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REPRESENTABILITY OF RECURSIVE P. MARTIN-LÖF TESTS

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The present paper, closely connected with [2], investigates the possibility of expressing P. Martin-Löf's complexity theory of strings in terms of Kolmogorov's complexity of strings which uses algorithms φ. We find for every recursive P. Martin-Löf test V an algorithm φ which in turn gives a P. Martin-Löf test $V(\varphi)$ such that $V \subset V(\varphi)$. The equality $V = V(\varphi)$ holds for some particular P. Martin-Löf tests called representable.

In this paper we continue our efforts to approach Kolmogorov's complexity theory of strings which uses algorithms φ (see [3]) and P. Martin-Löf's complexity theory of strings which uses $M - L$ tests V (see [5]). A very good up-to-date survey paper is [7]. The present authors have already done some attempts in this direction in [2]. We work within the general framework of a not necessarily binary alphabet (see [1]).

It has already been noticed that these theories are not equivalent (see [2]).

In this paper we find for every recursive $M - L$ test V an algorithm φ which in turn gives a $M - L$ test $V(\varphi)$ such that $V \subset V(\varphi)$ (see Theorem 2). The equality $V = V(\varphi)$ holds for some particular $M - L$ tests V which we call *representable* (see Theorem 3). Such an equality $V = V(\varphi)$ would be a good interpretation of the somewhat unprecise term "equivalence" between Kolmogorov's and P. Martin-Löf's theories. In this respect see also Theorem 4.

The last section of our paper contains remarks and open problems.

1. BASIC NOTIONS

Throughout the paper N will be the set of all natural numbers, i.e. $N = \{0, 1, 2, \ldots\}$. If A is a finite set, card A will be the number of elements in A.

For every non-empty sets A and B and for every function $f : A' \to B$ (where $A' \subset A$) we shall write $f : A \overset{\circ}{\to} B$. We shall say that f is a *partial function* from A to B. We consider that $f(x) = \infty$ in case f is not defined in the point x.

If $f : A \overset{\circ}{\to} B$ is a partial function, then the domain of f is the set $\mathrm{dom}(f) = \{x \in A \mid f(x) \neq \infty\}$; range $(f) = \{f(x) \mid x \in \mathrm{dom}(f)\}$; graph $(f) = \{(x, f(x)) : x \in \mathrm{dom}(f)\}$.

Let $X = \{a_1, a_2, \ldots, a_p\}$, $p \geqq 2$ be a finite alphabet. Denote by X^* the free monoid generated by X under concatenation, i.e. X^* consists of all strings $x = x_1x_2 \ldots x_n$, where the x_i belong to X; also the null string λ belongs to X^*. For every a in X and every natural $n > 0$, $a^n = aa \ldots a$ (n copies of a). We shall consider that $a^0 = \lambda$. For every x in X^*, $l(x)$ is the length of x, i.e. $l(x) = m$ in case $x = x_1x_2 \ldots x_m$ and $l(\lambda) = 0$. For Recursive Function Theory see [4] and [6]. We shall consider *partial recursive functions* (*p.r. functions* in the sequel) $\varphi : X^* \times N \overset{\circ}{\to} X^*$ or $g : N - \{0\} \overset{\circ}{\to} X^* \times N$.

For every p.r. function $\varphi : X^* \times N \overset{\circ}{\to} X^*$, the *Kolmogorov complexity* induced by φ is a function $K_\varphi : X^* \times N \to N \cup \{\infty\}$, defined by $K_\varphi(x \mid m) = \min\{l(y) \mid y \in X^*, \varphi(y, m) = x\}$ in case $x = \varphi(y, m)$ for some y in X^* and $K_\varphi(x \mid m) = \infty$, otherwise.

For every $W \subset X^* \times (N - \{0\})$ and for every natural $m \geqq 1$ we shall write $W_m = \{x \in X^* \mid (x, m) \in W\}$. We define the *critical level induced by* W to be the function $m_W : X^* \to N \cup \{\infty\}$ given by $m_W(x) = \sup\{m \in N \mid m \geqq 1,\ x \in W_m\}$ in case such m exists, and $m_W(x) = 0$, in the opposite case.

A non-empty recursively enumerable set $V \subset X^* \times (N - \{0\})$ will be called *Martin-Löf test* ($M - L$ test) if it possesses the following two properties:

1) For every natural $m \geqq 1$, $V_{m+1} \subset V_m$.

2) For every natural numbers m, n, $m \geqq 1$,

$$\mathrm{card}\{x \in X^* \mid l(x) = n, x \in V_m\} < p^{n-m}/(p - 1).$$

We agree upon the fact that the empty set is a $M - L$ test.

The second condition enables us to say that m_V takes only finite values for every $M - L$ test V, because in case $(x, m) \in V$, then $m \leqq l(x) - 1$ (directly from the definition).

For every p.r. function $\varphi : X^* \times N \overset{\circ}{\to} X^*$ we can obtain the particular $M - L$ test $V(\varphi) = \{(x, m) \in X^* \times (N - \{0\}) \mid K_\varphi(x \mid l(x)) < l(x) - m\}$, see [1]. We shall call a $M - L$ test V *representable* in case there exists a p.r. function $\varphi : X^* \times N \overset{\circ}{\to} X^*$ such that $V = V(\varphi)$, see [2].

The lexicographical order on X^* induced by $a_1 < a_2 < \ldots < a_p$ is given by $\lambda < a_1 < a_2 < \ldots < a_p < a_1a_1 < a_1a_2 < \ldots < a_1a_p < a_2a_1 < \ldots$. The enumeration of X^* in this order will be $y(1) = \lambda$, $y(2) = a_1$, $y(3) = a_2, \ldots, y(p + 1) = a_p$, $y(p + 2) = a_1a_1, \ldots$. It follows that $a_p^m = y(s(m))$, where $s(m) = 1 + p + p^2 + \ldots + p^m = (p^{m+1} - 1)/(p - 1)$. This enumeration of X^* is recursive. In the sequel, the *lexicographical order* will mean this lexicographical order on X^*.

2. RESULTS

It is easily seen that there exist $\boldsymbol{M} - \boldsymbol{L}$ tests which are not recursive. For instance, take $A \subset \{a_1\}^* = \{\lambda, a_1, a_1^2, a_1^3, \ldots\}$ which is recursively enumerable and not recursive. Then $V = (A - \{\lambda, a_1\}) \times \{1\}$ is a non-recursive $\boldsymbol{M} - \boldsymbol{L}$ test.

The following theorem gives necessary and sufficient conditions under which a $\boldsymbol{M} - \boldsymbol{L}$ test is recursive.

Theorem 1. A $\boldsymbol{M} - \boldsymbol{L}$ test V is recursive iff the function m_V is recursive.

Proof. If m_V is recursive we can compute $m_V(x)$ for every x in X^*. Let (x, m) be in $X^* \times (N - \{0\})$. If $m_V(x) \geqq m$, then $(x, m) \in V$; if $m_V(x) < m$, then $(x, m) \notin V$. Thus V is recursive.

Now suppose V is recursive which means that χ_V is a recursive function (χ_V is the characteristic function of V). It is easy to see that for every x in X^* we have $m_V(x) = \max\{m \in N \mid \chi_V(x, m) = 1\}$, in case $(x, 1) \in V$, and $m_V(x) = 0$, in case $(x, 1) \notin V$. This shows that m_V is recursive. □

Actually, the object of our paper will be the study of some properties of *recursive* $\boldsymbol{M} - \boldsymbol{L}$ tests.

Theorem 2. Let $V \subset X^* \times N$ be a recursive $\boldsymbol{M} - \boldsymbol{L}$ test. Then there exists a p.r. function $\varphi : X^* \times N \overset{\circ}{\rightarrow} X^*$ such that $V \subset V(\varphi)$.

The p.r. function φ can be taken to possess the following properties:

(a) The function φ is injective.

(b) The graph of φ is recursive.

(c) For every x in X^*, we have the equivalence: $(x, 1) \in V$ iff $(x, 1) \in V(\varphi)$.

Proof. The set $A = \{(x, m_V(x)) \mid x \in V_1\}$ is obviously recursive. We distinguish two cases: i) *V is infinite* and in this case there exists an injective recursive function $g : N - \{0\} \rightarrow X^* \times N$ such that $g(N - \{0\}) = A$; ii) *V is finite* and A has q elements, and in this case there exists an injective (p.r.) function $g : \{1, 2, \ldots, q\} \rightarrow X^* \times N$ such that $g(\{1, 2, \ldots, q\}) = A$. In all cases, if i is in the domain of g, we put $g(i) = (x_i, m_V(x_i))$. Moreover, due to the recursiveness of V, we can suppose that g has the following "lexicographical" property: for all natural $1 \leqq i < j$:

u) $l(x_i) \leqq l(x_j)$,

v) if $l(x_i) = l(x_j)$, then $m_V(x_i) \geqq m_V(x_j)$.

We can define the procedure for φ.

For $i = 1$, $g(1) = (x_1, m_V(x_1))$ and we put $\varphi(z_1, l(x_1)) = x_1$, where $z_1 = y(s(l(x_1) - m_V(x_1) - 1))$. See the definition of s in Section 1.

Next, let $i = 2$, so $g(2) = (x_2, m_V(x_2))$. In case $l(x_1) \neq l(x_2)$, we put $\varphi(z_2, l(x_2)) = x_2$, where $z_2 = y(s(l(x_2) - m_V(x_2) - 1))$. In case $l(x_1) = l(x_2)$, we consider the greatest element (according to the lexicographical order) of the set $\{y(1), y(2), \ldots$

$\ldots, y(s(l(x_2) - m_V(x_2) - 1))\} - \{z_1\}$, and we shall call this element z_2. Put $\varphi(z_2, l(x_2)) = x_2$.

Continuing the procedure we reach the step $i > 1$. There are two cases. In the first case $l(x_i) \neq l(x_j)$, for all $j < i$. In this case we put $\varphi(z_i, l(x_i)) = x_i$, where $z_i = y(s(l(x_i) - m_V(x_i) - 1))$. In the second (opposite) case let $j(1) < j(2) < \ldots < j(k) < i$ be all indices $j < i$ such that $l(x_i) = l(x_j)$. In fact, $j(2) = j(1) + 1$, $j(3) = j(2) + 1, \ldots$, due to the properties of the enumeration function g. We define z_i to be the greatest element (in lexicographical order) of the set $\{y(1), y(2), \ldots, y(s(l(x_i) - m_V(x_i) - 1))\} - \{z_{j(1)}, z_{j(2)}, \ldots, z_{j(k)}\}$ and put $\varphi(z_i, l(x_i)) = x_i$. Notice that φ acts as a function, because if $l(x_i) = l(x_j)$ we have $z_i \neq z_j$.

The construction is possible and the motivation follows. Put $l(x_i) = l(x_{j(1)}) = l(x_{j(2)}) = \ldots = l(x_{j(k)}) = l$. We have $m_i = m_V(x_i) \leqq m_k = m_V(x_{j(k)}) \leqq m_{k-1} = m_V(x_{j(k-1)}) \leqq \ldots \leqq m_1 = m_V(x_{j(1)})$. For every natural $t \in \{1, 2, \ldots, k\} \cup \{i\}$ let $\boldsymbol{B}_t = \{y(1), y(2), \ldots, y(s(l - m_t - 1))\}$. Notice that $\boldsymbol{B}_1 \subset \boldsymbol{B}_2 \subset \ldots \subset \boldsymbol{B}_k \subset \boldsymbol{B}_i$ and $\boldsymbol{B}_u = \boldsymbol{B}_v$(for $u < v$) iff $m_u = m_v$. We shall try to describe in a detailed manner the action of φ and this will complete the motivation.

Clearly, $z_{j(1)} = y(s(l - m_1 - 1))$. In order to obtain $z_{j(2)}$, we distinguish two possible cases: a) $m_1 > m_2$ (and in this case $z_{j(2)} = y(s(l - m_2 - 1))$; *b*) $m_1 = m_2$ (and in this case $\boldsymbol{B}_1 = \boldsymbol{B}_2$, so $z_{j(2)}$ must be $y(s(l - m_2 - 1) - 1)$). It is to be seen that in case b) one has $s(l - m_2 - 1) - 1 \geqq 1$ (in other words the construction is possible) because $2 \leqq \text{card}\,\{x \in \boldsymbol{X}^* \mid l(x) = l \text{ and } (x, m_2) \in \boldsymbol{V}\} \leqq (p^{l-m_2} - 1)/(p - 1) = s(l - m_2 - 1)$. The case when strict inclusion occurs between the $\boldsymbol{B}_t'$s being clearly favorable, we focus our attention to the "bad" situation $m_h = m_{h+1} = m_{h+2} = \ldots = m_r = m$ $(1 \leqq h \leqq r \leqq i)$. Here, in case $h > 1$, we consider $m_{h-1} < m_h$. We have $\boldsymbol{B}_h = \boldsymbol{B}_{h+1} = \ldots = \boldsymbol{B}_r$. The construction gives: $z_{j(h)} = y(s(l - m - 1))$, $z_{j(h+1)} = y(s(l - m - 1) - 1, \ldots$, $z_{j(r)} = y(s(l - m - 1) - (r - h))$. It remains to show that $s(l - m - 1) - (r - h) \geqq 1$, i.e. $r - h + 1 \leqq (p^{l-m} - 1)/(p - 1)$. This inequality follows from $r - h + 1 \leqq \text{card}\,\{x \in \boldsymbol{X}^*: l(x) = l,\ (x, m) \in \boldsymbol{V}\} \leqq (p^{l-m} - 1)/(p - 1)$.

It is worth to add the fact that in case $\boldsymbol{V}$ is finite and the set $\boldsymbol{A}$ (see the beginning of the proof) has q elements, the procedure stops at step q.

The injectivity of φ is derived from the injectivity of $g : (x_i, m_V(x_i)) \neq (x_j, m_V(x_j))$ iff $x_i \neq x_j$ or $m_V(x_i) \neq m_V(x_j)$. This implies that for different i and j one must obtain different values $\varphi(z_i, l(x_i)) = x_i$ and $\varphi(z_j, l(x_j)) = x_j$.

Our next task is to prove the inclusion $\boldsymbol{V} \subset \boldsymbol{V}(\varphi)$. Indeed, in case (x, m) is in $\boldsymbol{V}$ let $(x, m_V(x)) = (x_i, m_V(x_i))$ in the enumeration given by g. So $m \leqq m_V(x_i)$ and $x_i = \varphi(z_i, l(x_i))$ where the length of z_i is less than $l(x_i) - m_V(x_i) - 1$, which shows that $\boldsymbol{K}_\varphi(x \mid l(x)) \leqq l(x_i) - m_V(x_i) - 1 < l(x) - m_V(x)$, i.e. $(x, m_V(x)) \in \boldsymbol{V}(\varphi)$. Consequently, (x, m) is in the $\boldsymbol{M} - \boldsymbol{L}$ test $\boldsymbol{V}(\varphi)$.

Moreover, we can prove here also point (c), because it is seen that for every x in $\boldsymbol{X}^*$ such that $(x, 1) \in \boldsymbol{V}(\varphi)$ there exists a natural i such that $x = x_i$ and $(x_i, m_V(x_i)) \in \boldsymbol{V}$, which implies $(x, 1) \in \boldsymbol{V}$.

All it remains to prove is point (b), i.e. the recursiveness of the graph of φ. This is seen taking arbitrarily $((z, l), x) \approx (z, l, x)$ in $X^* \times N \times X^*$ and checking if (z, l, x) belongs to the graph of φ, according to the following decision algorithm:

1. If $m_V(x) = 0$, NO. Stop.
2. If $l(x) \neq l$, NO. Stop.
3. Choose i such that $g(i) = (x_i, m_V(x_i))$ and $x = x_i$.
4. Run the first i steps in the procedure defining φ in order to find z_i.
5. If $z = z_i$, YES. Stop.
6. If $z \neq z_i$, NO. Stop. □

Remark. It is obvious that for a given recursive $M - L$ test V there are many p.r. functions $\varphi : X^* \times N \overset{\circ}{\to} X^*$ such that $V \subset V(\varphi)$, e.g. our construction depends on the enumeration function g.

Theorem 3. Let V be a $M - L$ test. Consider the following assertions:
(1) V is representable.
(2) For every natural $m \geqq 1$, one has

$$(*) \qquad \text{for all} \quad n \geqq m + 1\,, \quad \text{card}\,\{x \in X^* \mid l(x) = n,\, m_V(x) = m\} \leqq p^{n-m-1}\,.$$

Then (1) ⇒ (2) and in case V is recursive the implication (2) ⇒ (1) holds too.

Proof. (1) ⇒ (2). The hypothesis is that $V = V(\varphi)$ for some p.r. function $\varphi : X^* \times N \overset{\circ}{\to} X^*$.

Fix the natural numbers $n > m > 0$. For every x in X^* with $l(x) = n$ and such that $m_V(x) = m$ there exists y in X^* with $l(y) < l(x) - m$ and $\varphi(y, l(x)) = x$. We have $l(y) \leqq n - m - 1$.

We shall show that $l(y) = n - m - 1$. Supposing by contradiction $l(y) \leqq n - m - 2$, let $l(y) = n - m - 1 - h$ with $h > 0$. This will lead us to the false relation $(x, m + h) \in V$. Indeed, $l(y) = n - m - h - 1 < n - m - h$ and $\varphi(y, l(x)) = x$ show that $(x, m + h) \in V(\varphi) = V$.

The just proved equality $l(y) = n - m - 1$ shows that

$$\text{card}\,\{x \in X^* \mid l(x) = n,\, m_V(x) = m\} \leqq \text{card}\,\{y \in X^* \mid l(y) = n - m - 1\} = p^{n-m-1}\,,$$

and the assertion (2) is proved.

Assuming that V is recursive we shall prove (2) ⇒ (1). The hypothesis is that $(*)$ holds for every $m \geqq 1$. We shall show that $V = V(\varphi)$, where φ is the p.r. function constructed in Theorem 2, namely we shall show that $V(\varphi) \subset V$.

Take (x, m) in $V(\varphi)$. In any case $(x, 1) \in V$ (see Theorem 2). We shall prove that $(x, m) \in V$ by proving that $m_V(x) = m_{V(\varphi)}(x)$. Since $V \subset V(\varphi)$ (see Theorem 2) we have $m_{V(\varphi)}(x) \geqq m_V(x)$ and all it remains to prove is that $m_V(x) \geqq m_{V(\varphi)}(x)$.

Supposing the contrary, it follows that $(x, m_V(x) + 1) \in V(\varphi)$, hence there exists z in X^* with $l(z) < l(x) - m_V(x) - 1$ and $\varphi(z, l(x)) = x$.

Let $g(i) = (x_i, m_V(x_i))$ where $x = x_i$ in the enumeration given by g (see the construction of φ in the proof of Theorem 2). We let the procedure giving φ run i steps and we obtain the string z_i such that $\varphi(z_i, l(x_i)) = x_i$. We shall show that $l(z_i) = l(x_i) - m_V(x_i) - 1 = l(x) - m_V(x) - 1$, thus deriving a contradiction (in view of the injectivity of φ).

Now the reader must remember the action of φ (see the proof of Theorem 2). In case $l(x_i) \neq l(x_j)$ for all $j < i$, we have $l(z_j) = l(x_i) - m_V(x_i) - 1$, and the proof is finished in this case. In case $l(x_{j(1)}) = l(x_{j(2)}) = \ldots = l(x_{j(k)}) = l(x_i)$, $1 \leqq j(1) < j(2) < \ldots < j(k) < i$, we have analysed several possibilities, according to the existence of some equalities in the sequence of inequalities: $m_V(x_{j(1)}) \geqq m_V(x_{j(2)}) \geqq \ldots \geqq m_V(x_{j(k)}) \geqq m_V(x_i)$. In the case of the strict inequality $m_V(x_i) < m_V(x_{j(k)})$ we saw that $l(z_i) = l(x_i) - m_V(x_i) - 1$, and again the proof is finished. The most complicated case is when $m_V(x_i) = m_V(x_{j(k)}) = m_V(x_{j(k-1)}) = \ldots = m_V(x_{j(k-r)})$, where $0 \leqq r < k$. In this case we must put $z_i = y(s(l(x_i) - m_V(x_i) - 1) - (r + 1))$. In any case we have $r + 2$ elements x such that $l(x) = n$ and $m_V(x) = m$ (we put $l(x_i) = n$ and $m_V(x_i) = m$) and the hypothesis gives $r + 2 \leqq p^{n-m-1} = \text{card}\,\{z \in X^* \mid l(z) = n - m - 1\}$. But $y(s(n - m - 1))$ is the last element (in lexicographical order) of the set $\{z \in X^* \mid l(z) = n - m - 1\} = H$. It follows that $z_i \in H$, which shows that the length of z_i is $n - m - 1$ and the proof is finished in this case too. □

The next result establishes a precise connection between the Kolmogorov complexity K_φ and the critical level m_V in case $V = V(\varphi)$.

Theorem 4. Let V be a representable $M - L$ test and let $\varphi : X^* \times N \xrightarrow{\circ} X^*$ be a p.r. function such that $V = V(\varphi)$.

The following assertions hold for all x in X^*:

(a) $m_V(x) = 0$ iff $K_\varphi(x \mid l(x)) \geqq l(x) - 1$.

(b) If $m_V(x) > 0$, then $K_\varphi(x \mid l(x)) = l(x) - m_V(x) - 1$.

In the particular case when φ has the additional property that $\text{range}(\varphi) = \{x \in X^* \mid (x, 1) \in V\} = V_1$, point (a) can be stated more precisely, namely:

(a′) $m_V(x) = 0$ iff $K_\varphi(x \mid l(x)) = \infty$.

Proof. (a) Assume $m_V(x) = 0$, therefore $(x, 1) \notin V = V(\varphi)$. This shows that for every y in X^* with $l(y) < l(x) - 1$ we have $\varphi(y, l(x)) \neq x$. Then, either $\varphi(y, l(x)) \neq x$ for all y in X^* (which shows that $K_\varphi(x \mid l(x)) = \infty$), or there exists y in X^* with $\varphi(y, l(x)) = x$, but this y must have $l(y) \geqq l(x) - 1$. So, $K_\varphi(x \mid l(x)) \geqq l(x) - 1$.

Assume now that $K_\varphi(x \mid l(x)) \geqq l(x) - 1$. There are two cases:

i) if $K_\varphi(x \mid l(x)) = \infty$, then $\varphi(y, l(x)) \neq x$ for all y in X^* and then $(x, 1) \notin V(\varphi)$ a.s.o.

ii) if $K_\varphi(x \mid l(x)) < \infty$, then there exists at least one y in X^* with $\varphi(y, l(x)) = x$ and one must have $l(y) \geqq l(x) - 1$. This shows that $(x, 1) \notin V(\varphi)$.

(b) According to the hypothesis, there exists y in X^* such that $\varphi(y, l(x)) = x$.

We have: $m_V(x) = m_{V(\varphi)}(x) = \max\{m \geqq 1 \mid$ there exists y in X^* with $l(y) < l(x) - m$ and $\varphi(y, l(x)) = x\} = \max\{m \geqq 1 \mid$ there exists y in X^* with $m < l(x) - l(y)$ and $\varphi(y, l(x)) = x\}$. The last maximum is attained for those y in X^* which have minimum length, i.e. for those y in X^* with $l(y) = K_\varphi(x \mid l(x))$. So, $m_V(x) = l(x) - K_\varphi(x \mid l(x)) - 1$.

In the particular case, all it remains to prove is the implication: $m_V(x) = 0 \Rightarrow K_\varphi(x \mid l(x)) = \infty$. Indeed, $m_V(x) = 0$ implies $(x, 1) \notin V = V(\varphi)$, so $x \notin$ range (φ). □

Corollary 5. Let V be a recursive representable $M - L$ test and let $\varphi : X^* \times N \overset{\circ}{\to} X^*$ be a p.r. function with the properties $V = V(\varphi)$ and range $(\varphi) = V_1$. Then the partial function $U_\varphi : X^* \overset{\circ}{\to} N$ given by $U_\varphi(x) = K_\varphi(x \mid l(x))$ is a p.r. function with recursive graph.

Proof. Relations (a') and (b) in Theorem 4 applied to the present function φ show that U_φ is a p.r. function. The graph of U_φ is recursive because the pair $(x, m) \in$ graph (U_φ) iff $m_V(x) > 0$ and $m = l(x) - m_V(x) - 1$. Here we made use of the recursiveness of the function m_V (see Theorem 1). □

Remark. The p.r. function φ given by the proof of Theorem 3 is a function satisfying the property that range $(\varphi) = V_1$.

Theorem 6. Let $\varphi : X^* \times N \overset{\circ}{\to} X^*$ be a p.r. function such that range $(\varphi) = (V(\varphi))_1$. Then the following assertions are equivalent:

(i) The partial function $U_\varphi : X^* \overset{\circ}{\to} N$ given by $U_\varphi(x) = K_\varphi(x \mid l(x))$ is a p.r. function with recursive graph.

(ii) The $M - L$ test $V(\varphi)$ is recursive.

Proof. (i) $\Rightarrow$ (ii). The proof will be given by the following equivalences: $((x, m) \in V(\varphi)) \Leftrightarrow (U_\varphi(x) < l(x) - m) \Leftrightarrow (U_\varphi(x) \in \{0, 1, 2, \ldots, l(x) - m - 1\}) \Leftrightarrow (U_\varphi(x) = 0$ or $U_\varphi(x) = 1$ or ... or $U_\varphi(x) = l(x) - m - 1) \Leftrightarrow ((x, 0) \in$ graph (U_φ) or $(x, 1) \in$ graph (U_φ) or ... or $(x, l(x) - m - 1) \in$ graph $(U_\varphi))$. We made the convention that in case $l(x) - m - 1 < 0$, the set $\{0, 1, 2, \ldots, l(x) - m - 1\}$ is empty.

(ii) $\Rightarrow$ (i). We put $V = V(\varphi)$ and apply Corollary 5 to this V and this φ. □

The following theorem will furnish an interesting class of recursive representable $M - L$ tests.

Theorem 7. Let $V \subset X^* \times N$ have the following properties:

(a) The set V is recursively enumerable.

(b) For every natural $m \geqq 1$ we have the inclusion $V_{m+1} \subset V_m$.

1. The following assertions are equivalent:

(i) For all natural $n > m \geqq 1$, we have:

$$\text{card}\,\{x \in X^* \mid l(x) = n\,,\quad (x, m) \in V\} = (p^{n-m} - 1)/(p - 1)\,.$$

(ii) For all natural $n > m \geqq 1$, we have:

$$\text{card}\,\{x \in X^* \mid l(x) = n\,,\ m_V(x) = m\} = p^{n-m-1}\,.$$

2. If one of the above conditions (i) or (ii) is fulfilled for a set V having properties (a) and (b), then V is a recursive representable $M - L$ test. Such $M - L$ tests will be called *full*.

Proof. 1. (i) $\Rightarrow$ (ii). The conditions (a), (b) and (i) insure that V is a $M - L$ test, hence m_V takes only finite values.

On the other hand, for every natural $j \geqq 0$ and $n \geqq j + 1$ one can see, using condition (b), that $\{x \in X^* \mid l(x) = n,\ m_V(x) = n - (j + 1)\} = \{x \in X^* \mid l(x) = n\,,\ (x, n - j - 1) \in V\} - \{x \in X^* \mid l(x) = n,\ (x, n - j) \in V\}$. Consequently, $\text{card}\,\{x \in X^* \mid l(x) = n, m_V(x) = n - (j + 1)\} = ((p^{n-(n-j-1)} - 1)/(p - 1)) - ((p^{n-(n-j)} - 1)/(p - 1)) = p^j$, using the hypothesis and condition (b). Taking $n - (j + 1) = m$, we obtain $\text{card}\,\{x \in X^* \mid l(x) = n, m_V(x) = m\} = p^{n-m-1}$.

(ii) $\Rightarrow$ (i). For every natural $n > m \geqq 1$ we have the equality

$$(**) \qquad \{x \in X^* \mid l(x) = n, (x, m) \in V\} = \{x \in X^* \mid l(x) = n, m_V(x) = m\} \cup$$
$$\cup \{x \in X^* \mid l(x) = n, m_V(x) = m + 1\} \cup \ldots \cup \{x \in X^* \mid l(x) = n, m_V(x) = n - 1\}\,.$$

In fact, $A_n = \{x \in X^* \mid l(x) = n,\ m_V(x) = n\} = \emptyset$, because $A_n \subset A_{n-1} = \{x \in X^* \mid l(x) = n,\ m_V(x) = n - 1\}$, according to condition (b) and $\text{card}\, A_{n-1} = 1$. If A_n were be non empty, then $\text{card}\, A_n = 1$, so $A_n = A_{n-1}$ and this is impossible. Again condition (b) guarantees also that $A_u = \emptyset$, where $u > n$. Thus the proof of (**) is complete.

Consequently, (**) yields

$$\text{card}\,\{x \in X^* \mid l(x) = n, (x, m) \in V\} =$$
$$= \sum_{j=m}^{n-1} \text{card}\,\{x \in X^* \mid l(x) = n, m_V(x) = j\} = \sum_{j=m}^{n-1} p^{n-j-1} = (p^{n-m} - 1)/(p - 1)\,.$$

2. All it remains to prove is that (i) implies the recursiveness of V (because in this case V will be a recursive $M - L$ test satisfying condition (2) in Theorem 3).

The case when V is finite is obvious.

Assume therefore that V is infinite and let $g : (N - \{0\}) \to X^* \times N$ be an injective recursive function such that $g(N - \{0\}) = V$. Put $g(i) = (x_i, m_i)$ for all natural $i \geqq 1$.

We take arbitrarily (x, m) in $X^* \times N$ and we describe an algorithm for testing if (x, m) is in V. Put $l(x) = n$. There exists a natural $q \geqq 1$ such that the set $G = \{g(1), g(2), \ldots, g(q)\}$ contains all the elements $(y, m) \in V$ with $l(y) = n$. Moreover, q can be effectively found. For instance, q can be taken to be the least natural number h such that the set $\{g(1), g(2), \ldots, g(h)\}$ contains exactly $(p^{n-m} - 1)/(p - 1)$ pairs (y, m) with $l(y) = n$. If $(x, m) \in G$, then $(x, m) \in V$ and if $(x, m) \notin G$, then $(x, m) \notin V$. □

Example 8. We shall exhibit an example of $M - L$ test V which is full and we shall also construct the associate p.r. function φ such that $V = V(\varphi)$ given by Theorem 3.

a) In order to give the $M - L$ test V we shall denote, for every $n > m \geqq 1$, by $A(n, m)$ the set $\{(x, m) \in V \mid l(x) = n\}$. It is clear that the $M - L$ test V will be completely determined if we shall give all the sets $A(n, m)$.

Put $A(n, m) = \{(y(s(n - 1) + i), m) \mid i = 1, 2, \ldots, s(n - m - 1)\}$ (see Section 1). It is seen that for every $m \geqq 1$ one has

$$V_m = \bigcup_{n=m+1}^{\infty} \{y(s(n - 1) + i) \mid i = 1, 2, \ldots, s(n - m - 1)\} .$$

The reader can see now that this V is a full $M - L$ test. Moreover, an elementary computation gives the form of the function m_V. We have for all $n \geqq 2$:

$$m_V(y(s(n - 1) + 1)) = n - 1 ,$$

and

$$m_V(y(s(n - 1) + i)) = n - k - 1 ,$$

for every $1 \leqq k \leqq n - 2$, where $i \in \{s(k - 1) + 1, s(k - 1) + 2, \ldots, s(k)\}$; also

$$m_V(x) = 0 ,$$

for the other x in X^*.

An inspection of $A(n, 1)$ shows that for $n \geqq 2$ one has:

$$\text{card}\,\{x \in X^* \mid l(x) = n, \text{ there exists an } m \geqq 1 \text{ such that } (x, m) \in V\} = s(n - 2) .$$

b) In order to do the construction indicated in the proof of Theorem 2, we shall choose an enumeration function g for the set $A = \{(x, m_V(x)) \mid x \in V_1\}$. This g will satisfy the conditions u), v) required in the proof of Theorem 2 and it possesses the supplementary property (which completely determines g):

w) if for $i < j$ one has $l(x_i) = l(x_j)$ and $m_V(x_i) = m_V(x_j)$, then $x_i > x_j$ in lexicographical order. This means that for every $n > m \geqq 1$, the set $\{x \in X^* \mid l(x) = n, m_V(x) = m\}$ is ordered by the inverse of the lexicographical order.

The p.r. function $\varphi : X^* \times N \overset{\circ}{\to} X^*$ produced by the proof of Theorem 2 is given by

$$\varphi(y(i), n) = y(s(n - 1) + i) ,$$

for every $n \geqq 2$ and $i = 1, 2, \ldots, s(n - 2)$.

An alternative of Theorem 7 (which was based upon the equalities (i) and (ii) guaranteeing the recursiveness of V) will be the following theorem. Here we shall actually replace the equalities (i) and (ii) in Theorem 7 by inequalities and we shall assume the recursiveness of V.

Theorem 9. Let $V \subset X^* \times N$ be a set having the following properties:

(a) The set V is recursive.

(b) For every natural $m \geqq 1$, we have the inclusion $V_{m+1} \subset V_m$.

(c) For all natural $n > m \geqq 1$, we have

$$\text{card}\,\{x \in X^* \mid l(x) = n,\, m_V(x) = m\} \leqq p^{n-m-1}\,.$$

Under these assumptions the set V is a representable $M - L$ test.

Proof. In view of Theorem 3, all it remains to prove is the fact that V is a $M - L$ test. This can be done using the equality (**) in the proof of Theorem 6, which yields

$$\text{card}\,\{x \in X^* \mid l(x) = n,\, (x, m) \in V\} = \sum_{j=m}^{n-1} \text{card}\,\{x \in X^* \mid l(x) = n,\, m_V(x) = j\} \leqq$$
$$\leqq \sum_{j=m}^{n-1} p^{n-j-1} = (p^{n-m} - 1)/(p - 1)\,. \qquad \square$$

3. REMARKS AND OPEN PROBLEMS

Our representability theory (see also [2]) is an attempt to compare Kolmogorov's complexity theory of strings which uses algorithms [3] with P. Martin-Löf's complexity theory of strings which uses $M - L$ tests [5]. We have already seen that there exist non-representable $M - L$ tests [2], i.e. these theories are not equivalent. For instance, take $p = 2$, $X = \{0, 1\}$ and $V = \{(000,1), (010,1), (111,1)\}$.

In this direction we could obtain the following result concerning recursive sets:

If we call ***K*-test** a set $V \subset X^* \times N$ having properties (a), (b) and (c) in Theorem 9, then Kolmogorov's complexity theory and P. Martin-Löf's complexity theory done only with K-tests are equivalent. This means that for every p.r. function $\varphi : X^* \times N \overset{\circ}{\rightarrow} X^*$ we can obtain the K-test $V(\varphi)$ and for every K-test $V \subset X^* \times N$ we can obtain a p.r. function $\varphi : X^* \times N \overset{\circ}{\rightarrow} X^*$ such that $V = V(\varphi)$ (see Example 10 in [1], Theorems 3, 4 and 9).

We set the following natural open problems:

A) Does the equivalence (1)$\Leftrightarrow$(2) in Theorem 3 hold also for non-recursive $M-L$ tests V? Equivalently, does the result in Theorem 9 hold also for non-recursive $M - L$ tests V?

B) Does the result in Theorem 2 hold also for non-recursive $M - L$ tests V?

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REFERENCES

[1] C. Calude and I. Chiţescu: Random strings according to A. N. Kolmogorov and P. Martin-Löf. Classical approach. Found. Control Engrg. *3* (1982), 73—85.

[2] C. Calude and I. Chiţescu: On representability of P. Martin-Löf tests. Kybernetika *19* (1983), 42—47.

[3] A. N. Kolmogorov: Three approaches to the quantitative definition of information. Problems Inform. Transmission *1* (1965), 1—7.

[4] M. Machtey and P. Young: An Introduction to the General Theory of Algorithms. North-Holland, Amsterdam 1978.

[5] P. Martin-Löf: The definition of random sequences. Inform. and Control *19* (1966), 602–619.
[6] H. Rogers, Jr.: The Theory of Recursive Functions and Effective Computability. McGraw-Hill, New York 1967.
[7] V. A. Uspensky and A. L. Semenov: What are the gains of the theory of algorithms. In: Algorithms in Modern Mathematics and Computer Science (A. P. Ersov, D. E. Knuth, eds.), Springer-Verlag, Berlin—Heidelberg—New York 1981, 100–234.

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