KYBERNETIKA — VOLUME 19 (1983), NUMBER 6

# NONLINEAR APPROXIMATION IN CONTROL PROBLEMS

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Estimators based on an output error are nonlinear in the model parameters and so they are closely connected with approximations of nonlinear functions. In the paper we show how this connection can be used for the solution of control problems. Some modifications of Newton's method are discussed and a new method for the approximation of signals is proposed. This method is based on the classical linear $L_2$-approximation.

## 1. INTRODUCTION

It is a generally accepted fact that no real system can be "exactly" described by a mathematical model. Unfortunately this fact is not respected in the theory of estimators. In the literature we can mostly find estimators starting from an equation error (EE estimators) ([3], [20]), which are based on the assumption that the structure of the considered system is known and respected [13]. Therefore the EE estimators can be used only for the identification of simple systems. The identification of high order or complex systems must be based on more robust methods – on the estimators which start from an output error (OE estimators) [20]. The OE estimators are not linear in the model parameters which causes practical and theoretical problems similar to that occurring in nonlinear approximation of functions. So the derivation of an effective nonlinear estimator must be necessarily based on a nonlinear approximation.

The historical background of the nonlinear approximation is connected with the Newton method. On one side there is a classical approximation theory ([1], [9]). On the other side there are heuristic numerical methods ([5], [11], [19]). Here the heuristics is valuable in making the methods perform more reliably and efficiently. To the first group we can range the formal derivation of Newton's method made by Kantorovich [7]. This classical alternative is repeated in many books (Faddeev, Faddeeva [4], Collatz [2]). Later it was clear that this classical approach is not suited for the solution of practical problems. Most of the effective methods for solving the least squares problem, which are currently in use, are the heuristic numerical methods of the second group. These methods can be found in the literature as the methods for unconstrained optimization ([5], [19]).

Let us turn our attention to the control problems. In the literature ([3], [20]) the classical version of the Gauss-Newton method is mostly considered. The practical implementation of the pertinent algorithm is not a simple matter [20]. In this paper we shall derive a new modification of the Gauss-Newton method which is more robust. We shall introduce a linear regression function which will be solved simultaneously with the nonlinear problem. This linear solution allows to demarcate the region in which the linearization of the pertinent nonlinear function can be used. A new interpretation of the varisolvence problem will be derived. The varisolvence was introduced by Rice [12]. We shall show that the signal approximation is a varisisolvent problem. So the difference between the original and the substitute function alone cannot serve as a criterion of the signal approximation.

## 2. DESCRIPTION OF SIGNALS

We shall consider an approximation of functions which are output signals of control systems. Therefore we start from a linear time-invariant single input-single output system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{\beta}\,u(t)\,, \\ y(t) &= \boldsymbol{\gamma}\,\boldsymbol{x}(t)\,, \end{aligned} \tag{2.1}$$

with $\boldsymbol{A}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ matrices having dimensions $n \times n$, $n \times 1$ and $1 \times n$ respectively.

For the approximation of the output signal the pertinent external description seems to be more useful. Using the Laplace transformation of (2.1) we obtain for the zero initial conditions

$$S : y(s) = \boldsymbol{\gamma}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\,\boldsymbol{\beta}\,u(s) = F(s)\,u(s) = \frac{M(s)}{N(s)}\,u(s)\,, \tag{2.2}$$

where $s$ is the complex variable, $y(s)$ resp. $u(s)$ are the Laplace transforms of $y(t)$ resp. $u(t)$. We shall limit our analysis to systems where $M(s)$ resp. $N(s)$ are Hurwitz polynomials of degree $m$ resp. $n$ without common factor.

The substitute signal $\bar{y}$ can be described with the help of the pertinent substitute transfer function $\bar{F}(s)$ (for $\bar{n} < n$, $\bar{m} < m$)

$$\bar{S} : \bar{y}(s) = \bar{F}(s, \boldsymbol{b}, \boldsymbol{a})\,u(s) = \frac{\bar{M}(s)}{\bar{N}(s)}\,u(s)\,, \tag{2.3}$$

where $\bar{M}(s) = 1 + \sum_{j=1}^{\bar{m}} b_j s^j$ and $\bar{N}(s) = \sum_{i=0}^{\bar{n}} a_i s^i$ are Hurwitz polynomials without common factor.

From the gradient of the substitute transfer function

(2.4)

$$\operatorname{grad} \bar{F}(s, \boldsymbol{b}, \boldsymbol{a}) = \frac{1}{\bar{N}^2(s)} \{s\, \bar{N}(s), s^2\, \bar{N}(s), \ldots, s^{\bar{m}}\, \bar{N}(s), -\, \bar{M}(s), -s\, \bar{M}(s), \ldots, -s^{\bar{n}}\, \bar{M}(s)\}$$

it follows that the substitute output signal $\bar{y}(t, \boldsymbol{b}, \boldsymbol{a})$ is nonlinear in coefficients $a_i$ $(i = 0, 1, \ldots, \bar{n})$. From this reason we shall limit our analysis to the nonlinear part of the problem, i.e. to $\bar{F}(s)$ with $\bar{M}(s) = 1$

$$\bar{F}(s, \boldsymbol{a}) = \frac{1}{\sum_{i=0}^{\bar{n}} a_i s^i}. \tag{2.5}$$

The considered approximation can be formulated in the following way: To the given $y \in V$ find the substitute function $\bar{y}(t, \boldsymbol{a}) \in \bar{V}(\bar{V} \subset V)$ such that

$$Q(\boldsymbol{a}) = \|y - \bar{y}\|^2 = \int_0^\infty [y(t) - \bar{y}(t, \boldsymbol{a})]^2\, \mathrm{d}t \tag{2.6}$$

takes the minimum value.

From the above formulation it follows that we shall consider the approximation in the Hilbert space $L_2(0, \infty)$ with the norm $\|\bar{y}\| = [\int_0^\infty \bar{y}^2(t)\, \mathrm{d}t]^{1/2}$ and the scalar product $(\bar{y}, z) = \int_0^\infty \bar{y}(t)\, z(t)\, \mathrm{d}t$. We consider the continuous case and in this way we obtain more comprehensive and more illustrative results. The time interval $t \in (0, \infty)$ allows to calculate the norms and the scalar products only from the coefficients of the transfer functions [16] and makes easy to verify the derived algorithms (for $y(\infty) = 0$ and $\bar{y}(\infty) = 0$).

Let us summarize the assumptions under which we shall solve the nonlinear approximation of signals:

(i) We start from the signals $y(t) \in V$ resp. $\bar{y}(t) \in \bar{V}(\bar{V} \subset V)$, which are the outputs of the systems $S$ resp. $\bar{S}$ to the same deterministic input signal $u(t) \in V_u$ for the zero initial conditions.

(ii) The structure of the substitute model $\bar{S}$ is known, i.e. the degree $\bar{n}$ of the polynomial $\bar{N}(s)$ is known.

(iii) The considered systems are stable in the sense of Routh-Hurwitz criterion $(\|y\| < \infty, \|\bar{y}\| < \infty)$.

## 3. MODIFICATIONS OF NEWTON'S METHOD

The least squares problem consists in minimizing (2.6). A necessary condition that $^*\boldsymbol{a}$ be a minimizer of $Q(\boldsymbol{a})$ is that

$$\partial_i Q(\boldsymbol{a}) = \emptyset \quad (i = 0, 1, \ldots, \bar{n}). \tag{3.1}$$

Let us denote the first partial derivatives of $\bar{y}(t, \boldsymbol{a})$ by

$$\partial_i \bar{y}(t, \boldsymbol{a}) = \frac{\partial}{\partial a_i} \bar{y}(t, \boldsymbol{a}) \quad (i = 0, 1, \ldots, \bar{n})$$

and the second partial derivatives by

$$\partial_i \, \partial_k \bar{y}(t, \boldsymbol{a}) = \frac{\partial^2}{\partial a_i \, \partial a_k} \bar{y}(t, \boldsymbol{a}) \quad (i, k = 0, 1, \ldots, \bar{n}) .$$

The Laplace transform of the first partial derivatives of $\bar{y}(t, \boldsymbol{a})$ pertinent to the transfer function (2.5) is given by

$$\mathscr{L}\{\partial_i \bar{y}(t, \boldsymbol{a})\} = \frac{\partial}{\partial a_i} \frac{u(s)}{\bar{N}(s)} = - \frac{s^i}{\bar{N}^2(s)} u(s) = \mathscr{L}\{-v^{(i)}(t)\} . \tag{3.2}$$

The function $v^{(i)}(t)$ is the sensitivity function pertinent to the coefficient $a_i$. Then the gradient vector $\boldsymbol{g}(t, \boldsymbol{a})$ of $\bar{y}(t, \boldsymbol{a})$ at $\boldsymbol{a}$ is defined by

$$\begin{aligned} \boldsymbol{g}(t, \boldsymbol{a}) &= [\partial_0 \bar{y}(t, \boldsymbol{a}), \partial_1 \bar{y}(t, \boldsymbol{a}), \ldots, \partial_{\bar{n}} \bar{y}(t, \boldsymbol{a})]^{\mathrm{T}} = \\ &= [-v^{(0)}(t), -v^{(1)}(t), \ldots, -v^{(\bar{n})}(t)]^{\mathrm{T}} . \end{aligned} \tag{3.3}$$

Condition (3.1) can be then written in the form

$$\partial_i Q(\boldsymbol{a}) = -2 \int_0^\infty [y(t) - \bar{y}(t, \boldsymbol{a})] \, \partial_i \bar{y}(t, \boldsymbol{a}) \, \mathrm{d}t = \emptyset . \tag{3.4}$$

The system of equations (3.1) is nonlinear. Then we must start from a linearization of $\bar{y}(t, \boldsymbol{a})$ ([5], [19]) so that we need

$$\begin{aligned} \partial_k \partial_i Q(\boldsymbol{a}) = 2 \Big\{ &- \int_0^\infty [y(t) - \bar{y}(t, \boldsymbol{a})] \, \partial_i \partial_k \bar{y}(t, \boldsymbol{a}) \, \mathrm{d}t + \\ &+ \int_0^\infty \partial_k \bar{y}(t, \boldsymbol{a}) \, \partial_i \bar{y}(t, \boldsymbol{a}) \, \mathrm{d}t \Big\} . \end{aligned} \tag{3.5}$$

Newton's method for solving the system of equations (3.1) consists in generating the sequence $\{{}^j\boldsymbol{a}\}$ from

$${}^{j+1}\boldsymbol{a} = {}^j\boldsymbol{a} + [\boldsymbol{J}({}^j\boldsymbol{a})]^{-1} \int_0^\infty \boldsymbol{g}(t, {}^j\boldsymbol{a}) \, [y(t) - {}^j\bar{y}(t, {}^j\boldsymbol{a})] \, \mathrm{d}t , \tag{3.6}$$

where the elements of $\boldsymbol{J}({}^j\boldsymbol{a})$ are given by (3.5)

$$J_{ik}({}^j\boldsymbol{a}) = \tfrac{1}{2} \, \partial_i \partial_k Q({}^j\boldsymbol{a}) . \tag{3.7}$$

Superscripts on the left indicate the iteration steps.

The Newton method as the identification method is not robust. This follows from the fact that the elements of $\boldsymbol{J}({}^j\boldsymbol{a})$ are given by the measured data, i.e. by data with a precision lower than two decimal places.

### 3.1. Gauss-Newton method

If $Q(^j\boldsymbol{a})$ is sufficiently small then we may neglect the part with the second derivatives and so we can write

$$\partial_k \partial_i Q(^j\boldsymbol{a}) \doteq 2 \int_0^\infty \partial_k \bar{y}(t, {}^j\boldsymbol{a})\, \partial_i \bar{y}(t, {}^j\boldsymbol{a})\, \mathrm{d}t\,. \tag{3.8}$$

If we approximate $\boldsymbol{J}(^j\boldsymbol{a})$ with the help of (3.8) we obtain the Gauss-Newton method given by the relation

$$^{j+1}\boldsymbol{a} = {}^j\boldsymbol{a} + \boldsymbol{G}^{-1}(^j\boldsymbol{a}) \int_0^\infty \boldsymbol{g}(t, {}^j\boldsymbol{a})\, [y(t) - {}^j\bar{y}(t, {}^j\boldsymbol{a})]\, \mathrm{d}t \tag{3.9}$$

with

$$\boldsymbol{G}(^j\boldsymbol{a}) = \int_0^\infty \boldsymbol{g}(t, {}^j\boldsymbol{a})\, \boldsymbol{g}^{\mathrm{T}}(t, {}^j\boldsymbol{a})\, \mathrm{d}t\,.$$

This method is numerically robust with respect to the inversion of $\boldsymbol{G}(^j\boldsymbol{a})$. But it is known from the literature ([5], [19]) that the classical Gauss-Newton method is too naive for the solution of realistic problems. However most of the effective methods, which are currently in use, are modifications of the Gauss-Newton method. Hartley [6] proposed the use of line search technique for the improvement of the convergence. Marquardt [8], Fletcher [5], Meyer-Roth [10] tried to improve the efficiency of nonlinear least squares method with different manipulations of the fundamental matrix. Nevertheless all these methods can perform badly on an exponential fitting problem. In this paper we shall try to analyze the causes of this situation.

The Gauss-Newton method given by (3.9) is effective, if the condition $\boldsymbol{g}(t, {}^{j+1}\boldsymbol{a}) \doteq \boldsymbol{g}(t, {}^j\boldsymbol{a})$ holds. At the present time there is no procedure how to predict $\boldsymbol{G}(^{j+1}\boldsymbol{a})$ or $\boldsymbol{g}(t, {}^{j+1}\boldsymbol{a})$ from the parameters of the $j$th iteration step. From the viewpoint of (3.9) the change of $\boldsymbol{g}(t, {}^j\boldsymbol{a})$ to $\boldsymbol{g}(t, {}^{j+1}\boldsymbol{a})$ is not under the computational control. This is the main reason why the applicability region of the Gauss-Newton method is so small and why different modifications are used.

### 3.2. The linear case

Signals are described by exponential functions. The first difference with respect to [5] and [19] is that we use the Laplace transforms. The second difference in the description of signals with respect to [3] and [20] is the use of the regression functions in the form

$$^j\bar{y}(t, {}^j\boldsymbol{a}) = \sum_{i=0}^{\bar{n}} {}^ja_i\, {}^jv^{(i)}(t)\,, \tag{3.10}$$

which essentially simplifies the solution of the signal approximation. Relation (3.10) follows from the pertinent Laplace transform [16]

$$\mathscr{L}\{^j\bar{y}(t, {}^j\boldsymbol{a})\} = \mathscr{L}\,\{\sum_{i=0}^{\bar{n}} {}^ja_i\, {}^jv^{(i)}(t)\} = \frac{\sum_{i=0}^{\bar{n}} {}^ja_i s^i}{{}^j\bar{N}^2(s)}\, u(s) = \frac{u(s)}{{}^j\bar{N}(s)}\,. \tag{3.11}$$

The first partial derivatives of this form ${}^j\bar{y}(t, {}^j\boldsymbol{a})$ is then

$$(3.12)\qquad \frac{\partial {}^j\bar{y}(t, {}^j\boldsymbol{a})}{\partial a_i} = {}^jv^{(i)}(t) + \sum_{k=0}^{\bar{n}} {}^ja_k \frac{\partial {}^jv^{(k)}}{\partial a_i} = {}^jv^{(i)}(t) - 2\,{}^jv^{(i)}(t) = -{}^jv^{(i)}(t)\,.$$

The influence of the sensitivity functions in (3.10) is twice greater as the influence of pure coefficients and the pertinent component has negative sign.

Relation (3.12) is the fundamental relation for deriving an effective nonlinear approximation of signals. With the help of (3.10) and (3.12) we can introduce the linear case. We can assume that the sensitivity functions ${}^jv^{(i)}(t)$ in (3.10) are known and then we can solve approximation of $y(t)$ with the linear regression function

$$(3.13)\qquad {}^jz(t, {}^j\boldsymbol{a}, {}^{j+1}\hat{\boldsymbol{a}}) = \sum_{i=0}^{\bar{n}} {}^{j+1}\hat{a}_i\, {}^jv^{(i)}(t) = {}^j\bar{y}(t, {}^j\boldsymbol{a}) + \sum_{i=0}^{\bar{n}} ({}^{j+1}\hat{a}_i - {}^ja_i)\, {}^jv^{(i)}(t) =$$

$$= {}^j\bar{y}(t, {}^j\boldsymbol{a}) + \sum_{i=0}^{\bar{n}} \Delta\, {}^ja_i\, {}^jv^{(i)}(t)\,.$$

A necessary condition that ${}^{j+1}\hat{\boldsymbol{a}}$ be a minimizer of ${}^jQ_L(\hat{\boldsymbol{a}}) = \|y - {}^jz\|^2$ is that

$$(3.14)\qquad \partial_i\, {}^jQ_L(\hat{\boldsymbol{a}})|_{\hat{\boldsymbol{a}}={}^{j+1}\hat{\boldsymbol{a}}} = -2(y - {}^jz, {}^jv^{(i)}) =$$

$$= -2\int_0^\infty \left[y(t) - {}^j\bar{y}(t, {}^ja) - \sum_{i=0}^{\bar{n}} \Delta\, {}^ja_i\, {}^jv^{(i)}(t)\right] {}^jv^{(i)}(t)\, \mathrm{d}t = \emptyset\,.$$

Then with regard to (3.4) and (3.9) we can write

$$(3.15)\qquad {}^{j+1}\hat{\boldsymbol{a}} = {}^j\boldsymbol{a} + \boldsymbol{G}^{-1}({}^j\boldsymbol{a}) \int_0^\infty \boldsymbol{g}_L(t, {}^j\boldsymbol{a}) \left[y(t) - {}^j\bar{y}(t, {}^j\boldsymbol{a})\right] \mathrm{d}t$$

with $\boldsymbol{g}_L(t, {}^j\boldsymbol{a}) = -\boldsymbol{g}(t, {}^j\boldsymbol{a})$ and with the nonsingular matrix

$$\boldsymbol{G}({}^j\boldsymbol{a}) = \int_0^\infty \boldsymbol{g}_L(t, {}^j\boldsymbol{a})\, \boldsymbol{g}_L^{\mathrm{T}}(t, {}^j\boldsymbol{a})\, \mathrm{d}t\,.$$

In accordance with (3.12) the linear case given by (3.15) differs from the nonlinear case given by (3.9) only in the sign.

### 3.2. The DNLS method

According to (3.9) and (3.15) the following modification of the Gauss-Newton method is considered

$$(3.16)\qquad {}^{j+1}\boldsymbol{a} = {}^j\boldsymbol{a} + {}^j\mu\, \boldsymbol{G}^{-1}({}^j\boldsymbol{a}) \int_0^\infty \boldsymbol{g}(t, {}^j\boldsymbol{a}) \left[y(t) - {}^j\bar{y}(t, {}^j\boldsymbol{a})\right] \mathrm{d}t =$$

$$= {}^j\boldsymbol{a} - {}^j\mu\, \boldsymbol{G}^{-1}({}^j\boldsymbol{a}) \int_0^\infty \boldsymbol{g}_L(t, {}^j\boldsymbol{a}) \left[y(t) - {}^j\bar{y}(t, {}^j\boldsymbol{a})\right] \mathrm{d}t\,,$$

(3.17) $$ {}^{j+1}\hat{\boldsymbol{a}} = {}^{j}\boldsymbol{a} - {}^{j}\mu\, \boldsymbol{G}^{-1}({}^{j}\boldsymbol{a}) \int_0^\infty \boldsymbol{g}(t, {}^{j}\boldsymbol{a}) \left[ y(t) - {}^{j}\bar{y}(t, {}^{j}\boldsymbol{a}) \right] \mathrm{d}t = $$

$$ = {}^{j}\boldsymbol{a} + {}^{j}\mu\, \boldsymbol{G}^{-1}({}^{j}\boldsymbol{a}) \int_0^\infty \boldsymbol{g}_L(t, {}^{j}\boldsymbol{a}) \left[ y(t) - {}^{j}\bar{y}(t, {}^{j}\boldsymbol{a}) \right] \mathrm{d}t $$

where ${}^{j}\mu \in (0, 1)$ is a damping factor.

In connection with (3.16) and (3.17) we shall speak about the Damped Nonlinear Least Squares (DNLS) method.

The factor ${}^{j}\mu$ must be selected as large as possible to get a rapid convergence. On the other side the solution given by (3.16) must be sufficiently near to the linear case given by (3.17) to be able to predict the characteristic parameters in the $(j + 1)$th iteration step without a laborious experimentation.

According to (3.13), (3.16) and (3.17) let us introduce the following functions

(3.18) $$ {}^{j}z(t, {}^{j}\boldsymbol{a}, {}^{j+1}\hat{\boldsymbol{a}}, {}^{j}\mu) = \sum_{i=0}^{\bar{n}} {}^{j+1}\hat{a}_i\, {}^{j}v^{(i)}(t) = {}^{j}\bar{y}(t, {}^{j}\boldsymbol{a}) + {}^{j}\mu \Delta\, {}^{j}\bar{y}(t) $$

and

(3.19) $$ {}^{j}\bar{\bar{y}}(t, {}^{j}\boldsymbol{a}, {}^{j+1}\boldsymbol{a}, {}^{j}\mu) = \sum_{i=0}^{\bar{n}} {}^{j+1}a_i\, {}^{j}v^{(i)}(t) = {}^{j}\bar{y}(t, {}^{j}\boldsymbol{a}) - {}^{j}\mu \Delta\, {}^{j}\bar{y}(t) $$

where

$$ \Delta\, {}^{j}\bar{y}(t) = \sum_{i=0}^{\bar{n}} \Delta\, {}^{j}a_i\, {}^{j}v^{(i)}(t) = \sum_{i=0}^{\bar{n}} ({}^{j+1}\hat{a}_i - {}^{j}a_i)\, {}^{j}v^{(i)}(t)\,. $$

Later we shall show that these two functions are closely connected and form the basis of the DNLS method. The linear case given by (3.17) and (3.18) serves as an etalon for the solution of the pertinent nonlinear case given by (3.16) and (3.19).

## 4. CONNECTION BETWEEN THE LINEAR AND NONLINEAR APPROXIMATION

In this section we shall consider the main topic of this paper – the case with a week nonlinearity, i.e. the case which is closely associated with the linear $L_2$-approximation. This case is important for the endsteps of the procedure given by (3.16), (3.17) and hence for the convergence to the global minimum. In this section we shall use the damping factor ${}^{j}\mu = 1$. Therefore we shall omit ${}^{j}\mu$ in all relations.

### 4.1. Linear $L_2$-approximation

Let us consider a linear, normed space $V \subset L_2(0, \infty)$ with elements $y, z, h, \ldots$ and let $\bar{V} \subset V$ be its finite dimensional subspace. Then the problem of the linear approximation is to find, for the given $y \in V$, an element $z \in \bar{V}$ such that

$$\|y - z\| \leqq \|y - h\| \tag{4.1}$$

holds for all $h \in \bar{V}$.

The space $L_2(0, \infty)$ is strictly convex, hence the following proposition must hold.

**Proposition 4.1.** If $V$ is strictly convex then there exists the unique best approximation of $y \in V$ in $\bar{V} \subset V$.

The proof is given in ([1], [9]).

For substitute function in the form

$$z(t) = \sum_{i=0}^{n} \lambda_i h_i(t) \tag{4.2}$$

the minimal solution can be obtained by an orthogonal projection, i.e. from the equations

$$(y - z, h_i) = 0 \quad (i = 0, 1, \ldots, n) \tag{4.3}$$

The error of the linear approximation is then given by the relation

$$\delta_m^2 = \|y - z\|^2 = (y - z, y) - (y - z, z) = (y, y) - \sum_{i=0}^{n} \lambda_i(y, h_i) . \tag{4.4}$$

### 4.2. Basic propositions

The nonlinear approximation will be analyzed with the help of relations (3.9) to (3.19) and (4.1) to (4.4). Relation (4.3) will be used in the form

$$(y - {}^jz, {}^jv^{(i)}) = \emptyset \quad (i = 0, 1, \ldots, \bar{n}) \tag{4.5}$$

First let us derive, with the help of (3.18) and (4.4), the relation for the error of coefficients

$${}^j\delta^2 = \|y - {}^jz\|^2 = (y, y) - \sum_{i=0}^{\bar{n}} {}^{j+1}\hat{a}_i(y, {}^jv^{(i)}) . \tag{4.6}$$

The error of the sensitivity functions is given by (3.12) and (3.18)

$${}^j\varphi^2 = \|{}^jz - {}^j\bar{y}\|^2 = \|{}^j\bar{y} - {}^j\bar{\bar{y}}\|^2 . \tag{4.7}$$

The considered functionals can be used if the pertinent systems are stable. Therefore let us introduce the following definition:

**Definition 4.1.** The vector of the coefficients ${}^j\boldsymbol{a}$ pertinent to the substitute system with the transfer function ${}^j\bar{F}(s, \boldsymbol{a}) = {}^j\bar{N}^{-1}(s)$ is an element of the subset $\Omega_a$ of stable vectors $\boldsymbol{a}$, if the polynomial ${}^j\bar{N}(s) = \sum_{i=0}^{\bar{n}} {}^ja_i s^i$ fulfils Routh-Hurwitz conditions of stability.

Now we can start our analysis and prove the following proposition.

**Proposition 4.2.** If $\{{}^j\boldsymbol{a}\} \in \Omega_a$ then for the differences of $y \in V$, ${}^jz, {}^j\bar{y} \in \bar{V}(\bar{V} \subset V)$ and ${}^j\bar{y} \in \bar{V}(\bar{V} \subset V)$ it holds

$$\|{}^j\bar{\bar{y}} - {}^jz\|^2 = 4\,{}^j\varphi^2\,, \tag{4.8}$$

$$\|y - {}^j\bar{\bar{y}}\|^2 = {}^j\delta^2 + 4\,{}^j\varphi^2\,, \tag{4.9}$$

$$\|y - {}^j\bar{y}\|^2 = {}^j\delta^2 + {}^j\varphi^2\,. \tag{4.10}$$

Proof. With regard to (3.18) and (3.19) we obtain

$$\|{}^j\bar{\bar{y}} - {}^jz\|^2 = \|{}^j\bar{\bar{y}} - {}^j\bar{y} + {}^j\bar{y} - {}^jz\|^2 = \|-2\Delta\,{}^j\bar{y}\|^2 = 4\,{}^j\varphi^2$$

Relation (4.9) resp. (4.10) can be arranged

$$\|y - {}^j\bar{\bar{y}}\|^2 = \|y - {}^jz + {}^jz - {}^j\bar{\bar{y}}\|^2 =$$

$$= \|y - {}^jz\|^2 + \|{}^jz - {}^j\bar{\bar{y}}\|^2 + 2[(y - {}^jz, {}^jz) - (y - {}^jz, {}^j\bar{\bar{y}})]$$

resp.

$$\|y - {}^j\bar{y}\|^2 = \|y - {}^jz + {}^jz - {}^j\bar{y}\|^2 =$$

$$= \|y - {}^jz\|^2 + \|{}^jz - {}^j\bar{y}\|^2 + 2[(y - {}^jz, {}^jz) - (y - {}^jz, {}^j\bar{y})]$$

and with respect to (4.5), (4.6), (4.8), resp. (4.5), (4.6), (4.7) we obtain (4.9), resp. (4.10). □

The main result of this paper is the relation (4.10). The separation of the errors ${}^j\varphi$ and ${}^j\delta$ according to (4.10) is decisive for an effective solution of the signal approximation. The condition ${}^j\varphi = \|{}^jz - {}^j\bar{y}\| \doteq \|{}^{j+1}\bar{y} - {}^j\bar{y}\|$ characterizes the fulfilment of the condition $\boldsymbol{g}({}^{j+1}\boldsymbol{a}) \doteq \boldsymbol{g}({}^j\boldsymbol{a})$ and can be governed by the damping factor ${}^j\mu$.

It follows from (4.10) that the solution of the signal approximation is not unique. Rice [12] speaks in this connection about varisolvent approximating functions. Therefore we must introduce an other functional, which helps to demarcate the solution of the nonlinear approximation problem. Let us start from the relations (4.5) for ${}^j\boldsymbol{a}$ resp. ${}^{j+1}\boldsymbol{a} = {}^j\boldsymbol{a} - \Delta\,{}^j\boldsymbol{a}$.

$$(y, {}^j\bar{y}) - ({}^j\bar{y}, {}^jz) = \emptyset \tag{4.11}$$

resp.

$$(y, {}^j\bar{\bar{y}}) - ({}^jz, {}^j\bar{\bar{y}}) = \emptyset\,. \tag{4.12}$$

Now we can introduce, with the help of (3.18), (3.19), (4.11) and (4.12), the following functional

$$ {}^j\varrho = ({}^j\bar{y}, \Delta\,{}^j\bar{y}) = ({}^j\bar{y}, y - {}^j\bar{y}) = ({}^j\bar{y}, {}^j\bar{y} - {}^j\bar{\bar{y}}) = ({}^j\bar{y}, {}^jz - {}^j\bar{y})\,. \tag{4.13}$$

With respect to (3.18), (3.19) and (4.13) we can write

$$\|{}^jz\|^2 = \|{}^j\bar{y}\|^2 + 2\,{}^j\varrho + {}^j\varphi^2\,, \tag{4.14}$$

$$\|{}^j\bar{\bar{y}}\|^2 = \|{}^j\bar{y}\|^2 - 2\,{}^j\varrho + {}^j\varphi^2\,. \tag{4.15}$$

The importance of the introduced functional ${}^j\varrho$ shows the following proposition.

**Proposition 4.3.** If $\{{}^j\boldsymbol{a}\} \in \Omega_a$ then for the functions $y \in V$, ${}^j\bar{y}$, ${}^jz \in \bar{V}(\bar{V} \subset V)$ and ${}^j\bar{\bar{y}} \in \bar{V}(\bar{V} \subset V)$ the following relations holds:

$$\|{}^jz\|^2 - (y, {}^j\bar{y}) = (y, {}^j\bar{y}) - (y, {}^j\bar{\bar{y}}) = (y, {}^jz) - (y, {}^j\bar{y}) = \tag{4.16}$$

$$= (y, \Delta\,{}^j\bar{y}) = {}^j\varrho + {}^j\varphi^2\,,$$

$$\|{}^j\bar{y}\|^2 - (y, {}^j\bar{\bar{y}}) = {}^j\varphi^2\,, \tag{4.17}$$

$$({}^j\bar{y}, {}^j\bar{\bar{y}}) - (y, {}^j\bar{\bar{y}}) = {}^j\varphi^2 - {}^j\varrho\,, \tag{4.18}$$

$$\|y\|^2 - \|{}^j\bar{y}\|^2 = {}^j\delta^2 + {}^j\varphi^2 + 2\,{}^j\varrho\,, \tag{4.19}$$

$$\|y\|^2 - (y, {}^j\bar{y}) = {}^j\delta^2 + {}^j\varphi^2 + {}^j\varrho\,. \tag{4.20}$$

Proof. With regard to (4.11) and (4.12) we can write (4.16) resp. (4.17) in the form

$$\|{}^jz\|^2 - (y, {}^j\bar{y}) = \|{}^j\bar{y} + \Delta\,{}^j\bar{y}\|^2 - ({}^j\bar{y}, {}^j\bar{y} + \Delta\,{}^j\bar{y}) = ({}^j\bar{y}, \Delta\,{}^j\bar{y}) + \|\Delta\,{}^j\bar{y}\|^2$$

resp.

$$\|{}^j\bar{y}\|^2 - (y, {}^j\bar{\bar{y}}) = \|{}^j\bar{y}\|^2 - ({}^jz, {}^j\bar{\bar{y}}) =$$

$$= \|{}^j\bar{y}\|^2 - ({}^j\bar{y} + \Delta\,{}^j\bar{y}, {}^j\bar{y} - \Delta\,{}^j\bar{y}) = \|\Delta\,{}^j\bar{y}\|^2\,.$$

Similarly for (4.18) resp. (4.19) we get

$$({}^j\bar{y}, {}^j\bar{\bar{y}}) - (y, {}^j\bar{\bar{y}}) = ({}^j\bar{y}, {}^j\bar{y} - \Delta\,{}^j\bar{y}) - ({}^j\bar{y} + \Delta\,{}^j\bar{y}, {}^j\bar{y} - \Delta\,{}^j\bar{y}) = \|\Delta\,{}^j\bar{y}\|^2 - ({}^j\bar{y}, \Delta\,{}^j\bar{y})$$

resp. with regard to (4.10) and (4.11)

$$\|y\|^2 - \|{}^j\bar{y}\|^2 = \|y - {}^j\bar{y}\|^2 + 2[(y, {}^j\bar{y}) - \|{}^j\bar{y}\|^2] =$$

$$= {}^j\delta^2 + {}^j\varphi^2 + 2[({}^j\bar{y}, {}^j\bar{y} + \Delta\,{}^j\bar{y}) - \|{}^j\bar{y}\|^2] = {}^j\delta^2 + {}^j\varphi^2 + 2\,{}^j\varrho\,.$$

Relation (4.20) follows directly from (4.13) and (4.19). □

### 4.3. Global minimum

The given propositions constitute the connection between the linear and nonlinear approximation. So we can essentially simplify the analysis of the global minimum. The asterisk on the left indicates the coefficients and functionals pertinent to the global minimum. If ${}^j\varrho = \emptyset$ and ${}^j\varphi = \emptyset$ then we reach the global minimum and the following proposition holds.

**Proposition 4.4.** If

(i) $^*\boldsymbol{a} \in \Omega_a$,

(ii) $^*\varphi = \emptyset$,

(iii) the structure of $\bar{S}$ is given,

then $^*\boldsymbol{a}$ is the global minimum with respect to (2.6).

The proof of this proposition follows directly from Proposition 4.1 for $^*\delta = \delta_m$, Proposition 4.2 and from (3.16), (3.17). The nonlinear and linear cases are identical for $^*\varphi = \emptyset$.

Let us add that in the considered case $^*\varrho = \emptyset$. This follows easily from the Schwarz-Buniakovski inequality

$$|^j\varrho| = |(^j\bar{y}, \Delta\, ^j\bar{y})| \leqq \|^j\bar{y}\|\, \|\Delta\, ^j\bar{y}\| = \|^j\bar{y}\|\, ^j\varphi\,. \tag{4.21}$$

### 4.4. The ideal case

The practical use of the nonlinear approximation must be based on some procedure for the prediction of functionals in the next step. Let us consider the ideal case which is given by the following conditions

$$^{j+1}\delta = {}^j\delta\,, \tag{4.22}$$

$$(y, {}^{j+1}\bar{y}) = (y, {}^jz)\,, \tag{4.23}$$

$$\|^{j+1}\bar{y}\|^2 = \|^jz\|^2\,. \tag{4.24}$$

These conditions follows from relations (3.12) (3.18), (3.19) and from Propositions 4.1 and 4.2.

Now we can prove the following proposition.

**Proposition 4.5.** If

(i) $\{^j\boldsymbol{a}\} \in \Omega_a$,

(ii) relations (4.22) to (4.24) hold, then we reach the global minimum in one iteration step, i.e. $^{j+1}\varphi = \emptyset$ and $^{j+1}\varrho = \emptyset$.

Proof. If condition (ii) holds then according to (4.5) and (4.13) resp. (4.5) and (4.16) we obtain

$$^{j+1}\varrho = (y - {}^{j+1}\bar{y}, {}^{j+1}\bar{y}) = (y, {}^{j+1}\bar{y}) - \|^{j+1}\bar{y}\|^2 = (y, {}^jz) - \|^jz\|^2 = \emptyset$$

resp.

$$^{j+1}\varphi^2 + {}^{j+1}\varrho = \|^{j+1}z\|^2 - (y, {}^{j+1}\bar{y}) = \|^jz\|^2 - (y, {}^jz) = \emptyset\,. \qquad \square$$

This proposition shows the connection of the **DNLS method** with the linear $L_2$-approximation. With respect to the sensitivity functions $^jv^{(i)}(t)$ $(i = 0, 1, \ldots, \bar{n})$ the function $^jz(t)$ is the best approximation of $y(t)$ (Proposition 4.1). Therefore Proposition 4.5 holds approximately for cases with a small error $^j\varphi$, i.e. near the global minimum. The pertinent region of the vectors ${}^{j}\boldsymbol{a}$ can be demarcated by

(4.25) $$\Phi_0 = \{{}^{j}\boldsymbol{a} : {}^{j+1}\delta \doteq {}^{j}\delta \doteq {}^{*}\delta, {}^{j}\varphi < {}^{j}\delta\} .$$

In this way the region of applicability of the DNLS method for ${}^{j}\mu = 1$ is given. In practical problems we seldom have the initial vector ${}^{0}\boldsymbol{a} \in \Phi_0$. This region $\Phi_0$ is mostly so small that it is impossible to get ${}^{0}\boldsymbol{a} \in \Phi_0$ only with the help of experiments. Deriving the method, which reduces the probability of failure, is therefore desirable.

## 5. EXAMPLE

To illustrate the derived propositions the system

$$S : F(s) = 1/N(s) =$$
$$= 1/(1 + 17s + 87{\cdot}24s^2 + 190{\cdot}84s^3 + 193{\cdot}04s^4 + 87{\cdot}84s^5 + 14{\cdot}4s^6)$$

was considered. The response of the closed control loop to the unit step given by

$$\mathscr{L}\{y_{st}(t)\} = F_w(s) = \frac{K\,F(s)}{s(1 + K\,F(s))} = \frac{K}{s(K + N(s))}$$

was approximated with $\bar{F}_w(s) = K/(s\,\bar{N}_c(s))$ for $\bar{n} = 3$. Here $K$ is a given gain coefficient ($K = 3$). The starting function ${}^{0}\bar{F}_w(s) = 3/(s\,{}^{0}\bar{N}(s)) = 3/(s(4 + 17s + 87{\cdot}24s^2 + 190{\cdot}84s^3))$ was used. According to Section 2 the condition $y(\infty) = \emptyset$ must be fulfilled, i.e. $y(t) = K/(K+1) - y_{st}(t)$ resp. ${}^{j}\bar{y}(t) = K/({}^{j}a_0) - {}^{j}\bar{y}_{st}(t)$ with $\mathscr{L}\{{}^{j}\bar{y}_{st}(t)\} = {}^{j}\bar{F}_w(s)$. Table 1 contains important parameters for ${}^{j}\mu = 1$ ($j = 1, 2, \ldots, 6$) calculated with a double precision PL 1 program based upon the procedure given by (3.16) and (3.17).

**Table 1.**

| $j$ | ${}^{j}\varphi^2$ | ${}^{j}\delta^2$ | $\nu_1({}^{j}\mu)$ | $\nu_2({}^{j}\mu)$ | $\Delta\,{}^{j}a_3$ |
|---|---|---|---|---|---|
| 1 | $2{\cdot}073 \,.\, 10^{-1}$ | $2{\cdot}792 \,.\, 10^{-2}$ | $-1{\cdot}8 \,.\, 10^{-1}$ | $-3{\cdot}1 \,.\, 10^{-1}$ | $-61{\cdot}9$ |
| 2 | $6{\cdot}437 \,.\, 10^{-2}$ | $8{\cdot}033 \,.\, 10^{-3}$ | $-2{\cdot}8 \,.\, 10^{-2}$ | $-4{\cdot}5 \,.\, 10^{-2}$ | $47{\cdot}1$ |
| 3 | $5{\cdot}202 \,.\, 10^{-3}$ | $1{\cdot}399 \,.\, 10^{-2}$ | $-3{\cdot}9 \,.\, 10^{-3}$ | $-7{\cdot}4 \,.\, 10^{-3}$ | $-12{,}2$ |
| 4 | $1{\cdot}388 \,.\, 10^{-4}$ | $1{\cdot}417 \,.\, 10^{-2}$ | $-2{\cdot}2 \,.\, 10^{-5}$ | $-3{\cdot}2 \,.\, 10^{-5}$ | $2{\cdot}57$ |
| 5 | $2{\cdot}052 \,.\, 10^{-6}$ | $1{\cdot}418 \,.\, 10^{-2}$ | $-1{\cdot}7 \,.\, 10^{-6}$ | $-3{\cdot}1 \,.\, 10^{-6}$ | $-3{\cdot}3 \,.\, 10^{-1}$ |
| 6 | $3{\cdot}269 \,.\, 10^{-8}$ | $1{\cdot}418 \,.\, 10^{-2}$ | $-1{\cdot}9 \,.\, 10^{-8}$ | $-3{\cdot}5 \,.\, 10^{-8}$ | $4{\cdot}6 \,.\, 10^{-2}$ |

Errors $\nu_1({}^{j}\mu)$ and $\nu_2({}^{j}\mu)$ characterize the quality of the prediction of the substitute function ${}^{j+1}\bar{y}(t)$ on the basis of the linear case given by (3.17), i.e. on the basis of ${}^{j}z(t)$. They are given by the relations

(5.1) $$(y, {}^{j+1}\bar{y}) = (y, {}^{j}z) + \nu_1({}^{j}\mu) ,$$

(5.2) $$\|{}^{j+1}\bar{y}\|^2 = \|{}^{j}z\|^2 + \nu_2({}^{j}\mu) .$$

So the values in Table 1 for the iteration step $j = 6$ corroborate Proposition 4.4.

The substitute function is given by ${}^6\bar{F}_w(s) = 3/(s(3{\cdot}75 + 15{\cdot}8s + 70{\cdot}62s^2 + 166s^3))$. The solutions in the iteration steps $j \geqq 4$ are practically given by the linear case. Only numerical errors make impossible the solution in these iterations according to Proposition 4.5. The result in the iteration step $j = 4$ can be taken in many practical tasks as the final result.

The solution of the first iteration step is varisolvent. We obtain better results for ${}^1\mu < 1$, e.g. for ${}^1\mu = 0{\cdot}7$ (${}^2\varphi^2({}^1\mu = 0{\cdot}7) = 3{\cdot}9 \,.\, 10^{-2}$, ${}^2\delta^2({}^1\mu = 0{\cdot}7) = 1{\cdot}39 \,.\, 10^{-2}$, $v_1(0{\cdot}7) = -7{\cdot}686 \,.\, 10^{-2}$, $v_2(0{\cdot}7) = -1{\cdot}472 \,.\, 10^{-1}$). Deriving the optimal factor ${}^j\mu_{opt}$ is the topic of [18].

Finally, let us discuss the problems connected with identification procedures. Here we expect in the first place the applications of the DNLS method. Recalling that the transfer function (2.2) is unknown in the identification problems, a more general problem – the approximation from a curve of the function $y(t)$ – must be solved. So deriving a "good" starting function is the key problem of the identification procedures. To illustrate this problem the example used in Table 1 with different starting functions given by ${}^0\bar{F}_w(s) = 3/(s\,{}^0\bar{N}_c(s))$ was considered (${}^1\mu = 1$).

**Table 2.**

| | ${}^0N_c(s)$ | ${}^1\varphi^2$ | ${}^1\delta^2$ | $v_1({}^1\mu)$ | $v_2({}^1\mu)$ |
|---|---|---|---|---|---|
| 1 | $4 + 17s + 87{\cdot}24s^2 + 190{\cdot}84s^3$ | $2{\cdot}073 \,.\, 10^{-1}$ | $2{\cdot}792 \,.\, 10^{-2}$ | $-1{\cdot}8 \,.\, 10^{-1}$ | $-3{\cdot}1 \,.\, 10^{-1}$ |
| 2 | $4{\cdot}5 + 14s + 60s^2 + 100s^3$ | $1{\cdot}268$ | $9{\cdot}490 \,.\, 10^{-2}$ | $-4{\cdot}1 \,.\, 10^{-1}$ | $-4{\cdot}3 \,.\, 10^{-1}$ |
| 3 | $4 + 17s + 40s^2 + 68s^3$ | $1{\cdot}110$ | $6{\cdot}470 \,.\, 10^{-1}$ | $-2{\cdot}8 \,.\, 10^{-1}$ | $-4{\cdot}0 \,.\, 10^{-1}$ |
| 4 | $4 + 17s + 2s^2 + 2s^3$ | $2{\cdot}876 \,.\, 10^{-1}$ | $1{\cdot}913$ | $6{\cdot}2 \,.\, 10^{-1}$ | $1{\cdot}4$ |

The total error ${}'\eta^2 = {}^1\delta^2 + {}^1\varphi^2$ is not very important for the appreciation of the suitability of ${}^1\bar{y}(t)$ (cf. No. 3 and 4 in Table 2). Examples No. 2 and 3 can be solved with the help of the DNLS method in few steps (${}^j\boldsymbol{a} \in \Phi_0$ for $j \leqq 5$). If a sufficiently good solution of the linear case does not exist (${}^j\delta \gg {}^*\delta$) then the sequence $\{{}^j\boldsymbol{a}\}$ becomes undefined i.e. $\{{}^j\boldsymbol{a}\} \notin \Omega_a$ (No. 4 in Table 2). To avoid this case the classification of substitute systems on the basis of ${}^j\delta$, ${}^j\varphi$ and ${}^j\varrho$ must be derived [18]. In this connection let us emphasize that the matrix $\boldsymbol{G}({}^j\boldsymbol{a})$ must be nonsingular. Only in this case the solution of the linear approximation exists ([1], [9]). Further let us remark that the problem of the local minima is not very important. The key problem is how to obtain the stable sequence $\{{}^j\boldsymbol{a}\}$, i.e. $\{{}^j\boldsymbol{a}\} \in \Omega_a$ [18].

## 6. CONCLUSION

Let us summarize the main results. The signal approximation has not a unique solution, i.e. the infinite number of solutions can exist with the same total error ${}^j\eta$ (${}^j\eta^2 = \|y - {}^j\bar{y}\|^2 = {}^j\delta^2 + {}^j\varphi^2$) but with a different ratio ${}^j\delta/{}^j\varphi$ (Proposition 4.2).

Here ${}^j\delta = \|y - {}^jz\|$ is the error of the pertinent linear case given by (3.17). These two solutions are closely connected (Propositions 4.2 and 4.3) and form the basis of the DNLS (Damped Nonlinear Least Squares) method. All other modifications of the Gauss-Newton method are based on the total error only.

The convergence of the sequence $\{{}^j\boldsymbol{a}\}$ is governed in the endsteps (i.e. ${}^j\varphi \to \emptyset$) by the linear case (Table 1). The global minimum of the nonlinear case (for ${}^j\varphi = {}^*\varphi = \emptyset$ in Proposition 4.4) and the unique minimum of the pertinent linear case (Proposition 4.1) are identical.

(Received December 31, 1981.)

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