r_3, r_2, r_5, \ldots, r_m) = \\ = \ldots \overset{(m-1)\text{th}}{=} F_m(p_1, p_m, \ldots, p_{m-1}, p_2; q_1, q_m, \ldots, q_{m-1}, q_2; r_1, r_m, \ldots, r_{m-1}, r_2)\,.$$

Using (1) of (2.20) = (2) of (2.20) (i.e. replacement of 1st element of each distribution with third element of each distribution) in (3), (4), ..., $(m-1)$th of (2.22), we get

$$(2.23)\qquad \overset{(3)}{F_m}(p_3, p_4, p_1, p_2, p_5, \ldots, p_m;\ q_3, q_4, q_1, q_2, q_5, \ldots \\ \ldots, q_m;\ r_3, r_4, r_1, r_2, r_5, \ldots, r_m) = \\ = \overset{(4)}{F_m}(p_3, p_5, p_1, p_4, p_2, p_6, \ldots, p_m;\ q_3, q_5, q_1, q_4, q_2, q_6, \ldots \\ \ldots, q_m;\ r_3, r_5, r_1, r_4, r_2, r_6, \ldots, r_m) = \\ = \ldots = \overset{(m-1)\text{th}}{F_m}(p_3, p_m, \ldots, p_2;\ q_3, q_m, \ldots, q_2;\ r_3, r_m, \ldots, r_2)\,.$$

Similarly, use of (1) of (2.20) = (3) of (2.20) (i.e. replacement of first element of each distribution with fourth element of each distribution) in (4), (5), ..., $(m-1)$th of (2.22), we get

$$(2.24)\qquad \overset{(4)}{F_m}(p_4, p_5, \ldots, p_m;\ q_4, q_5, \ldots, q_m;\ r_4, r_5, \ldots, r_m) = \\ = \overset{(5)}{F_m}(p_4, p_6, \ldots, p_m;\ q_4, q_6, \ldots, q_m;\ r_4, r_6, \ldots, r_m) = \\ = \ldots = \overset{(m-1)\text{th}}{F_m}(p_4, p_m, \ldots;\ q_4, q_m, \ldots;\ r_4, r_m, \ldots)$$

and so on.

In the end, use (1) of (2.20) = $(m-2)$th of (2.20) in $(m-1)$th of (2.22), we get

$$(2.25)\qquad F_m(p_{m-1}, p_m, \ldots;\ q_{m-1}, q_m, \ldots;\ r_{m-1}, r_m, \ldots)\,.$$

Using Postulate I_n $(n \geqq 5)$, symmetry of F_2 and F_{m-1} in (2.22), (2.21), (2.23), (2.24), and so on, and (2.25) then each F_m in these would yield $2(m-2)!$ permutations of F_m and (2.22), (2.21), (2.23), (2.24), and so on, and (2.25) would yield $2(m-1)(m-2)!$, $2(m-2)(m-2)!, \ldots, 2(m-2)!$ permutations of F_m respectively. Therefore, the algebraic sum of all these permutations of F_m is $2(m-1)(m-2)! + 2(m-2)(m-2)! + \ldots + 2(m-2)! = m!$, which implies that F_m, $m \geqq 5$ is a symmetric function. Again, we may come across various cases similar to the one, as discussed in the symmetry of F_3 and F_4. They can be easily verified for the symmetry of F_m, $m \geqq 5$. Hence we conclude the symmetry of F_n, $n \geqq 2$.

Thus Lemma 3 is proved. □

Proof of the main theorem

Now Postulates I_n $(n = 3, 4)$ and VI_n $(n = 3, 4)$ gives $3!$ permutations of $F_3 \Rightarrow$ $\Rightarrow$ symmetry of F_3. Kannappan and Rathie [3] has also taken symmetry of F_3 as one of the postulate in their proof. Replacing 3-symmetry of F_3 by our Postulate VI_n $(n \geqq 3)$, the proof of the theorem follows from their lines of action. Hence the theorem is proved. □

Remarks.

1. The authors have proved in this paper that the symmetry of generalized directed divergence (1.1) for $n \geqq 2$ follows from Postulates I_n $(n \geqq 3)$ and VI_n $(n \geqq 3)$

and thus have proved that (1.1) can be characterized without symmetry postulate.

2. It has been analytically proved that F_n has $n!$, $(n \geqq 2)$ permutations $\Rightarrow F_n$, $(n \geqq 2)$ is a symmetric function.

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