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ON THE DECOUPLING OF ONE CLASS OF MULTIVARIABLE SYSTEMS

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In the paper a sufficient condition for the decoupling of one class of linear multi-input, multi-output systems is derived. It is much easier to verify this condition than to calculate the Falb-Wolovich's test for decoupling.

1. INTRODUCTION

The problem of transforming a multivariable system to a decoupled form can be principally treated by two approaches. The first one employs the knowledge of the internal system structure, and an appropriate state feedback compensating the undesirable interactions is constructed. Obviously, such a feedback also affects the dynamics of the considered system. The second approach is based on the external frequency domain description of the system. The internal structure of the system remains unchanged, only suitable compensators are constructed in such a way that the overall system is decoupled from the external point of view.

In this paper we focus our attention on the simplest problem of decoupling by static state feedback into single-input single-output subsystems.

2. BASIC PROBLEM

Let us consider an r-input, r-output, n-dimensional linear system

$$(1) \qquad S(\mathbf{A}, \mathbf{B}, \mathbf{C}): \quad \begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \qquad \mathbf{x}(t_0) = \mathbf{0} \end{aligned}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are real constant matrices, respectively $n \times n$, $n \times r$, $r \times n$ and denote by

(2) $$\mathbf{S}(p) = \mathbf{C}(p\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}$$

its transfer matrix.

We say that $S(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is decoupled if $\mathbf{S}(p)$ is diagonal and nonsingular, i.e. a component u_i of $\mathbf{u}$ does not affect any component $y_j, j \neq i$, of $\mathbf{y}$ but it does affect the y_i in the sense that

(3) $$\operatorname{rank} \mathbf{C}[\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}] = r\,.$$

The decoupling thus means non-interaction and output controllability, and it is not a generic property of the system.

The problem of decoupling by static state feedback (Morgan's problem, see [7]) then consists in determining a control law

(4) $$\mathbf{u}(t) = \mathbf{Rx}(t) + \mathbf{Qv}(t)$$

such that the close-loop system with the transfer matrix

(5) $$\mathbf{S}(p) = \mathbf{C}(p\mathbf{I} - \mathbf{A} - \mathbf{BR})^{-1}\,\mathbf{BQ}$$

be decoupled.

For every component y_i of the vector $\mathbf{y}$ it is possible to find a number d_i defined as

(6) $$d_i = \min \{\varkappa : \mathbf{C}_i\mathbf{A}^{\varkappa}\,.\,\mathbf{B} \neq 0,\ \varkappa = 0, 1, \ldots, n - 1\}$$

(6a) $$d_i = n - 1 \ldots \text{ if } s_i(t) = 0\,.$$

Physically the coefficient d_i denotes the lowest non-zero derivative of the ith coordinate of the impulse response vector $\mathbf{s}(t)$ at the time $t = 0$. When introducing the constants $\bar{d}_i$ analogically for the closed-loop system, then it can be proved that

(7) $$d_i = \bar{d}_i$$

The criterion of decoupling derived by Falb and Wolovich can be summarized as follows:

Theorem $V_{\mathrm{F-W}}$. Let the system S be output controllable. Then S can be decoupled by static state feedback if and only if

(8) $$\det \mathbf{B}_D = \det \begin{bmatrix} \mathbf{C}_1\mathbf{A}^{d_1}\mathbf{B} \\ \vdots \\ \mathbf{C}_r\mathbf{A}^{d_r}\mathbf{B} \end{bmatrix} \neq 0\,.$$

The decoupled system $\bar{S}$ can be described by the transfer matrix

(9) $$\bar{\mathbf{S}}(p) = \operatorname{diag}\,[\bar{S}_{ii}(p)]_{i=1,\ldots,r}$$

and it always holds

(10) $$\sum_{i=1}^{r} d_i + r \leqq n\,.$$

3. CONSTRUCTION OF A MORE SIMPLE CRITERION

The test for decoupling by means of V_{F-W} is difficult especially in the case of higher order systems. So we shall try to simplify the above general criterion for some special cases.

We make the following assumptions:

(a) system S is of nth order with r inputs and r outputs

(b) $d_i \neq n - 1$ for $i = 1, 2, \ldots, r$

(c) S is output controllable that means

$$\text{rank}\, \mathbf{P}_y = \text{rank}\left[\mathbf{CB}, \mathbf{CAB}, \ldots, \mathbf{CA}^{n-1}\mathbf{B}\right] = r\,. \tag{11}$$

Relation (11), enabling us to determine the constants $d_1, d_2, \ldots, d_r$ by inspection, is obviously equivalent to the condition

$$\det \mathbf{S}(p) \neq 0\,. \tag{12}$$

As $\mathbf{S}(p)$ represents a matrix of dimension $r \times r$, then it holds

$$\text{rank}\, \mathbf{S}(p) = r = \text{rank}\, \mathbf{C}(p\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \leqq \min\{\text{rank}\, \mathbf{C}, \text{rank}\, \mathbf{B}\} \tag{13}$$

and so

(13a) $\text{rank}\, \mathbf{C} = r \ldots$ corresponds to the requirement of independent outputs ,

(13b) $\text{rank}\, \mathbf{B} = r \ldots$ corresponds to the requirement of independent inputs .

The relation (8) in the Falb-Wolovich's test of decoupling can be expressed not only in the form

$$\det \mathbf{B}_D = \begin{bmatrix} \mathbf{C}_1\mathbf{A}^{d_1}\mathbf{B} \\ \vdots \\ \mathbf{C}_r\mathbf{A}^{d_r}\mathbf{B} \end{bmatrix} \neq 0$$

but also as

$$\text{rank}\left\{\underbrace{\begin{bmatrix} \mathbf{C}_1 & \mathbf{0} & \mathbf{0}\ \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 & \mathbf{0}\ \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{0}\ \mathbf{0} & \ldots & \mathbf{C}_r \end{bmatrix}}_{\hat{\mathbf{C}}} \cdot \begin{bmatrix} \mathbf{A}^{d_1} \\ \mathbf{A}^{d_2} \\ \vdots \\ \mathbf{A}^{d_r} \end{bmatrix} \cdot \mathbf{B}\right\} = r\,. \tag{14}$$

As in (14)

$$\text{rank}\, \hat{\mathbf{C}} = r \tag{15}$$

and taking into account the necessary condition (13b) it also holds

$$\text{rank} \begin{bmatrix} \mathbf{A}^{d_1} \\ \mathbf{A}^{d_2} \\ \vdots \\ \mathbf{A}^{d_r} \end{bmatrix} \geqq r\,. \tag{16}$$

This condition represents a necessary but not sufficient condition for decoupling. Notice that for

(17) $$d_1 = d_2 = \ldots = d_r = d^*$$

condition (16) can be written as

(18) $$\operatorname{rank} \mathbf{A}^{d^*} \geqq r .$$

The simplification of the general Falb-Wolovich's test for some special cases can be carried out on the basis of the matrix analysis. Using some manipulations (11) can be written as

(19) $$\mathbf{P}_y = \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{C}_1\mathbf{B} & \mathbf{C}_1\mathbf{AB} & \ldots & & \mathbf{C}_1\mathbf{A}^{d_1}\mathbf{B} & \ldots & \mathbf{C}_1\mathbf{A}^{n-1}\mathbf{B} \\ \vdots & \vdots & & & & & \vdots \\ \mathbf{C}_r\mathbf{B} & & \ldots & \mathbf{C}_r\mathbf{A}^{d_r}\mathbf{B} & \ldots\ldots\ldots & & \mathbf{C}_r\mathbf{A}^{n-1}\mathbf{B} \end{array}\right]$$

The case when the equality takes in (10), i.e. when the state feedback can affect the layout of all poles of the system, can be considered as a special case within multi-variable systems.

Let us suppose that besides the equality in (10) also (17) holds. Then (19) takes the form

(20) $$\mathbf{P}_y = \begin{bmatrix} 0 & 0 & 0 & \mathbf{C}_1\mathbf{A}^{d^*}\mathbf{B} & \ldots & \mathbf{C}_1\mathbf{A}^{n-1}\mathbf{B} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \mathbf{C}_r\mathbf{A}^{d^*}\mathbf{B} & \ldots & \mathbf{C}_r\mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

Obviously in (20) the column vectors of the matrix $\mathbf{B}$ of the system $S(\mathbf{A}, \mathbf{B}, \mathbf{C})$ are orthogonal to all row vectors of the matrices $\mathbf{C}, \mathbf{CA}, \ldots, \mathbf{CA}^{d^*-1}$. For a decoupled system it must hold

(21) $$\operatorname{rank} \mathbf{C} = \operatorname{rank} \mathbf{CA} = \operatorname{rank} \mathbf{CA}^2 = \ldots = \operatorname{rank} \mathbf{CA}^{d^*-1} = r .$$

Now let us prove the following statement.

Lemma. Let the following hold for the system $S(\mathbf{A}, \mathbf{B}, \mathbf{C})$ given by (1)

(i) $\operatorname{rank} \mathbf{P}_y = r$

(ii) $\sum_{i=1}^{r} d_i + r = n$

(iii) relation (17) is satisfied.

Then

(22) $$\operatorname{rank} \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \vdots \\ \mathbf{C}_r \\ \hline \vdots \\ \hline \mathbf{C}_1\mathbf{A}^{d_1-1} \\ \vdots \\ \mathbf{C}_r\mathbf{A}^{d_r-1} \end{bmatrix} = d^* \cdot r$$

Proof. As (21) holds it suffices to prove the linear independence of row vectors associated with different powers of the matrix $\boldsymbol{A}$.

Let us suppose that the row vector $\boldsymbol{C}_i\boldsymbol{A}^j$ of the matrix on the left hand side of (22) can be expressed as

$$\boldsymbol{C}_i\boldsymbol{A}^j = \sum_{q=1}^{k} \lambda_q \boldsymbol{C}_f A^h \tag{23}$$

$$f \in \langle 1, \ldots, r \rangle\,, \quad h \neq j\,, \quad h \leqq d^* - 1\,.$$

Then, according to (8), each row of the matrix $\boldsymbol{B}_D$ can be expressed as

$$\boldsymbol{B}_{D_i} = \boldsymbol{C}_i\boldsymbol{A}^{d^*}\boldsymbol{B} = \boldsymbol{C}_i\boldsymbol{A}^{j+m}\boldsymbol{A} = \sum_{q=1}^{k} (\lambda_q \boldsymbol{C}_f \boldsymbol{A}^h) \,.\, \boldsymbol{A}^m\boldsymbol{B} \tag{24}$$

$$j + m = d^*\,.$$

However

$$h + m \neq d^* \tag{25}$$

that contradicts to (ii). Hence relation (22) holds. □

Recalling the rows in (22) are linearly independent (so they represent $d^* . r$ vectors of a basis of the linear n-dimensional vector space), the columns of the matrix $\boldsymbol{B}$ are orthogonal to all basis vectors and due to (13b) they can also represent the rest of r basis vectors. The same role can be performed also by the r row vectors $\boldsymbol{C}_1\boldsymbol{A}^{d^*}, \ldots$ $\ldots, \boldsymbol{C}_r\boldsymbol{A}^{d^*}$. Both above mentioned sets of r vectors describe the same sub-space of the dimension r. Hence no set with more than r linearly independent elements can be found in this space and some entries of the matrix $\boldsymbol{B}_D$ (those which are scalar products of the row vectors $\boldsymbol{C}_i\boldsymbol{A}^{d^*}$, $i = 1, 2, \ldots, r$, with the column vectors of the matrix $\boldsymbol{B}$) must be non-zero.

Theorem 1. Let the assumptions (i), (ii), (iii) of the Lemma be fulfilled. Then the system $S(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ can be decoupled.

Proof. If (22) holds then the column of the matrix $\boldsymbol{B}$ are orthogonal to the row vectors of the matrix

$$\begin{bmatrix} \boldsymbol{C}_1 \\ \vdots \\ \boldsymbol{C}_r \\ \hline \vdots \\ \hline \boldsymbol{C}_1\boldsymbol{A}^{d^*-1} \\ \vdots \\ \boldsymbol{C}_r\boldsymbol{A}^{d^*-1} \end{bmatrix}$$

Moreover the elements of the columns of the matrix $\boldsymbol{B}$ in the subspace of dimension

$d^* \,.\, r$ (generated by the vectors $\mathbf{C}_1, \ldots, \mathbf{C}_r \mathbf{A}^{d^*-1}$) are uniquely determined. Denote

$$
(26) \qquad \begin{bmatrix} \mathbf{C}_1 \mathbf{A}^{d^*} \\ \mathbf{C}_2 \mathbf{A}^{d^*} \\ \vdots \\ \mathbf{C}_r \mathbf{A}^{d^*} \end{bmatrix} \overset{!}{=} \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \\ \boldsymbol{\alpha}_r \end{bmatrix} \qquad \mathbf{B} \overset{!}{=} [\mathbf{b}_1 \mid \mathbf{b}_2 \mid \ldots \mid \mathbf{b}_r]
$$

Thus $\boldsymbol{\alpha}_i$'s correspond to the row vectors, $\mathbf{b}_i$'s to the column vectors $(i = 1, \ldots, r)$ and each element of the matrix $\mathbf{B}_D$ can be expressed as

$$
(27) \qquad B_{D_{kl}} = \boldsymbol{\alpha}_k \,.\, \mathbf{b}_l \,.
$$

Denoting the columns of the matrix $\mathbf{B}_D$ by

$$
(28) \qquad \mathbf{B}_D = [\mathbf{b}_{D_1} \mid \mathbf{b}_{D_2} \mid \ldots \mid \mathbf{b}_{D_r}]
$$

then it can be written

$$
(29) \qquad \mathbf{b}_{D_j} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_r \end{bmatrix} . \, \mathbf{b}_j \,.
$$

Now we shall investigate the singularity condition of the matrix $\mathbf{B}_D$. In case $\mathbf{B}_D$ is non-singular the columns $\mathbf{b}_{D_i}$, $i = 1, \ldots, r$ must be linearly independent and so

$$
(30) \qquad \mathbf{b}_{D_j} = \sum_{\substack{f=1 \\ f \neq j}}^{r} a_f \,.\, \mathbf{b}_{D_f} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \vdots \\ \boldsymbol{\alpha}_r \end{bmatrix} . \, (a_1 \,.\, \mathbf{b}_1 + \ldots + a_r \,.\, \mathbf{b}_r) \,.
$$

a_f = linear combination coefficients

Comparing (29) and (30) we get

$$
(31) \qquad \mathbf{b}_j = \sum_{\substack{f=1 \\ f \neq j}}^{r} a_f \,.\, \mathbf{b}_f
$$

and this contradicts to the assumption of output controllability and the independent of inputs. The theorem has been proved. □

The proof can be carried out also by using the rows of $\mathbf{B}_D$.

It can be shown that Theorem 1 holds even in the more general case, when the relation (17) is not satisfied. The proof can be done in the same way and is based on the equation

$$
(32) \qquad \operatorname{rank} \begin{bmatrix} \mathbf{C}_1 \\ \vdots \\ \mathbf{C}_1 \mathbf{A}^{d_1 - 1} \\ \hline \vdots \\ \hline \mathbf{C}_r \\ \vdots \\ \mathbf{C}_r \mathbf{A}^{d_r - 1} \end{bmatrix} = \sum_{i=1}^{r} d_i \,.
$$

Let us then summarize:

Theorem 2. Every output controllable system $S(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ satisfying $\sum_{i=1}^{r} d_i + r = n$ can be decoupled.

Theorems 1 and 2 substantially simplify the test of decoupling of one class of multivariable systems.

Note. It results from the text that the test for decoupling expressed by Theorem 1 or Theorem 2 is effective only in special case of the multivariable systems. If the system under study does not meet the assumptions of Theorem 1 or 2, the Falb-Wolovich's criterion, which is more difficult to use, must be applied.

4. CONCLUSION

Theorem 2 gives a sufficient condition for decoupling of one class of multivariable systems. When applicable, this condition is much easier to verify than the general criterion formulated by Falb and Wolovich in [1].

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