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CONTROLLABILITY OF A CLASS OF NONLINEAR SYSTEMS WITH DISTRIBUTED DELAYS IN CONTROL

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Sufficient conditions for global relative controllability of a class of nonlinear time varying systems with distributed delays in the control and implicit derivative are given. The results are obtained by using the measure of noncompactness of a set and the Darbo fixed point theorem.

1. INTRODUCTION

In the study of economic, biological and physiological systems as well as electromagnetic systems composed of subsystems interconnected by hydraulic, mechanical and various other linkages, one encounters phenomena which cannot be readily modelled unless relations involving time delays are admitted. Examples of such models have been given in [8], which also includes the systems considered here. In particular, models for systems with delay in the control occur in the study of gas-pressurized bipropellant rocket systems, in population models and in some complex economic systems [3]. More specifically, models for systems with distributed delays in the control occur in the study of agricultural economics [2] and in population dynamics [1]. We shall consider the nonlinear systems which are perturbations of linear systems.

Using the measure of noncompactness of a set and Darbo's fixed point theorem, controllability of nonlinear systems with implicit derivative has been studied by Dacka [4]. In [5] Dacka has extended the method to other types of nonlinear systems with delays in control and implicit derivative. Controllability of nonlinear systems with distributed delays in control has been considered by Klamka [6, 7], with the aid of Schauder's fixed point theorem. In this paper we shall examine the controllability of nonlinear systems with distributed delays in control and implicit derivative by the method of Dacka.

2. MATHEMATICAL PRELIMINARIES

Let $(X, \|.\|)$ be a Banach space and E a bounded subset of X. In this paper the following definition of the measure of noncompactness of a set E is used [4, 9]

$$\mu(E) = \inf\{r > 0;\ E \text{ can be covered by a finite number of balls whose radii are smaller than } r\}.$$

The following version of Darbo's fixed point theorem being a generalisation of Schauder's fixed point theorem shows the usefulness of the measure of noncompactness. "If S is a nonempty bounded closed convex subset of X and $P : S \to S$ is a continuous mapping such that for any set $E \subset S$ we have

$$\mu(PE) \leqq k\,\mu(E) \tag{1}$$

where k is a constant $0 \leqq k < 1$, then P has a fixed point".

For the space of continuous functions $C_n[t_0, t_1]$ the measure of noncompactness of a set E is given by

$$\mu(E) = \tfrac{1}{2}\, w_0(E) = \tfrac{1}{2} \lim_{h \to 0^+} w(E, h) \tag{2}$$

where $w(E, h)$ is the common modulus of continuity of the functions which belong to the set E, that is

$$w(E, h) = \sup_{x \in E}\,[\sup |x(t) - x(s)| : |t - s| \leqq h] \tag{3}$$

where as in the space of differentiable functions $C_n^1[t_0, t_1]$ we have

$$\mu(E) = \tfrac{1}{2}\, w_0(DE) \tag{4}$$

where

$$DE = \{\dot{x} : x \in E\}$$

If the space X is the Cartesian product $X = X_1 \times X_2$ of two spaces X_1 and X_2, then for any set $E \subset X$

$$\mu(E) = \max\,[\mu(E_1), \mu(E_2)]$$

where E_1 and E_2 denote the natural projections of E onto X_1 and X_2 respectively.

Let $h > 0$ be a given real number. For functions $u : [t_0 - h, t_1] \to \mathbb{R}^m$ and $t \in [t_0, t_1]$, let u_t denote the function on $[-h, 0)$ defined by $u_t(s) = u(t + s)$ for $s \in [-h, 0)$. Furthermore if the function $g(s)$ is of bounded variation on $[-h, 0]$, then the symbol $\operatorname{Var}_{[-h,0]} g(s)$ will denote the variation of the function $g(s)$ on the interval $[-h, 0]$. The integrals are in the Lebesque-Stieltjes sense [6].

3. STATEMENT OF THE PROBLEM

Let us consider the nonlinear time varying system with distributed delays in the control represented by the following differential equation

$$\dot{x}(t) = A(t)\,x(t) + \int_{-h}^{0} \mathrm{d}_s\, B(t, s)\, u(t + s) + f(t, x, \dot{x}, u) \tag{5}$$

satisfied everywhere on the interval $[t_0, t_1]$, and where $x(t) \in \mathbb{R}^n$, $u(t)$ is an m-dimensional control vector and $u \in C_m[t_0 - h, t_1]$, $A(t)$ is the $n \times n$ matrix whose elements are continuous in t and $f(t, x, \dot{x}, u)$ is an n-dimensional continuous function in its arguments. The matrix $B(t, s)$ is an $n \times m$ dimensional, continuous in t for fixed s and is of bounded variation in s on $[-h, 0]$ for each $t \in [t_0, t_1]$ and continuous from left in s on the interval $(-h, 0)$. The symbol d_s in (5) denotes that the integral is in the Lebesque-Stieltjes sense. Assume that

$$\|A(t)\| \leqq M\,, \quad \|B(t, s)\| \leqq N \quad \text{for each} \quad s \in [-h, 0] \tag{6}$$

$$|f(t, x, y, u)| \leqq K \quad \text{for} \quad t \in [t_0, t_1] \quad \text{and} \quad x, y \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

where M, N and K are positive constants. Further for every y, $\bar{y} \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $t \in [t_0, t_1]$

$$|f(t, x, y, u) - f(t, x, \bar{y}, u)| \leqq k|y - \bar{y}| \tag{7}$$

where k is a positive constant such that $0 \leqq k < 1$. The following definition of the complete state of the system and global relative controllability are assumed [6].

Definition 1. The set $z(t) = \{x(t), u_t\}$ is said to be the complete state of the system (5) at time t.

Definition 2. The system (5) is globally relatively controllable on $[t_0, t_1]$ if for every complete state $z(t_0)$ and every $x_1 \in \mathbb{R}^n$ there exist a control $u(t)$ defined on $[t_0, t_1]$ such that the corresponding trajectory of the system (5) satisfies the condition $x(t_1) = x_1$.

The solution of the system (5) is given by

$$x(t) = F(t, t_0)\,x(t_0) + \int_{t_0}^{t} F(t, \tau)\left(\int_{-h}^{0} \mathrm{d}_s\, B(\tau, s)\; u(\tau + s)\right)\mathrm{d}\tau + \int_{t_0}^{t} F(t, \tau)\,f(\tau, x, \dot{x}, u)\,\mathrm{d}\tau \tag{8}$$

where $F(t, t_0)$ is the transition matrix of the system $\dot{x}(t) = A(t)\,x(t)$ with $F(t_0, t_0) = I$.

The second term in the right hand side of (8) contains the values of $u(t)$ for $t < t_0$ as well as for $t > t_0$. The values of the control $u(t)$ for $t \in [t_0 - h, t_0)$ enter into the definition of the initial complete state $z(t_0)$. This can be separated by changing the order of integration and using the unsymmetric Fubini theorem, we have the following equalities [6, 7]

$$
\begin{aligned}
x(t) &= F(t, t_0)\, x(t_0) + \int_{t_0}^{t} F(t, \tau) f(\tau, x, \dot{x}, u)\, \mathrm{d}\tau + \\
&+ \int_{-h}^{0} \mathrm{d}_{Bs} \left(\int_{t_0}^{t} \left(F(t, \tau)\, B(\tau, s)\, u(\tau + s)\, \mathrm{d}\tau \right) = \right. \\
&= F(t, t_0)\, x(t_0) + \int_{t_0}^{t} F(t, \tau) f(\tau, x, \dot{x}, u)\, \mathrm{d}\tau + \\
&+ \int_{-h}^{0} \mathrm{d}_{Bs} \left(\int_{t_0+s}^{t_0} F(t, \tau - s)\, B(\tau - s, s)\, u_{t_0}\, \mathrm{d}\tau \right) + \\
&+ \int_{-h}^{0} \mathrm{d}_{Bs} \left(\int_{t_0}^{t+s} F(t, \tau - s)\, B(\tau - s, s)\, u(\tau)\, \mathrm{d}\tau \right)
\end{aligned} \tag{9}
$$

where d_{Bs} denotes that the integration is in the Lebesque-Stieltjes sense with respect to the variable s in the function $B(t, s)$.

Let us introduce the following notation.

$$
B_t(\tau, s) = \begin{cases} B(\tau, s) & \text{for} \quad \tau \leqq t \\ 0 & \text{for} \quad \tau > t \end{cases} \tag{10}
$$

Hence $x(t)$ can be expressed in the following form

$$
\begin{aligned}
x(t) &= F(t, t_0)\, x(t_0) + \int_{t_0}^{t} F(t, \tau) f(\tau, x, \dot{x}, u)\, \mathrm{d}\tau + \\
&+ \int_{-h}^{0} \mathrm{d}_{Bs} \left(\int_{t_0+s}^{t_0} F(t, \tau - s)\, B(\tau - s, s)\, u_{t_0}\, \mathrm{d}\tau \right) + \\
&+ \int_{-h}^{0} \mathrm{d}_{Bs} \left(\int_{t_0}^{t} F(t, \tau - s)\, B_t(\tau - s, s)\, u(\tau)\, \mathrm{d}\tau \right)
\end{aligned} \tag{11}
$$

Using again unsymmetric Fubini's theorem, the equality (11) can be rewritten as follows

$$
\begin{aligned}
x(t) &= F(t, t_0)\, x(t_0) + \int_{t_0}^{t} F(t, \tau) f(\tau, x, \dot{x}, u)\, \mathrm{d}\tau + \\
&+ \int_{-h}^{0} \mathrm{d}_{Bs} \left(\int_{t_0+s}^{t_0} F(t, \tau - s)\, B(\tau - s, s)\, u_{t_0}\, \mathrm{d}\tau \right) + \\
&+ \int_{t_0}^{t} \left(\int_{-h}^{0} F(t, \tau - s)\, \mathrm{d}_s\, B_t(\tau - s, s) \right) u(\tau)\, \mathrm{d}\tau
\end{aligned} \tag{12}
$$

Now let us consider the solution $x(t)$ of system (5) for $t = t_1$

$$x(t_1) = F(t_1, t_0)\,x(t_0) + \int_{t_0}^{t_1} F(t_1, \tau) f(\tau, x, \dot{x}, u)\,d\tau + \tag{13}$$
$$+ \int_{-h}^{0} d_{Bs}\left(\int_{t_0+s}^{t_0} F(t_1, \tau - s)\,B(\tau - s, s)\,u_{t_0}\,d\tau\right) +$$
$$+ \int_{t_0}^{t_1}\left(\int_{-h}^{0} F(t_1, \tau - s)\,d_s\,B_{t_1}(\tau - s, s)\right) u(\tau)\,d\tau$$

For brevity let us introduce the following notations

$$S(t_1, \tau) = \int_{-h}^{0} F(t_1, \tau - s)\,d_s\,B_{t_1}(\tau - s, s) \tag{14}$$

and

$$q(z(t_0), x(t_1)) = x_1 - F(t_1, t_0)\,x(t_0) - \int_{t_0}^{t_1} F(t_1, \tau) f(\tau, x, \dot{x}, u)\,d\tau - \tag{15}$$
$$- \int_{-h}^{0} d_{Bs}\left(\int_{t_0+s}^{t_0} F(t_1, \tau - s)\,B(\tau - s, s)\,u_{t_0}\,d\tau\right)$$

Define the controllability matrix $W(t_0, t_1)$ by

$$W(t_0, t_1) = \int_{t_0}^{t_1} S(t_1, \tau)\,S'(t_1, \tau)\,d\tau \tag{16}$$

where the prime indicates the matrix transpose.

Remark. It should be noted that under the assumption that the function f satisfies the Lipschitz condition with respect to the state variables, the response is uniquely determined by any control.

4. MAIN RESULT

Theorem. Given the system (5) with conditions (6) and (7). Assume that the matrix $W(t_0, t_1)$ is non-singular for $t_1 > t_0$. Then the system is globally relatively controllable on $[t_0, t_1]$.

Proof. The proof of the theorem is similar to the proof given in [4] and hence it will be only sketched. Let us consider the Banach space.

$$B[t_0, t_1] = C_m[t_0, t_1] \times C_n^1[t_0, t_1]\,.$$

Define the following nonlinear mapping of the space B.

$$T([u, x])(t) = [T_1([u, x])(t), T_2([u, x])(t)]$$

where the pair of operators T_1 and T_2 is defined as follows

(17) $$T_1([u, x])(t) = S'(t_1, t)\, W^{-1}(t_0, t_1)\, q(z(t_0), x_1)$$

and

$$T_2([u, x])(t) = F(t, t_0)\, x(t_0) + \int_{t_0}^{t} F(t, \tau) f(\tau, x, x, T_1([u, x]))\, d\tau +$$
$$+ \int_{-h}^{0} d_{Bs}\left(\int_{t_0+s}^{t_0} F(t, \tau - s)\, B(\tau - s, s)\, u_{t_0}\, d\tau\right) +$$
$$+ \int_{t_0}^{t} S(t, \tau)\, T_1([u, x])(\tau)\, d\tau$$

(here $z(t_0)$ and x_1 are chosen arbitrarily). It is easy to see that the operator T is continuous, since all the functions involved in the definition of T are continuous and it transforms the space $B[t_0, t_1]$ into itself.

Consider the closed convex subset of B

(19) $$H = \{[u, x] : \|u\| \leqq N_1,\ \|x\| \leqq N_2,\ \|Dx\| \leqq N_3\}$$

since the Lebesque-Stieltjes integral is finite, the positive constants N_1, N_2 and N_3 are defined by

(20) $$N_1 = K_2 \|W^{-1}(t_0, t_1)\| \left[|x_1| + \exp(M(t_1 - t_0))\, |x(t_0)| + \right.$$
$$\left. + (t_1 - t_0)\, K \exp(2M(t_1 - t_0)) + K_1\right]$$

(21) $$N_2 = \exp(M(t_1 - t_0))\, |x(t_0)| + K(t_1 - t_0) \exp(M(t_1 - t_0)) +$$
$$+ K_1 + (t_1 - t_0)\, K_2 N_1$$

(22) $$N_3 = MN_2 + NN_1K_3 + K$$

$$K_1 = \left\|\int_{-h}^{0} d_{Bs}\left(\int_{t_0+s}^{t_0} F(t_1, \tau - s)\, B(\tau - s, s)\, u_{t_0}\, d\tau\right)\right\|$$

$$K_2 = \max_{t_0 \leqq \tau \leqq t_1} \|S(t_1, \tau)\|\,, \quad K_3 = \max_{t_0 \leqq t \leqq t_1} \operatorname*{Var}_{s \in [-h, 0)} B(t, s)$$

The set H is bounded, closed and convex in B. The operator T transform H into H. It is easily seen that for each pair $[u, x] \in H$ we have

(23) $$w(T_1[u, x]), h) \leqq w(S', h)\, a$$

where

$$a = \sup_{[u,x] \in H} \{\|W^{-1}(t_0, t_1)\|\, q(z(t_0), x_1)\}$$

Since the function S does not depend on the choice of the points in H, all the functions $T_1([u, x)]\,(t)$ have a uniformly bounded modulus of continuity, hence, they are equicontinuous.

All the functions $T_2([u, x])(t)$ are also equicontinuous, since they have uniformly bounded derivatives. Now we shall find an estimate of the modulus of continuity of the functions $DT_2([u, x])(t)$.

$$\begin{aligned} |DT_2([u, x])(t) - DT_2([u, x])(\bar{t})| &\leqq |A(t)\, T_2([u, x])(t) - A(\bar{t})\, T_2([u, x])(\bar{t})| + \\ (24) \qquad &+ \left| \int_{-h}^{0} \mathrm{d}_s B(t, s)\, T_1([u, x])(t + s) - \int_{-h}^{0} \mathrm{d}_s B(\bar{t}, s)\, T_1([u, x])(\bar{t} + s) \right| + \\ &+ |f(t, x(t), \dot{x}(t), T_1([u, x])(t)) - f(\bar{t}, x(\bar{t}), \dot{x}(\bar{t}), T_1([u, x])(\bar{t}))| \end{aligned}$$

For the first two terms of the right hand side of inequality (24) we may give the upper estimate as $\beta_0(|t - \bar{t}|)$, where β_0 is a non-negative function such that $\lim_{h \to 0^+} \beta_0(h) = 0$ and that it may be chosen independently of the choice of the element $[u, x] \in H$. In the same manner we find that the last term on the right of (24) can be estimated from above by $k|\dot{x}(t) - \dot{x}(\bar{t})| + \beta_1(|t - \bar{t}|)$. Letting $\beta = \beta_0 + \beta_1$ we finally obtain

$$w(DT_2([u, x]), h) \leqq kw(Dx, h) + \beta(h) \tag{25}$$

Hence by (2) and (3) we conclude that for any set $E \subset H$

$$w_0(T_1E) = 0 \quad \text{and} \quad w_0(T_2E) \leqq kw_0(DE_2)$$

where E_2 is the natural projection of the set E on the space $C_n^1[t_0, t_1]$. Hence, it follows that

$$\mu(TE) \leqq k\, \mu(E)\,.$$

By the Darbo fixed point theorem the mapping T has at least one fixed point, therefore there exists functions $u^* \in C_m[t_0, t_1]$ and $x^* \in C_n^1[t_0, t_1]$ such that

$$u^*(t) = T_1([u^*, x^*])(t) \tag{26}$$

$$x^*(t) = T_2([u^*, x^*])(t) \tag{27}$$

Substituting this fixed point into (17) and (18), a direct differentiation of (18) with respect to t shows that $x^*(t)$ is a solution to the system (5) for the control $u^*(t)$. It is easy to verify that the control $u^*(t)$ steer the system (5) from the complete initial state $z(t_0)$ to $x_1 \in \mathbb{R}^n$, on the interval $[t_0, t_1]$ and since $z(t_0)$ and x_1 have been chosen arbitrarily, then by Definition 2, system (5) is globally relatively controllable on $[t_0, t_1]$. □

5. CONCLUSION

Using the measure of noncompactness of a set and Darbo's fixed point theorem sufficient conditions fot global relative controllability of nonlinear systems with distributed delays in control and implicit derivative have been derived.

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