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# STOCHASTIC MULTIVARIABLE TRACKING
## A Polynomial Equation Approach

MICHAEL ŠEBEK

A new technique to design optimal multivariable tracking systems is presented for stochastic plants described by rational transfer matrices. The technique is based on polynomial matrices. The optimal controller is shown to consist of feedback and feedforward parts and it is designed essentially by solving two linear matrix polynomial equations whose coefficients are obtained by spectral factorization.

## INTRODUCTION

The stochastic tracking problem is one of the most significant problems in optimal control. It can be solved in either frequency or time domain. The frequency domain solution based on Wiener-Hopf technique was given in [12] and [3] for the case of error-actuated controller. However, the error-actuated controller is only sub-optimal in general so that better results can be achieved when using a more general structure. On the other hand, the time domain solution described in [7] is applicable only for stable reference generators and only for dynamical systems.

Recently, an alternative solution based on operations with *polynomials* was developed for single-input single-output plants [8] which covers both stable and unstable reference generators. The aim of this paper is to generalize this approach for multivariable plants. The *polynomial matrix* techniques, employed recently by Kučera [6] when solving regulator problems, are extended here to accomodate the tracking problems.

By a systematic use of matrix fractions, the design procedure is reduced to spectral factorization and the solution of two linear equations in polynomial matrices. This is believed to be computationally superior to existing methods and general enough to handle unstable and/or nonminimum-phase plants and reference generators with improper and rectangular transfer matrices, singular noise intensities and singular weighting matrices in the measure of performance

For a slightly less general discrete time version of this approach the reader is referred to [9].

A prominent tole throughout this paper will play polynomial matrices. They are treated in detail, e.g., in the books by Kailath [4], Kučera [5] and Wolovich [11].

## PROBLEM FORMULATION

Consider a linear time-invariant multivariable stochastic plant modeled by the equation (all quantities are Laplace transform)

$$y(s) = R(s)\,u(s) + S(s)\,w(s) \tag{1}$$

where $y$ is the vector output of the plant, $u$ is the vector input of the plant and $w$ is the background noise. Let the measured output of the plant be corrupted by an observation noise $v$.

Further consider a reference vector output $r$, represented by an output of a linear system driven by a noise $\bar{w}$,

$$r(s) = \bar{S}(s)\,\bar{w}(s) \tag{2}$$

Let the available version of the reference be corrupted by an observation noise $\bar{v}$.

All four vector random sources $v$, $w$ and $\bar{v}$, $\bar{w}$ are pairwise independent zero-mean covariance-stationary white vector random processes with intensities $\Lambda$, $\Omega$ and $\bar{\Lambda}$, $\bar{\Omega}$. respectively, which all are real nonnegative definite matrices. The $R$, $S$ and $\bar{S}$ are real rational (not necessarily proper) matrices of appropriate dimensions.

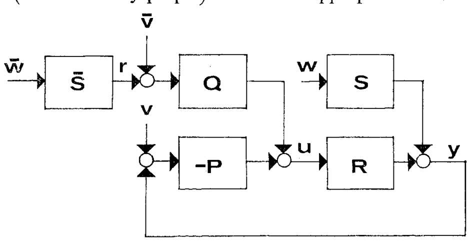


**Fig. 1.** Block diagram of a tracking system.

For a given plant the design of the optimal controller

$$u(s) = -P(s)\,(y(s) + v(s)) + Q(s)\,(r(s) + \bar{v}(s)) \tag{3}$$

evolves from minimization of the weighted sum of steady-state variances of the input and the tracking error, i.e., of the cost

$$J = \frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} \mathscr{E}\{u_*(s)\,\Phi\,u(s) + (r_*(s) - y_*(s))\,\Psi(r(s) - y(s))\}\,\mathrm{d}s \tag{4}$$

with nonnegative definite weighting matrices $\Phi$ and $\Psi$, where for any matrix $H$ the $H_*$ is defined by $H_*(s) = H^T(-s)$ and $\mathscr{E}\{.\}$ denotes ensemble average.

Thus our *design objective is to minimize* (4) *subject to the constraint that the tracking system* defined by (1) through (3) and shown in Fig. 1 *be asymptotically stable.* To this effect, we shall assume that all the matrices $\Phi + R_*\Psi R$, $\Lambda + S\Omega S_*$ and $\bar{\Lambda} + \bar{S}\bar{\Omega}\bar{S}_*$ are positive definite for $s$ on imaginary axis and, consequently, that all the spectral factors defined below by (5)–(7) exist.

## DESIGN PROCEDURE

Let us set aside for the moment all questions of solvability and let us attempt to construct $P(s)$ and $Q(s)$ starting with the primary data $R(s)$, $S(s)$, $\bar{S}(s)$ and $\Phi$, $\Omega$, $\bar{\Lambda}$, $\bar{\Omega}$, $\Phi$, $\Psi$. For simplicity, the function arguments are omitted wherever convenient. The optimal design is carried out in the following steps:

1) Find any left coprime polynomial matrix fraction decomposition of $R$ and $S$

$$A^{-1}[B \ C] = [R \ S]$$

such that the matrix $[A\Lambda, C\Omega]$ is row reduced, then calculate any left and right coprime polynomial matrix fraction decomposition of $R$

$$A_0^{-1}B_0 = B_1A_1^{-1} = R$$

such that $\begin{bmatrix} \Phi A_1 \\ \Psi B_1 \end{bmatrix}$ is column reduced, and a left coprime decomposition to $\bar{S}$

$$\bar{A}^{-1}\bar{C} = \bar{S}$$

such that $[\bar{A}\bar{\Lambda}, \bar{C}\bar{\Omega}]$ is row reduced.

2) Perform the spectral factorization to obtain stable polynomial matrices $F$, G, and $\bar{G}$ satisfying

$$A_{1*}\Phi A_1 + B_{1*}\Psi B_1 = F_*F \tag{5}$$

$$A\Lambda A_* + C\Omega C_* = GG_* \tag{6}$$

$$\bar{A}\bar{\Lambda}\bar{A}_* + \bar{C}\bar{\Omega}\bar{C}_* = \bar{G}\bar{G}_* \tag{7}$$

3) Calculate a left coprime polynomial matrix fraction

$$\bar{A}_3^{-1}A_3 = A_0\bar{A}^{-1}$$

and right coprime polynomial matrix fractions

$$B_2G_1^{-1} = G^{-1}B \qquad A_2G_2^{-1} = G^{-1}A$$

$$DE^{-1} = (A_3\bar{G})^{-1}\bar{A}_3B_0 \quad \bar{A}_2\bar{G}_2^{-1} = \bar{G}^{-1}\bar{A}$$

4) Find any polynomial matrix solutions $X$, $Y$, $Z$ and $\bar{X}$, $\bar{Y}$, $\bar{Z}$ of the equations

$$F_*[X,\ Y] - Z_*[B_2,\ -A_2] = [A_{1*}\Phi G_1,\ B_{1*}\Psi G_2] \tag{8}$$

$$F_*[\bar{X},\ \bar{Y}] - \bar{Z}_*[D,\ -\bar{A}_2] = [A_{1*}\Phi E,\ B_{1*}\Psi \bar{G}_2] \tag{9}$$

5) Perform the two-sided division algorithms

$$(AA\Psi B_1 = GZ) = GU_* F_* + V \tag{10}$$

$$(\bar{A}\bar{A}\Psi B_1 = \bar{G}\bar{Z}) = \bar{G}\bar{U}_* F_* + \bar{V} \tag{11}$$

to obtain polynomial quotients $U, \bar{U}$ and reminders $V, \bar{V}$ (with elements $v_{ij}$, $\bar{v}_{ij}$) such that

$\deg v_{ij} < \deg_{ri} G + \deg_{cj} F$ and $\deg \bar{v}_{ij} < \deg_{ri} \bar{G} + \deg_{cj} F$ where $\deg_{ri}$. ($\deg_{cj}$.) denotes the maximum degree occurring in $i$th row ($j$th column) of a matrix, and put

$$X_1 = X + UB_2\,,\quad Y_1 = Y - UA_2\,,\quad \bar{Y} = \bar{Y} - \bar{U}\bar{A}_2 \tag{12}$$

6) Calculate left coprime matrix fractions

$$G_0^{-1}[X_0,\ Y_0] = [X_1 G_1^{-1},\ Y_1 G_2^{-1}] \tag{13}$$

$$\bar{G}_0^{-1}[\bar{X}_0,\ \bar{Y}_0] = [X_1 G_1^{-1},\ \bar{Y}_1 \bar{G}_2^{-1}] \tag{14}$$

7) The optimal controller is then defined by the transfer matrices

$$P = X_0^{-1} Y_0\,,\quad Q = \bar{X}_0^{-1} \bar{Y}_0 \tag{15}$$

and must be realized as a single dynamical system of least order having two (vector) inputs $y$ and $r$ and the (vector) output $u$.

There are efficient algorithms to implement all steps of this design procedure. The spectral factors can be calculated by means of various recurrent schemes. A matrix version of the polynomial algorithm developed by Vostrý [10] is to be preferred for it combines efficiency with quadratic convergence. The other algorithms can be found gathered together, e.g., in [5], [6]. Besides, an alternative method of solution of matrix polynomial equations has been reported recently by Emre [1] and Emre and Silvermann [2].

## EXISTENCE AND UNIQUENESS

The design procedure above produces the optimal controller whenever one exists. This section is devoted to the proof of this claim.

**Theorem.** The optimal tracking problem is solvable iff

1) the polynomial matrix $X_1$ is nonsingular;

2) the eight rational matrices

(16) $$\Phi A_1 N A\,, \quad \Phi A_1 N S \Omega\,, \quad \Psi B_1 N A\,, \quad \Psi(I - B_1 N)\, S \Omega$$

(17) $$\Phi A_1 \bar{N} \bar{A}\,, \quad \Phi A_1 \bar{N} \bar{S} \bar{\Omega}\,, \quad \Psi B_1 \bar{N} \bar{A}\,, \quad \Psi(I - B_1 \bar{N})\, \bar{S} \bar{\Omega}$$

with $N$, $\bar{N}$ given by (20), (21), are strictly proper;

3) the greatest common left divisor of $A$ and $B$ is a stable polynomial matrix;

4) the unstable part of $\bar{A}$ is a right divisor of $A$ (i.e., $\bar{A}_3$ is a stable polynomial matrix).
The optimal controller has the unique transfer functions (15).

Proof. Defining rational matrices $M$, $N$ and $\bar{N}$ by relations

$$P = M^{-1}N\,, \quad MA_1 + NB_1 = I\,, \quad Q = M^{-1}\bar{N}\,,$$

simple algebraic manipulations yield

(18) $$u = -A_1 N v - A_1 N A^{-1} C w + A_1 \bar{N} \bar{v} + A_1 \bar{N} \bar{A}^{-1} \bar{C} \bar{w}$$

(19) $$r - y = B_1 N v - (I - B_1 N)\, A^{-1} C w - B_1 \bar{N} \bar{v} + (I - B_1 \bar{N})\, \bar{A}^{-1} \bar{C} \bar{w}$$

and the cost can be then expressed in the form

$$J = \frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} (W_1 + \bar{W}_1)\, \mathrm{d}s$$

where

$$W_1 = (\Phi A_1 N \bar{A} N_* A_{1*} + \Phi A_1 N A^{-1} C \Omega C_* A_*^{-1} N_* A_{1*}$$
$$+ \Psi B_1 N A N_* B_{1*} + \Psi(I - B_1 N)\, A^{-1} C \Omega C_* A_*^{-1} (I - B_1 N)_*)$$
$$\bar{W}_1 = \mathrm{tr}\,(\Phi A_1 \bar{N} \bar{A} \bar{N}_* A_{1*} + \Phi A_1 \bar{N} \bar{A}^{-1} \bar{C} \bar{\Omega} \bar{C}_* \bar{A}_*^{-1} \bar{N}_* A_{1*}$$
$$+ \Psi B_1 \bar{N} \bar{A} \bar{N}_* B_{1*} + \Psi(I - B_1 \bar{N})\, \bar{A}^{-1} \bar{C} \bar{\Omega} \bar{C}_* \bar{A}_*^{-1} (I - B_1 \bar{N})_*)$$

Thus the cost splits into two independent parts: $W_1$ and $W_2$. The former part is independent of the reference and is related to optimal regulation. When solving the corresponding regulator problem, Kučera [6] derived the minimizing values

(20) $$M = F^{-1} X_1 G_1^{-1} \quad \text{and} \quad N = F^{-1} Y_1 G_2^{-1}$$

The latter part is related to optimal tracking. It does depend on the reference and can be analysed analogously:

Substituting (5) and (7), employing the identity $\mathrm{tr}\, UV = \mathrm{tr}\, VU$ and completing the sqares, $\bar{W}_1$ has a form

$$\bar{W}_1 = \mathrm{tr}\, \bar{W}\bar{W} + \mathrm{tr}\,(\Psi \bar{A}^{-1} \bar{C} \bar{\Omega} \bar{C}_* \bar{A}_*^{-1} -$$
$$- F_* B_{1*} \Psi \bar{A}^{-1} \bar{C} \bar{\Omega} \bar{C}_* \bar{G}_*^{-1} \bar{G}^{-1} \bar{C} \bar{\Omega} \bar{C}_* \bar{A}_*^{-1} \Psi B_1 F^{-1})$$

where

$$\bar{W} = F \bar{N} \bar{A}^{-1} \bar{G} + F_*^{-1} B_{1*} \Psi \bar{A} \bar{A}_* \bar{G}_*^{-1} - F_*^{-1} B_{1*} \Psi \bar{G}_2 \bar{A}_2^{-1}$$

Equation (8) yields $F_*^{-1}B_{1*}\Psi\bar{G}_2\bar{A}_2^{-1} = \bar{Y}\bar{A}_2^{-1} + F_*^{-1}\bar{Z}_*$ so that, substituting (11) and (12), we finally get

$$\bar{W} = \bar{T} + F_*^{-1}\bar{V}_*\bar{G}_*^{-1}$$

where

$$\bar{T} = (F\bar{N}\bar{G}_2 - \bar{Y}_1)\,\bar{A}_2^{-1}$$

Repeating the discussion from [6], the second part of the cost attains its minimum when $\bar{T} = 0$, that is, for

$$\bar{N} = F^{-1}\bar{Y}_1\bar{G}_2^{-1} \tag{21}$$

and, combining it with (14), the feedforward part of the optimal controller (15) results.

As to the solvability conditions, (1) is clear. In order for the cost (4) to represent a finite weighted sum of steady-state variances it is necessaty and sufficient that the rational matrices (16), (17) appearing in (18), (19) be strictly proper (condition (condition 2)) as well as analytic in Re $s \geqq 0$. Matrices (16) are analytic in Re $s \geqq 0$ iff 3) holds and, analogously, so are (17) iff 4) holds. Asymptotic stability the resulting tracking system is assured by analycity of $M$, $N$ in Re $s \geqq 0$ (see [6]). □

It is possible to show that when all rational matrices $R$, $S$ and $\bar{S}$ are strictly proper and real matrices $\Phi$, $\Lambda$ and $\bar{\Lambda}$ are positive definite, then conditions 1) and 2) are satisfied automatically so that the problem is solvable iff 3) and 4) hold. In addition, instead of divisions (10), (11), simply the solutions of (8) and (9) are to be found for which $Y_1A_2^{-1}$ and $\bar{Y}_1\bar{A}_2^{-1}$ are strictly proper.

## CONCLUDING REMARKS

A complete yet relatively simple solution to the steady-state minimum variance tracking problem has been presented for systems described by rational transfer matrices and zero-mean covariance-stationary random inputs.

The design procedure involves spectral factorization and the solution of linear equations in polynomial matrices. Compared to the Wiener-Hopf approach, these algorithms manipulate polynomial rather then rational matrices and obviate the need to calculate partial fraction expansions. As to the state space approach, input-output models are easier to obtain than state-variable ones and our design is restricted neither to strictly proper $R$, $S$ and $\bar{S}$ nor to positive definite $\Phi$, $\Lambda$ and $\bar{\Lambda}$. The computational complexity of the used algorithms is compared in [6]. The numerical feasibility of our design procedure is particulary pronounced for single-input-output systems [8].

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*Ing. Michael Šebek, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Prague 8. Czechoslovakia.*