

A FEW REMARKS ON VECTOR OPTIMIZATION FROM THE COALITION-GAME THEORETICAL POINT-OF-VIEW

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Some aspects of the multicriterial decision-making problem are similar to the ones solved by the coalition-game theory. The similarity is described in this paper, and some properties of the vector optimization connected with game theoretical concepts are derived. The main results concern the mutual relation between the solution of a game and an acceptable compromise multicriterial decision, and some properties of the grouped criteria.

0. INTRODUCTION

There exist more general attitudes to the multicriterial decision-making problem. One of them is the optimization of complete utilities vectors and the construction of some compromise between the demands of criteria and their combinations. The usual vector optimization result is represented by a set of utility vectors connected with the set of reasonable outcomes of the multicriterial decision-making problem. The final result of the decision-making can be chosen from that set by using more sophisticated methods.

The problem of the simultaneous satisfaction of more demands is investigated in both, the multicriterial decision-making and the cooperative games, theories. This means that there generally exists a similarity between the models used in both theories, and a potential possibility to transfer some methods and concepts from the coalition-games theory to the multicriterial decision-making.

The main goal of this paper is to show some possibilities of such applications of the game theoretical tools to the vector optimization model, and to derive some results concerning those applications. Two conclusive sections are subjected to the optimization under an incomplete class of sets of criteria and to some elementary results on the elimination of redundant criteria.

1. PROBLEM FORMULATION

In this section we describe the multicriterial decision-making problem investigated in the presented paper.

We denote by \mathbb{R} the set of all real numbers. By $I = \{1, \dots, n\}$ we denote a finite set of indices of criteria, and we suppose that $n \geq 2$. Let us consider a non-empty set of admissible decisions \mathbf{D} . Let \mathcal{C} be an abstract topological space containing the consequences of decisions from \mathbf{D} . The consequences are connected with decisions by means of a result function $q: \mathbf{D} \rightarrow \mathcal{C}$, and we suppose that the set

$$C = \{c \in \mathcal{C} : \text{there exists } d \in \mathbf{D} \text{ such that } c = q(d)\}$$

is a compact subset of \mathcal{C} . Let us suppose that the criteria are represented by utility functions $u_i: C \rightarrow \mathbb{R}$, $i = 1, \dots, n$, which are continuous.

Remark 1. The compactness of C and continuity of u_i imply that the sets $\{x \in \mathbb{R} : \exists c \in C, u_i(c) = x\}$ are compact subsets of \mathbb{R} for any $i \in I$.

The utility functions u_i , $i \in I$, form a numerical representation of the criteria preferences. The higher value of the function u_i is achieved, the more the respective criterion is satisfied.

The goal of the multicriterial decision-making is to find optimal utility vectors $x = (x_i)_{i \in I}$ achievable by the decisions from \mathbf{D} . As the criteria are not independent, it is useful to represent not only the possibilities and demands of single criteria but also the correlated demands of their groups.

If $K \subset I$ is a group of criteria then the utility vectors achievable from the point-of-view of K are the vectors from the set

$$(1.1) \quad \begin{aligned} W(K) &= \{x = (x_i)_{i \in I} \in \mathbb{R}^I : \exists c \in C, \forall i \in K, x_i \leq u_i(c)\} = \\ &= \{x \in \mathbb{R}^I : \exists d \in \mathbf{D}, \forall i \in K, x_i \leq u_i(q(d))\}. \end{aligned}$$

Lemma 1. The sets $W(K)$ are non-empty for all $K \subset I$.

Proof. The statement follows from the non-emptiness of \mathbf{D} and from (1.1). \square

Lemma 2. The equality $W(K) = \mathbb{R}^I$ holds iff $K = \emptyset$.

Proof. If $K = \emptyset$ then by (1.1) $x \in W(K)$ for any $x \in \mathbb{R}^I$. If $K \neq \emptyset$ then the projection of $W(K)$ into \mathbb{R}^K , i.e. the set

$$\mathcal{P}(W(K), \mathbb{R}^K) = \{x_K = (x_i)_{i \in K} \in \mathbb{R}^K : \exists c \in C, \forall i \in K, x_i \leq u_i(c)\}$$

is a compact subset of \mathbb{R}^K as follows from the compactness of C and from the continuity of all, u_i , $i \in I$. It means that C is a bounded set, and there exists a $y_K = (y_i)_{i \in K} \in \mathbb{R}^K$ such that $y_i > u_i(c)$ for some $i \in K$ and for all $c \in C$. Then $y_K \notin \mathcal{P}(W(K), \mathbb{R}^K)$, $y' \notin W(K)$ for $y' \in \mathbb{R}^I$ such that $y'_i = y_i$ for all $i \in K$. \square

Lemma 3. The sets $W(K)$ are closed for all $K \subset I$.

Proof. It was shown in the proof of Lemma 2 that $\mathcal{P}(W(K), \mathbb{R}^K)$ is a compact subset of \mathbb{R}^K for all $K \subset I, K \neq \emptyset$. As $W(K)$ is a cylindric set in \mathbb{R}^I over $\mathcal{P}(W(K), \mathbb{R}^K)$, it is closed, as well. For $K = \emptyset$ the statement follows from Lemma 2. \square

Lemma 4. If $x \in W(K), y \in \mathbb{R}^I$ and $x_i \geq y_i$ for all $i \in K$ then $y \in W(K)$.

Proof. The statement follows immediately from (1.1). \square

It is obvious that in real and interesting situations there usually do not exist decisions maximizing all utility functions simultaneously. It is necessary then to find certain form of compromise between the demands of criteria and their groups, and to describe the boundaries of such compromises. Investigation of the sets of acceptable compromise decisions and their consequences is the main purpose of this paper.

Let us suppose that for any $K \subset I$ there is given a set $U(K) \subset \mathbb{R}^I$ such that

$$(1.2) \quad U(K) \text{ is closed,}$$

$$(1.3) \quad U(K) \subset W(K),$$

$$(1.4) \quad \text{if } x \in U(K), y \in \mathbb{R}^I, x_i \geq y_i \text{ for all } i \in K, \text{ then } y \in U(K),$$

$$(1.5) \quad U(K) \neq \emptyset,$$

$$(1.6) \quad U(I) = W(I), \quad U(\emptyset) = W(\emptyset) = \mathbb{R}^I.$$

The sets $U(K)$ help us to represent the unacceptability of decisions and their consequences. The boundary of $U(K)$ is also the boundary between the acceptable and unacceptable utility vectors for the set of criteria $K \subset I$. As the criteria are considered to be generally dependent, the sets $U(K)$ are defined not only for single criteria but also for all their groups. If in some actual applications some of criteria are not dependent, then their independency can be expressed by choosing sufficiently small sets $U(K)$ for their groups.

The difference between the set $W(K)$ and the interior part of $U(K)$ represents the space for finding a compromise acceptable for the group of criteria K , the interior part of $U(K)$ contains unacceptable utility vectors.

Before introducing the notion of commonly acceptable utility vectors, it is useful to mention the domination relation. If $x, y \in \mathbb{R}^I$ and $K \subset I, K \neq \emptyset$, then we say that x dominates y via K and write $x \text{ dom}_K y$ iff

$$(1.7) \quad x_i \geq y_i \text{ for all } i \in K, \quad x_j > y_j \text{ for some } j \in K.$$

Some utility vector from \mathbb{R}^I can be accepted iff it is achievable by some decision from D , and if it cannot be dominated by any unacceptable utility vector via any group of criteria. Exactly formulated, we say that a utility vector $x \in \mathbb{R}^I$ is acceptable iff

$$(1.8) \quad \exists d \in D, \quad \forall i \in I, \quad x_i \leq u_i(d),$$

$$(1.9) \quad \forall K \subset I, \quad \forall y \in U(K), \quad \text{non } (y \text{ dom}_K x).$$

If it will be useful for explanation, we shall use the terms of *acceptable decision* $d \in D$ or *acceptable consequence* $c \in C$ iff the utilities $u_i(c)$, i.e. $u_i(g(d))$, $i \in I$, form an acceptable utility vector.

The main goal of this paper is to study the *multicriterial decision-making problem* described by the triplet

$$(1.10) \quad (I, W, U), \text{ where } W = (W(K))_{K \in I}, \quad U = (U(K))_{K \in I},$$

described above. Some methods and results derived in the general coalition-games theory will be applied for this purpose.

2. GENERAL COALITION-GAMES

The general coalition-games concept was introduced in [2] and investigated in some other papers including [3] and [4]. Its application to the multicriterial decision-making is presented in this paper. For this purpose, it is useful to recall here some notions and results concerning the general coalition-games and related to the topic of this paper.

The *general coalition-game* is a pair (I, V) where I is a non-empty and finite set of *players*, and V is a class of subsets of \mathbb{R}^I , $V = (V(K))_{K \in I}$. We suppose that for any set of players $K \subset I$

$$(2.1) \quad V(K) \text{ is closed,}$$

$$(2.2) \quad \text{if } x \in V(K), \quad y \in \mathbb{R}^I, \quad x_i \geq y_i \text{ for all } i \in K, \text{ then } y \in V(K),$$

$$(2.3) \quad V(K) \neq \emptyset,$$

$$(2.4) \quad V(K) = \mathbb{R}^I \text{ iff } K = \emptyset.$$

The subsets of I are called *coalitions*, every partition of I into disjoint coalitions is called a *coalition structure*. The class V is called a *general characteristic function* and vectors from \mathbb{R}^I are called *imputations*. For every $K \subset I$ we denote

$$(2.5) \quad \begin{aligned} V^*(K) &= \{y \in \mathbb{R}^I: \text{there exists no } x \in V(K) \text{ such that } x \text{ dom}_K y\} = \\ &= \{y \in \mathbb{R}^I: \text{for all } x \in V(K) \text{ either there exists } i \in K \\ &\quad \text{such that } x_i < y_i \text{ or } x_i = y_i \text{ for all } i \in K\}. \end{aligned}$$

If $\mathcal{M} \subset 2^I$ is a non-empty class of coalitions then we denote

$$V(\mathcal{M}) = \bigcap_{K \in \mathcal{M}} V(K), \quad V^*(\mathcal{M}) = \bigcap_{K \in \mathcal{M}} V^*(K).$$

For any $K \subset I$ we denote by $P(K)$ the set of imputations called the *Pareto optimum* and defined by

$$(2.6) \quad P(K) = V(K) \cap V^*(K).$$

Remark 2. It is obvious from (2.4) and (2.5) that for the empty coalition $V(K) = V^*(K) = P(K) = \mathbb{R}^I$ holds.

Theorem 1. If $K \subset I$, $K \neq \emptyset$, then the Pareto optimum $P(K)$ is empty if and only if for every imputation $x \in V(K)$ there exists $i \in K$ such that every $y \in \mathbb{R}^I$ such that $y_i > x_i$ and $y_j = x_j$ for all $j \in K$, $j \neq i$, is also in $V(K)$. In symbols $P(K) = \emptyset$ iff

$$(2.7) \quad \forall x \in V(K), \exists i \in K, \forall y \in \mathbb{R}^I ((\forall j \in K, j \neq i, y_j = x_j \text{ and } y_i > x_i) \Rightarrow y \in V(K)).$$

Proof. If condition (2.7) is fulfilled then for any $x \in V(K)$ there exists $y \in V(K)$ such that $y \text{ dom}_K x$ (cf. (1.7)), and consequently $V(K) \cap V^*(K) = \emptyset$. If $P(K) = \emptyset$ then for every $x \in V(K)$ there exists $z \in V(K)$ such that $z \text{ dom}_K x$. This means that $z_j \geq x_j$ for all $j \in K$ and $z_i > x_i$ for some $i \in K$. As the same relation is true for z , (2.2) holds and I is finite, the statement of the theorem and relation (2.7) is fulfilled. \square

Theorem 2. If $\{i\}$ is a one-player coalition then there exists a real number $a \in \mathbb{R}$ such that

$$P(\{i\}) = \{x \in \mathbb{R}^I : x_i = a\} \neq \emptyset.$$

Proof. Assumptions (2.1), (2.2), (2.3), (2.4) and (2.5) imply that there exists $a \in \mathbb{R}$ such that

$$V(\{i\}) = \{x \in \mathbb{R}^I : x_i \leq a\}, \quad V^*(\{i\}) = \{x \in \mathbb{R}^I : x_i \geq a\}. \quad \square$$

Theorem 3. If a coalition K contains at least two players then the Pareto optimum is empty if there exists $L \subset K$, $L \neq \emptyset$, and $a = (a_i)_{i \in L} \in \mathbb{R}^L$ such that

$$(2.8) \quad V(K) = \bigcup_{i \in L} \{x \in \mathbb{R}^I : x_i \leq a_i\}$$

Proof. If (2.8) holds then for every $x \in V(K)$ there exists $j \in L$ such that $x_j \leq a_j$. Let us choose $z \in \mathbb{R}^I$ such that $z_j = x_j$, $z_i < x_i$ for $i \in K$, $i \neq j$. Then (2.8) implies that $z \in V(K)$ and $z \text{ dom}_K x$. Hence, $x \notin V^*(K)$, and $P(K) = \emptyset$. \square

The solution of a general coalition-game is represented by the concept of the strong stability. We say that an imputation $x \in \mathbb{R}^I$ is *strongly stable* in a general coalition-game (I, V) iff

$$(2.9) \quad x \in V^*(K) \text{ for all } K \subset I,$$

$$(2.10) \quad x \in V(\mathcal{K}) = \bigcap_{K \in \mathcal{K}} V(K) \text{ for some coalition structure } \mathcal{K}.$$

A coalition structure \mathcal{K} is strongly stable iff there exists a strongly stable imputation $x \in V(\mathcal{K})$.

In the following sections we shall frequently consider some special types of general colition-games. They were investigated in [3] and here we briefly recall their definitions.

We say that a general coalition-game (I, V) is *superadditive* iff for any $K, L \subset I$, $K \cap L = \emptyset$,

$$V(K \cup L) \supset V(K) \cup V(L),$$

we say that it is *subadditive* iff for any $K, L \subset I$, $K \cap L = \emptyset$,

$$V^*(K \cup L) \supset V^*(K) \cup V^*(L),$$

and we say that it is *additive* iff it is superadditive and subadditive.

Let us consider a one-to-one transformation T of \mathbb{R}^I onto \mathbb{R}^I . If $S \subset \mathbb{R}^I$ then we denote

$$TS = \{x \in \mathbb{R}^I : \exists y \in S, x = Ty\}.$$

The transformation T is called a game-preserving one iff for any general coalition-game (I, V) the pair (I, TV) is also a general coalition-game. It was shown in [4] that an imputation $\bar{x} \in \mathbb{R}^I$ is strongly stable in (I, V) iff $T\bar{x}$ is strongly stable in (I, TV) .

3. MULTICRITERIAL DECISION-MAKING PROBLEM AND GENERAL COALITION-GAMES

In this section we show a correspondence between the multicriterial decision-making problem presented in Section 1 and general coalition-games.

Let us consider a multicriterial decision-making problem (I, W, U) consisting of the set of criteria I and of classes of sets of utility vectors $(W(K))_{K \subset I}$ and $(U(K))_{K \subset I}$ possessing properties introduced in Section 1.

Theorem 4. If (I, W, U) is a multicriterial decision-making problem then the pair (I, W) forms a general coalition-game.

Proof. The statement follows immediately from Lemma 1, Lemma 2, Lemma 3 and Lemma 4 and from the definition of a general coalition-game. \square

Theorem 5. If (I, W, U) is a multicriterial decision-making problem then the pair (I, U) forms a general coalition-game.

Proof. The statement follows immediately from the assumptions about the sets $U(K)$ introduced in Section 1. \square

Theorem 6. The general coalition-game (I, W) is subadditive.

Proof. Let us consider $K, L \subset I$, $K \cap L = \emptyset$, and the corresponding sets $W(K)$, $W(L)$, $W(K \cup L)$ given by (1.1). If $x \in W(K \cup L)$ then there exists $c \in C$ such that $x_i \leq u_i(c)$ for all $i \in K \cup L$. Hence $x_i \leq u_i(c)$ for all $i \in K$ and all $i \in L$, and $W(K \cup L) \subset W(K) \cup W(L)$. It was shown in [3], that this inclusion implies the subadditivity of the game (I, W) . \square

As the pairs (I, W) and (I, U) are general coalition-games, it is possible to apply (2.5) and to define the sets $W^*(K)$ and $U^*(K)$ for any $K \subset I$. The Pareto optima for both games will be denoted by

$$(3.1) \quad P_W(K) = W(K) \cap W^*(K), \quad P_U(K) = U(K) \cap U^*(K)$$

for all $K \subset I$.

Lemma 5. If $x \in P_W(I)$ then there exists: $d \in D$ and $c \in C$ such that $x_i = u_i(c) = u_i(d)$ for all $i \in I$.

Proof. Let us suppose that $x \in W(I)$ and there is no $c \in C$ such that $x_i = u_i(c)$ for all $i \in I$. Then (1.1) implies that there exists $c' \in C$ such that $x_i \leq u_i(c')$ for all $i \in I$ and $x_i < u_i(c')$ for some $i \in I$. If we denote $x' \in \mathbb{R}^I$ where $x'_i = u_i(c')$ for all $i \in I$ then $x' \in W(I)$ and $x' \text{ dom}_I x$. It means that $x \notin W^*(I) \supset P_W(I)$. \square

Lemma 6. The Pareto optimum $P_W(K)$ is non-empty for any $K \subset I$.

Proof. If $K = \emptyset$ then obviously $P_W(K) = W(K) = W^*(K) = \mathbb{R}^I$. If $K \neq \emptyset$ then the set

$$(3.2) \quad \{x \in \mathbb{R}^I : \exists c \in C, \forall i \in K, x_i = u_i(c)\}$$

is a compact subset of \mathbb{R}^I as C is compact and u_i are continuous functions. It means that the set (3.2) is bounded. Formula (1.1) implies that for any $y \in W(K)$ there exists x in (3.2) such that $x_i \geq y_i$ for all $i \in K$. Theorems 1, 2 and 3 imply that $P_W(K) \neq \emptyset$. \square

Lemma 7. The Pareto optimum $P_U(K)$ is non-empty for any $K \subset I$.

Proof. The statement follows from Lemma 6 and from (1.3) and (1.2) in connection with Theorem 1. \square

4. ACCEPTABLE UTILITY VECTORS AND STRONGLY STABLE IMPUTATIONS

In this section we shall investigate some mutual relations between the solutions of the vector optimization problem and the corresponding general coalition-games.

It means that we shall study some common properties of the acceptable utility vectors defined by (1.8) and (1.9), and the strongly stable imputations defined by (2.8) and (2.9). First, we shall introduce an equivalent formulation of the acceptability of the utility vectors and its immediate consequence.

Remark 3. A utility vector $x \in \mathbb{R}^I$ is acceptable in the multicriterial decision-making problem (I, W, U) iff

$$(4.1) \quad x \in W(I) \cap \left(\bigcap_{K \subset I} U^*(K) \right).$$

Remark 4. Relations (4.1) and (1.6) immediately imply that any acceptable utility vector belongs to the Pareto optimum

$$P_v(I) = P_w(I).$$

Mutual relations between both types of solutions are described by the following statements.

Theorem 7. If $x \in \mathbb{R}^I$ is an acceptable utility vector then there exist $d \in D$ and $c \in C$ such that for all $i \in I$ $u_i(c) = u_i(\varrho(d)) = x_i$.

Proof. The statement follows immediately from Lemma 5 and Remark 4.

Theorem 8. A real-valued vector $x \in \mathbb{R}^I$ is an acceptable utility vector in a multicriterial decision-making problem (I, W, U) if and only if it is a strongly stable imputation in the general coalition-game (I, U) and $x \in U(I)$.

Proof. The statement follows from Remark 3 and from (1.6) immediately. \square

Theorem 9. There exists an acceptable utility vector in the multicriterial decision-making problem (I, W, U) if and only if the coalition structure I containing exactly the coalition I is strongly stable in the general coalition-game (I, U) .

Proof. The statement follows immediately from Theorem 8. \square

The following statements concern the existence of acceptable utility vectors in some special cases of the multicriterial decision-making problem and the related general coalition-games.

Theorem 10. If the general coalition-game (I, U) is additive then there exists exactly one acceptable utility vector y in the multicriterial decision-making problem (I, W, U) , and

$$(4.2) \quad y = (y_i)_{i \in I}, \quad y_i = \max \{x_i : x \in U(\{i\})\}, \quad i \in I.$$

Proof. As the game (I, U) is subadditive then $y \in \mathbb{R}^I$ fulfilling (4.2) belongs to $U^*(K)$ for all $K \subset I$. Moreover, $y \in U(K)$ for any coalition K as follows from the superadditivity of the game (I, U) . It means that y is strongly stable in (I, U) , and by Theorem 8 y is an acceptable utility vector in (I, W, U) . Let us suppose, now, that there exists another $z \in \mathbb{R}^I$, $z \neq y$, being an acceptable utility vector in (I, W, U) . It means that z is a strongly stable imputation in the game (I, U) , hence $z \in U^*(\{i\})$ for all $i \in I$. It means $z_i \geq y_i$ for all $i \in I$. If $z \neq y$ then there exists $j \in I$ such that $z_j > y_j$. Then $z \text{ dom}_I y$, and consequently $z \notin U(I)$, as follows from the relation $y \in U^*(I)$ shown above. But this fact contradicts to the assumption that z is an acceptable utility vector and to Theorem 8. \square

Theorem 11. If the game (I, U) is subadditive then a utility vector $x \in \mathbb{R}^I$ is accept-

able if and only if

$$x \in P_v(I) \cap \left(\bigcap_{i \in I} U^*(\{i\}) \right).$$

Proof. The statement follows from Remark 3 and Remark 4 and from the sub-additivity assumption implying for all $K \subset I$

$$U^*(K) \subset \bigcap_{i \in K} U^*(\{i\}). \quad \square$$

Theorem 12. Let us suppose that the set I contains exactly two criteria. Then there exists an acceptable utility vector if and only if the game (I, U) is superadditive.

Proof. Let $I = \{i, j\}$, and let (I, U) be superadditive. Then

$$(4.3) \quad U(I) = W(I) \supset U(\{i\}) \cap U(\{j\}).$$

As

$$\begin{aligned} U(\{i\}) &= \{x = (x_i, x_j) : x_i \leq a_i\}, & U^*(\{i\}) &= \{x = (x_i, x_j) : x_i \geq a_i\}, \\ U(\{j\}) &= \{x = (x_i, x_j) : x_j \leq a_j\}, & U^*(\{j\}) &= \{x = (x_i, x_j) : x_j \geq a_j\}, \end{aligned}$$

where a_i, a_j are real numbers, and as (4.3) holds, then there exist $d \in D$ and $y \in P_v(I)$ such that

$$y_i = u_i(q(d)) \geq a_i, \quad y_j = u_j(q(d)) \geq a_j.$$

Then $y \in U^*(\{i\}) \cap U^*(\{j\})$, and $y \in P_v(I)$. It means that y is an acceptable utility vector. On the other hand, let (I, U) be not superadditive. Then there exists $z \in U(\{i\}) \cap U(\{j\})$ such that $z \notin U(I) = W(I)$, $z = (z_i, z_j)$. It means that $z_i \leq a_i$, $z_j \leq a_j$, and by (2.2) none utility vector $x \in U^*(\{i\}) \cap U^*(\{j\})$ can belong to $W(I)$. Consequently, there is no acceptable utility vector. \square

There exists an important and often investigated type of coalition-games called coalition-games with side-payments. A coalition-game with side-payments can be described as a general coalition-game (I, V) such that for every coalition $K \subset I$ there exists a real number v_K such that

$$(4.4) \quad V(K) = \{x \in \mathbb{R}^I : \sum_{i \in K} x_i \leq v_K\}.$$

It is usually supposed that for $K, L \subset I$, $K \cap L = \emptyset$, the inequality $v_{K \cup L} \geq v_K + v_L$ holds. The set

$$C = \{x \in \mathbb{R}^I : \sum_{i \in I} x_i = v_I \text{ and } \forall K \subset I, \sum_{i \in K} x_i \geq v_K\}$$

is called a core of the game (I, V) . It is not difficult to see (cf. [2]) that C is identical with the set of all strongly stable imputations of the general coalition-game (I, V) .

Theorem 8 implies that the existing results on core of the coalition-games with side-payments can be used for the investigation of the multicriterial decision-making problem in which the sets $U(K)$ can be described by relations analogous to (4.4).

Moreover, it is possible to apply the results on transformations of general coali-

tion-games and to transform certain types of the sets $U(K)$ in multicriterial decision-making problems into some sets of the (4.4) type. It concerns namely the situations in which for every $i \in I$ there exists an increasing and continuous function $f_i: \mathbb{R} \rightarrow \mathbb{R}$ such that for all $K \subset I$

$$(4.5) \quad U(K) = \{x \in \mathbb{R}^I : \sum_{i \in K} f_i(x_i) \leq v_K\}$$

for some real constants v_K .

5. SUBOPTIMAL SOLUTION

It is obvious that there exist multicriterial decision-making problems (I, W, U) in which no acceptable utility vector can be found. If some rational decision is desirable even in those cases then the original decision-making problem must be modified in some proper way. Some of these modifications are briefly mentioned in this section.

It is, for example, possible to reduce the sets $U(K)$ of unacceptable utility vectors. In such case the set of acceptable utility vectors generally increases.

Theorem 13. Let (I, W, U_0) and (I, W, U_1) be two multicriterial decision-making problems such that for any $K \subset I$, $U_0(K) \subset U_1(K)$. If $x \in \mathbb{R}^I$ is a utility vector acceptable in (I, W, U_1) then it is acceptable in (I, W, U_0) as well.

Proof. It follows from (2.5) and (2.2) that for any $K \subset I$

$$U_0^*(K) \cup U_0(K) = \mathbb{R}^I = U_1^*(K) \cup U_1(K).$$

It means that for any $K \subset I$ the inclusion $U_0(K) \subset U_1(K)$ implies $U_0^*(K) \supset U_1^*(K)$. As $U_0(I) = U_1(I) = W(I)$ by (1.6), the statement follows from Remark 3. \square

Another possibility is to omit some criteria. It is usually too strong to ignore some criterion completely, but it is possible to consider it only in combination with some other criteria. In our model, it is possible to consider a subclass \mathcal{M} of the class 2^I of the groups of criteria. This class \mathcal{M} needs not contain all one-element groups $\{i\}$, $i \in I$, nor some larger groups $K \subset I$. The preliminary idea given above can be formulated in the following way.

Let $\mathcal{M} \subset 2^I$ be a class of subsets of I such that

$$(5.1) \quad \mathcal{M} \neq \emptyset \quad \text{and} \quad \bigcup_{K \in \mathcal{M}} K = I,$$

and let $x \in \mathbb{R}^I$ be a utility vector. Then we say that x is \mathcal{M} -acceptable in the decision-making problem (I, W, U) iff

$$(5.2) \quad \exists d \in D, \quad \forall i \in I, \quad x_i \leq u_i(q(d)),$$

$$(5.3) \quad \forall K \in \mathcal{M}, \quad \forall y \in U(K), \quad \text{non}(y \text{ dom}_K x).$$

Remark 5. It can be easily seen that a utility vector x is \mathcal{M} -acceptable iff there exists $d \in D$ such that $x_i = u_i(\varphi(d))$ and $x \in U^*(K)$ for all $K \in \mathcal{M}$, i.e. iff

$$x \in W(I) \cap \left(\bigcap_{K \in \mathcal{M}} U^*(K) \right)$$

(cf. Theorem 7 and Remark 3).

Remark 6. If $I \in \mathcal{M} \subset 2^I$ then every \mathcal{M} -acceptable utility vector belongs to the Pareto optima $P_W(I)$ and $P_U(I)$.

The introduction of the \mathcal{M} -acceptability of utility vectors is motivated by the intention to substitute the acceptability of utility vectors given in Section 1 by its weaker analogy. The modification mostly prefers the demands of larger groups of criteria to the demands of the single ones. The existence of the \mathcal{M} -acceptable utility vectors and the extent of their set obviously depend on the form of the class \mathcal{M} . The following results concern that dependence.

Remark 7. Let $\mathcal{M} \subset 2^I$ and $\mathcal{N} \subset 2^I$ fulfil (5.1) and let $\mathcal{M} \subset \mathcal{N}$. If $x \in \mathbb{R}^I$ is an \mathcal{N} -acceptable utility vector then it is also \mathcal{M} -acceptable, as follows from (5.2) and (5.3).

Remark 8. If $\mathcal{M} = 2^I$ and $x \in \mathbb{R}^I$ is a utility vector then x is \mathcal{M} -acceptable iff it is acceptable.

Remark 9. If the class \mathcal{M} contains exactly the set I of all criteria and if $x \in \mathbb{R}^I$ is a utility vector then x is \mathcal{M} -acceptable iff x belongs to the Pareto optima $P_U(I) = P_W(I)$ (cf. Remarks 4 and 6). These Pareto optima are always non-empty as follows from Lemma 6.

Theorem 14. Let us consider groups of criteria $K, L \subset I, K \cap L = \emptyset$, such that

$$(5.4) \quad U^*(K \cup L) \supset U^*(K) \cap U^*(L),$$

let $\mathcal{M} \subset 2^I$ and $\mathcal{N} \subset 2^I$ where

$$(5.5) \quad K, L, K \cup L \in \mathcal{M} \quad \text{and} \quad \mathcal{N} = \mathcal{M} - \{K \cup L\},$$

and let $x \in \mathbb{R}^I$ be a utility vector. Then x is \mathcal{M} -acceptable if and only if it is \mathcal{N} -acceptable.

Proof. If (5.4) holds then (5.5) implies

$$(5.6) \quad W(I) \cap \left(\bigcap_{M \in \mathcal{N}} U^*(M) \right) = W(I) \cap \left(\bigcap_{M \in \mathcal{M}} U^*(M) \right).$$

The statement of this theorem follows immediately from (5.6) and Remark 5. \square

Corollary. If $K, L \subset I, K \cap L = \emptyset$, if (5.4) holds, and if $\mathcal{N} = 2^I - \{K \cup L\}$ then a utility vector x is \mathcal{N} -acceptable iff it is acceptable.

Theorem 15. If (I, W, U) is a multicriterial decision-making problem, if the general coalition-game (I, U) is subadditive, and if we denote by $\mathcal{M} = \{\{i\}\}_{i \in I}$ the class of all one-element subsets of I , then a utility vector $x \in \mathbb{R}^I$ is \mathcal{M} -acceptable if and only if it is acceptable.

Proof. The statement follows from Theorem 14. If $K \subset I$, $K \neq \emptyset$, and if the game (I, U) is subadditive then

$$U^*(K) \supset \bigcap_{i \in K} U^*(\{i\}).$$

Consequently, for

$$\mathcal{M} = \{\{i\}\}_{i \in I} \quad \text{and} \quad \mathcal{N} = \mathcal{M} \cup \{K\},$$

a utility vector $x \in \mathbb{R}$ is \mathcal{M} -acceptable iff it is \mathcal{N} -acceptable. Repeating the procedure for all $K \subset I$, it can be shown that any $x \in \mathbb{R}^I$ is \mathcal{M} -acceptable iff it is 2^I -acceptable, and Remark 8 completes the proof. \square

It is obvious that in case of subadditivity of the game (I, U) no real reduction of demands of criteria was done, and the game (I, U) does not differ from (I, W) .

6. REDUNDANT CRITERIA

In the previous section, we have investigated the situation in which some groups of criteria are omitted in order to increase the possibility of finding some acceptable utility vectors at least in the weaker sense.

Here we shall consider a rather opposite problem. There often exist some groups of criteria which do not influence the acceptability of utility vectors, or which influence it in a minimal degree. Then it is useful to eliminate those groups of criteria in order to simplify the considered multicriterial decision-making problem and the necessary manipulation with its elements.

In this section, we introduce a few results concerning the groups of criteria not influencing the acceptability of utility vectors at all. It can be easily seen that the reduction of the redundant sets of criteria is a reduction of a class $\mathcal{M} \subset 2^I$ (including $\mathcal{M} = 2^I$) to some subclass $\mathcal{N} \subset \mathcal{M}$. In fact, the first result of this type is presented in Theorem 14. Other theorems concerning the situation in which the class $\mathcal{M} \subset 2^I$ can be reduced follow.

Lemma 8. Let us consider $K \subset I$, $K \neq \emptyset$, $\mathcal{M} \subset 2^I$, $\mathcal{N} \subset 2^I$, $K \in \mathcal{N}$, $\mathcal{M} = \mathcal{N} - \{K\}$. If $x \in \mathbb{R}^I$ is a utility vector such that $x \in W(K) - U(K)$ then x is \mathcal{M} -acceptable iff it is \mathcal{N} -acceptable.

Proof. By Remark 7, any \mathcal{N} -acceptable $x \in \mathbb{R}^I$ is also \mathcal{M} -acceptable. If x is \mathcal{M} -acceptable and if $x \in W(K) - U(K)$ then $x \in U^*(K)$, and x is \mathcal{N} -acceptable. \square

Theorem 16. Let us consider a multicriterial decision-making problem (I, W, U) . Let $K \in 2^I$, $K \neq \emptyset$, $\mathcal{M} \subset 2^I$, $\mathcal{N} \subset 2^I$, $K \in \mathcal{N}$, $\mathcal{M} = \mathcal{N} - \{K\}$. If for any \mathcal{N} -accept-

able utility vector y there holds $y \in W(K) - U(K)$ then the sets of \mathcal{M} -acceptable and \mathcal{N} -acceptable utility vectors are identical.

Proof. The statement follows immediately from Lemma 8. \square

Theorem 17. Let us consider a multicriterial decision-making problem (I, W, U) . Let $K, L \subset I, K \subset L, \mathcal{M} \subset 2^I, \mathcal{N} \subset 2^I, K, L \in \mathcal{N}, \mathcal{M} = \mathcal{N} - \{K\}$. If $U(K) \subset U(L)$ then the sets of \mathcal{M} -acceptable and \mathcal{N} -acceptable utility vectors are identical.

Proof. If $x \in R^I$ is \mathcal{N} -acceptable then it is also \mathcal{M} -acceptable as follows from Remark 7. Let us consider an \mathcal{M} -acceptable utility vector $x \in R^I$. It means that $x \in U^*(L)$. If $U(L) \supset U(K)$ then also $U^*(L) \subset U^*(K)$ as follows from (2.5). This means that $x \in U^*(K)$. \square

The last theorem of this section concerns a rather degenerated case of the multicriterial decision-making problem.

Theorem 18. Let us consider a multicriterial decision-making problem (I, W, U) , and let the Pareto optimum $P_W(I)$ be a one-element set. Then there exist $d \in D$ and $c \in C, c = \varrho(d)$, such that the utility vector $(u_i(c))_{i \in I}$ is the exactly one acceptable utility vector in (I, W, U) , and the vectors $(u_i(c))_{i \in K}$ are single acceptable utility vectors in multicriterial decision-making problems (K, W_0, U_0) , where $K \subset I, K \neq \emptyset$, and for any $L \subset K$ is $W_0(L) = W(L), U_0(L) = U(L)$.

Proof. If

$$(6.1) \quad P_W(I) = \{x\}, \quad x = (x_i)_{i \in I} \in R^I$$

then by Lemma 5 there exist $d \in D$ and $c \in C$ such that

$$c = \varrho(d), x_i = u_i(c) \quad \text{for all } i \in I.$$

Moreover, (6.1) and the relation $P_W(I) = W(I) \cap W^*(I)$ imply that if $y \in W(I)$ then

$$(6.2) \quad y_i \leq x_i \quad \text{for all } i \in I.$$

The same is true for all utility vectors $y_K = (y_i)_{i \in K}$ with regard to $x_K = (x_i)_{i \in I}$. As for any $y_K \neq x_K, K \subset I, K \neq \emptyset$, then at least one of the inequalities in (6.2) is strict. It means that

$$P_W(K) = \{z \in R^I : z_i = x_i \text{ for all } i \in K\},$$

and for the reduced multicriterial decision-making problems (K, W_0, U_0) described in the statement of this theorem

$$P_{W_0}(K) = \{x_K\}.$$

It means that $x \in W^*(K)$ for all $K \subset I$, and $x \in U^*(K)$ for all $K \subset I$, as $U(K) \subset W(K)$. Consequently, x is acceptable in (I, W, U) and in (K, W_0, U_0) of the described type, and there does not exist any $y_K \in R^I$ acceptable in (K, W_0, U_0) for any $K \subset I, K \neq \emptyset$. \square

7. NUMERICAL EXAMPLES

The theoretical concepts presented in the previous sections are illustrated here by simple examples of the multicriterial decision-making problems.

Example 1. Let us suppose that the set \mathbf{D} of decisions is the set of all probability distributions over four pure decisions $\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}$, i.e.

$$\mathbf{D} = \{d = (d_1, d_2, d_3, d_4) : 0 \leq d_i, i = 1, \dots, 4, \sum_{i=1}^4 d_i = 1\}.$$

For every decision $d \in \mathbf{D}$, the probability of acception of the pure decision $\delta^{(i)}$ is $d_i, i = 1, \dots, 4$. The set of consequences is naturally the set of all probability distributions over four pure consequences $c^{(1)}, c^{(2)}, c^{(3)}, c^{(4)}$, where $c^{(i)} = \varrho(\delta^{(i)})$. It means that

$$\mathbf{C} = \{c = (c_1, c_2, c_3, c_4) : 0 \leq c_i, i = 1, \dots, 4, \sum_{i=1}^4 c_i = 1\},$$

and for every $c \in \mathbf{C}$ such that $c = \varrho(d)$ is $c_i = d_i, i = 1, \dots, 4$. The consequences and the corresponding decisions are ordered by means of three criteria, $I = \{1, 2, 3\}$, represented by the utility functions u_1, u_2, u_3 . Let

$$\begin{aligned} u_1(c^{(1)}) &= 6, & u_2(c^{(1)}) &= 2, & u_3(c^{(1)}) &= 4, \\ u_1(c^{(2)}) &= 2, & u_2(c^{(2)}) &= 6, & u_3(c^{(2)}) &= 4, \\ u_1(c^{(3)}) &= 2, & u_2(c^{(3)}) &= 2, & u_3(c^{(3)}) &= 8, \\ u_1(c^{(4)}) &= 3, & u_2(c^{(4)}) &= 3, & u_3(c^{(4)}) &= 5, \end{aligned}$$

and, of course,

$$u_i(c) = \sum_{j=1}^4 c_j u_i(c^{(j)}) = \sum_{j=1}^4 d_j u_i(\varrho(d^{(j)})),$$

for $i \in I$.

Then $\mathbb{R}^I = \mathbb{R}^3$ is the space of the utility vectors $x = (x_1, x_2, x_3)$, and obviously

$$\begin{aligned} \mathcal{W}(I) &= \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 \leq 12, x_1 + x_2 \leq 8, \\ &\quad x_2 + x_3 \leq 10, x_1 + x_3 \leq 10, x_1 \leq 6, \\ &\quad x_2 \leq x_3 \leq 8\}, \end{aligned}$$

$$\mathcal{W}(\{1, 2\}) = \{x \in \mathbb{R}^3 : x_1 + x_4 \leq 8, x_1 \leq 6, x_2 \leq 6\},$$

$$\mathcal{W}(\{1, 3\}) = \{x \in \mathbb{R}^3 : x_2 + x_3 \leq 10, x_2 \leq 6, x_3 \leq 8\},$$

$$\mathcal{W}(\{1, 3\}) = \{x \in \mathbb{R}^3 : x_1 + x_3 \leq 10, x_1 \leq 6, x_3 \leq 8\},$$

$$\mathcal{W}(\{1\}) = \{x \in \mathbb{R}^3 : x_1 \leq 6\}, \quad \mathcal{W}(\{2\}) = \{x \in \mathbb{R}^3 : x_2 \leq 6\},$$

$$\mathcal{W}(\{3\}) = \{x \in \mathbb{R}^3 : x_3 \leq 8\}.$$

The set of the achievable utility vectors $\mathcal{W}(I)$ is presented in Figure 1. It can be easily

seen that the utility vectors

$$x^{(1)} = (6, 2, 4), \quad x^{(2)} = (2, 6, 4), \quad x^{(3)} = (2, 2, 8), \quad x^{(4)} = (3, 3, 5)$$

representing the utilities of $c^{(1)}, c^{(2)}, c^{(3)}, c^{(4)}$, respectively, fully determine the mentioned sets. The utility vector $x^{(4)}$ is dominated, e.g. by the utility vector $x = (3, 4,$

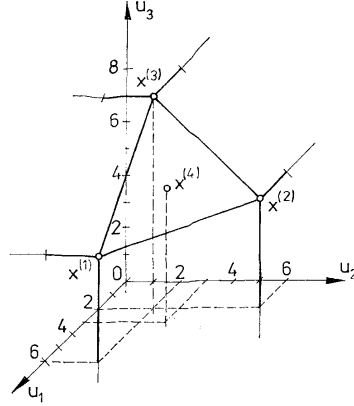


Fig. 1. The set $W(I)$ of the multicriterial decision-making problem in Examples 1, 2 and 3.

$3, 4, 5, 2) \in W(I)$ and it cannot belong to $W^*(K)$ for any $K \subset I, K \neq \emptyset$. It is not difficult to see that

$$\begin{aligned} W^*(I) &= (\mathbb{R}^3 - W(I)) \cup \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 12\} = \\ &= \{x \in \mathbb{R}^3 : (x_1 + x_2 + x_3 \geq 12) \text{ or } (x_1 + x_2 > 8) \text{ or } \\ &\quad (x_2 + x_3 > 10) \text{ or } (x_1 + x_3 > 10) \text{ or } (x_1 > 6) \text{ or } \\ &\quad (x_2 > 6) \text{ or } (x_3 > 8)\}, \end{aligned}$$

and analogously

$$\begin{aligned} W^*({1, 2}) &= (\mathbb{R}^3 - W({1, 2})) \cup \{x \in \mathbb{R}^3 : x_1 + x_2 = 8\} = \\ &= \{x \in \mathbb{R}^3 : (x_1 + x_2 \geq 8) \text{ or } (x_1 > 6) \text{ or } (x_2 > 6)\}, \\ W^*({2, 3}) &= (\mathbb{R}^3 - W({2, 3})) \cup \{x \in \mathbb{R}^3 : x_2 + x_3 = 10\} = \\ &= \{x \in \mathbb{R}^3 : (x_2 + x_3 \geq 10) \text{ or } (x_2 > 6) \text{ or } (x_3 > 8)\}, \\ W^*({1, 3}) &= (\mathbb{R}^3 - W({1, 3})) \cup \{x \in \mathbb{R}^3 : x_1 + x_3 = 10\} = \\ &= \{x \in \mathbb{R}^3 : (x_1 + x_3 \geq 10) \text{ or } (x_1 > 6) \text{ or } (x_3 > 8)\}, \\ W^*({1}) &= \{x \in \mathbb{R}^3 : x_1 \geq 6\}, \quad W^*({2}) = \{x \in \mathbb{R}^3 : x_2 \geq 6\}, \\ W^*({3}) &= \{x \in \mathbb{R}^3 : x_3 \geq 8\}. \end{aligned}$$

It means that

$$\begin{aligned} P_W(I) &= W(I) \cap W^*(I) = \\ &= \{x \in \mathbb{R}^3 : x_1 = 2 + 4\lambda, x_2 = 2 + 4\mu, x_3 = 8 - 4\lambda - 4\mu, \\ &\quad 0 \leq \lambda, 0 \leq \mu, \lambda + \mu \leq 1\}, \end{aligned}$$

and

$$\begin{aligned} P_W(\{1, 2\}) &= \{x \in \mathbb{R}^3 : x_1 = 2 + 4\lambda, x_2 = 6 - 4\lambda, 0 \leq \lambda \leq 1\}, \\ P_W(\{2, 3\}) &= \{x \in \mathbb{R}^3 : x_2 = 6 - 4\lambda, x_3 = 2 + 6\lambda, 0 \leq \lambda \leq 1\}, \\ P_W(\{1, 3\}) &= \{x \in \mathbb{R}^3 : x_1 = 6 - 4\lambda, x_3 = 4 + 4\lambda, 0 \leq \lambda \leq 1\}, \\ P_W(\{1\}) &= \{x \in \mathbb{R}^3 : x_1 = 6\}, \quad P_W(\{2\}) = \{x \in \mathbb{R}^3 : x_2 = 6\}, \\ P_W(\{3\}) &= \{x \in \mathbb{R}^3 : x_3 = 8\}. \end{aligned}$$

The game (I, W) is described, now, and it is desirable to introduce the sets $U(K)$, $K \subset I$, and the game (I, U) as well as the multicriterial decision-making problem (I, W, U) .

Let us suppose that a consequence $c \in C$ is unacceptable iff

$$u_1(c) < 2 \quad \text{or} \quad u_2(c) < 2 \quad \text{or} \quad u_3(c) < 6$$

or if for its common utility for pairs of criteria the following inequalities hold

$$u_1(c) + u_2(c) < 4 \quad \text{or} \quad u_2(c) + u_3(c) < 9 \quad \text{or} \quad u_1(c) + u_3(c) < 6.$$

It means that the sets $U(K)$ for $K \subset I$ are described by the formulas

$$\begin{aligned} U(I) &= W(I), \\ U(\{1, 2\}) &= \{x \in \mathbb{R}^3 : x_1 + x_2 \leq 4, x_1 \leq 6, x_2 \leq 6\}, \\ U(\{2, 3\}) &= \{x \in \mathbb{R}^3 : x_2 + x_3 \leq 9, x_2 \leq 6, x_3 \leq 8\}, \\ U(\{1, 3\}) &= \{x \in \mathbb{R}^3 : x_1 + x_3 \leq 6, x_1 \leq 6, x_3 \leq 8\}, \\ U(\{1\}) &= \{x \in \mathbb{R}^3 : x_1 \leq 2\}, \quad W(\{2\}) = \{x \in \mathbb{R}^2 : x_2 \leq 2\}, \\ U(\{3\}) &= \{x \in \mathbb{R}^3 : x_3 \leq 6\}, \end{aligned}$$

and

$$\begin{aligned} U^*(I) &= W^*(I), \\ U^*(\{1, 2\}) &= \{x \in \mathbb{R}^3 : (x_1 + x_2 \geq 4) \text{ or } (x_1 > 6) \text{ or } (x_2 > 6)\}, \\ U^*(\{2, 3\}) &= \{x \in \mathbb{R}^3 : (x_2 + x_3 \geq 9) \text{ or } (x_2 > 6) \text{ or } (x_3 > 8)\}, \\ U^*(\{1, 3\}) &= \{x \in \mathbb{R}^3 : (x_1 + x_3 \geq 6) \text{ or } (x_1 > 6) \text{ or } (x_3 > 8)\}, \\ U^*(\{1\}) &= \{x \in \mathbb{R}^2 : x_1 \geq 2\}, \quad W^*(\{2\}) = \{x \in \mathbb{R}^2 : x_2 \geq 2\}, \\ U^*(\{3\}) &= \{x \in \mathbb{R}^3 : x_3 \geq 6\}. \end{aligned}$$

Then the set of all acceptable utility vectors is the following one (cf. (4.1))

$$W(I) \cap U^*(I) \cap U^*({1, 2}) \cap U^*({1, 3}) \cap U^*({2, 3}) \cap U^*({1}) \cap U^*({2}) \cap U^*({3}).$$

The set is equal to

$$W(I) \cap U^*(I) \cap U^*({2, 3}) \cap U^*({3}) = \\ = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 12, x_2 + x_3 \geq 9, x_1 \geq 2, x_2 \geq 2, x_3 \geq 6\}.$$

This set is fully determined by the following utility vectors

$$x^{(3)} = (2, 2, 8), \quad y^{(1)} = (2, 4, 6), \quad y^{(2)} = (3, 2, 7), \quad y^{(3)} = (3, 3, 6)$$

as their convex hull, and it is presented in Figure 2. It means that the set of all acceptable utility vectors can be also expressed in the following form

$$\{x \in \mathbb{R}^3 : x_1 = 3 - \lambda - \mu, x_2 = 3 + \lambda - \mu - v, \\ x_3 = 6 + 2\mu + v, 0 \leq \lambda, 0 \leq \mu, 0 \leq v, \lambda + \mu + v \leq 1\}.$$

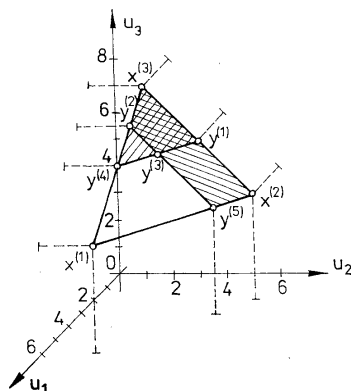


Fig. 2. The sets of acceptable utility vectors in Examples 1, 2 and 3.

It is obvious that in this example the groups of criteria $\{1, 2\}$, $\{1, 3\}$, $\{1\}$, $\{2\}$, are redundant in the sense described in Section 6.

Example 2. Let us consider the decision-making problem described by Example 1 with rather modified sets $U(K)$ for some $K \subset I$. Such situation is described by Theorem 13. Let us preserve the sets $U(K)$ from Example 1 for $K \subset I$, $K \neq \{2, 3\}$, and let

$$U(\{2, 3\}) = \{x \in \mathbb{R}^3 : x_2 + x_3 \leq 8, x_2 \leq 6, x_3 \leq 8\}.$$

Then the set of acceptable utility vectors is equal to

$$\begin{aligned} & W(I) \cap U^*(I) \cap U^*({3}) = \\ & = \{x \in \mathbb{R}^2 : x_1 + x_2 + x_3 = 12, x_1 \geq 2, x_2 \geq 2, x_3 \geq 6\}. \end{aligned}$$

It is the convex hull of the utility vectors

$$x^{(3)} = (2, 2, 8), \quad y^{(1)} = (2, 4, 6), \quad y^{(4)} = (4, 2, 6),$$

(see Figure 2).

Example 3. The decision-making problem described by Example 1 can be also considered for the construction of a suboptimal solution in the sense of Section 5. Let us consider the multicriterial decision-making problem (I, W, U) introduced in Example 1, and the class of sets of criteria $\mathcal{M} = \{I, \{2, 3\}\}$, where

$$U(I) = W(I),$$

$$U(\{2, 3\}) = \{x \in \mathbb{R}^3 : x_2 + x_3 \leq 9, x_2 \leq 6, x_3 \leq 8\}.$$

Then the set of all \mathcal{M} -acceptable utility vectors is the following one

$$\begin{aligned} & W(I) \cap U^*(I) \cap U^*(\{2, 3\}) = \\ & = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 12, x_2 + x_3 \geq 9, x_1 \geq 2, x_2 \geq 2, x_3 \geq 6\}. \end{aligned}$$

It is the convex hull of the utility vectors

$$x^{(2)} = (2, 6, 4), \quad x^{(3)} = (2, 2, 8), \quad y^{(2)} = (3, 2, 7), \quad y^{(5)} = (3, 5, 2)$$

(see Figure 2).

Example 4. Another decision-making problem appears if we do not admit the probabilistic mixture of the given pure decisions, it means if we consider only pure decisions $\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}$ and their pure consequences. Let us consider the decision-making problem with $I = \{1, 2, 3\}$ and

$$D = \{\delta^{(1)}, \delta^{(2)}, \delta^{(3)}, \delta^{(4)}\}, \quad C = \{c^{(1)}, c^{(2)}, c^{(3)}, c^{(4)}\},$$

$$c^{(i)} = q(\delta^{(i)}), \quad i = 1, 2, 3, 4.$$

Let the criteria be represented by the utility functions $u_i, i = 1, 2, 3$, with the values $u_i(c^{(j)}), i = 1, 2, 3, j = 1, 2, 3, 4$, given in Example 1.

Then the sets $W(K), K \subset I$, are determined by the utility vectors $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}$ known from Example 1. Namely

$$\begin{aligned} W(I) &= \{x \in \mathbb{R}^3 : (x_1 \leq 6, x_2 \leq 2, x_3 \leq 4) \text{ or } (x_1 \leq 2, \\ & \quad x_2 \leq 6, x_3 \leq 4) \text{ or } (x_1 \leq 2, x_2 \leq 2, x_3 \leq 8) \\ & \quad \text{or } (x_1 \leq 3, x_2 \leq 3, x_3 \leq 5)\}, \end{aligned}$$

$$\begin{aligned}
W(\{1, 2\}) &= \{x \in \mathbb{R}^3 : (x_1 \leq 6, x_2 \leq 2) \text{ or } (x_1 \leq 2, x_2 \leq 6) \\
&\quad \text{or } (x_1 \leq 3, x_2 \leq 3)\}, \\
W(\{2, 3\}) &= \{x \in \mathbb{R}^3 : (x_2 \leq 6, x_3 \leq 4) \text{ or } (x_2 \leq 2, x_3 \leq 8) \\
&\quad \text{or } (x_2 \leq 3, x_3 \leq 5)\}, \\
W(\{1, 3\}) &= \{x \in \mathbb{R}^3 : (x_1 \leq 6, x_3 \leq 4) \text{ or } (x_1 \leq 2, x_3 \leq 8) \\
&\quad \text{or } (x_1 \leq 3, x_3 \leq 8)\}, \\
W(\{1\}) &= \{x \in \mathbb{R}^3 : x_1 \leq 6\}, \quad W(\{2\}) = \{x \in \mathbb{R}^3 : x_2 \leq 6\}, \\
W(\{3\}) &= \{x \in \mathbb{R}^3 : x_3 \leq 8\}.
\end{aligned}$$

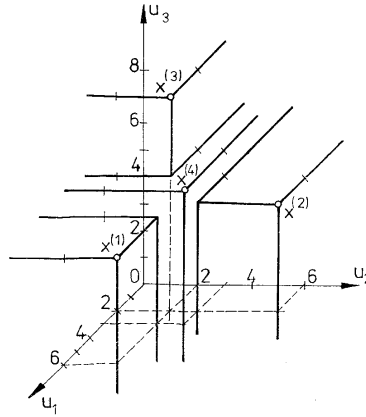


Fig. 3. The set $W(I)$ of the multicriterial decision-making problem in Example 4.

Those sets can be seen in Figure 3. Then

$$\begin{aligned}
W^*(I) &= (\mathbb{R}^3 - W(I)) \cup \{x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\}, \\
W^*(\{1, 2\}) &= (\mathbb{R}^3 - W(\{1, 2\})) \cup \{x \in \mathbb{R}^3 : (x_1 = 6, x_2 = 2) \\
&\quad \text{or } (x_1 = 2, x_2 = 6) \text{ or } (x_1 = 3, x_2 = 3)\}, \\
W^*(\{2, 3\}) &= (\mathbb{R}^3 - W(\{2, 3\})) \cup \{x \in \mathbb{R}^3 : (x_2 = 6, x_3 = 4) \\
&\quad \text{or } (x_2 = 2, x_3 = 8) \text{ or } (x_2 = 3, x_3 = 5)\}, \\
W^*(\{1, 3\}) &= (\mathbb{R}^3 - W(\{1, 3\})) \cup \{x \in \mathbb{R}^3 : (x_1 = 6, x_3 = 4) \\
&\quad \text{or } (x_1 = 2, x_3 = 8) \text{ or } (x_1 = 3, x_3 = 5)\}, \\
W^*(\{1\}) &= \{x \in \mathbb{R}^3 : x_1 \geq 6\}, \quad W^*(\{2\}) = \{x \in \mathbb{R}^3 : x_2 \geq 6\}, \\
W^*(\{3\}) &= \{x \in \mathbb{R}^3 : x_3 \geq 8\},
\end{aligned}$$

and the Pareto optima are

$$\begin{aligned}
P_W(I) &= \{x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\}, \\
P_W(\{1, 2\}) &= \{x \in \mathbb{R}^3 : (x_1 = 6, x_2 = 2) \text{ or } (x_1 = 2, x_2 = 6) \\
&\quad \text{or } (x_1 = 3, x_2 = 3)\}, \\
P_W(\{2, 3\}) &= \{x \in \mathbb{R}^3 : (x_2 = 6, x_3 = 4) \text{ or } (x_2 = 2, x_3 = 8) \\
&\quad \text{or } (x_2 = 3, x_3 = 5)\}, \\
P_W(\{1, 3\}) &= \{x \in \mathbb{R}^3 : (x_1 = 6, x_3 = 4) \text{ or } (x_1 = 2, x_3 = 8) \\
&\quad \text{or } (x_1 = 3, x_3 = 5)\}, \\
P_W(\{1\}) &= \{x \in \mathbb{R}^3 : x_1 = 6\}, \quad P_W(\{2\}) = \{x \in \mathbb{R}^3 : x_2 = 6\}, \\
P_W(\{3\}) &= \{x \in \mathbb{R}^3 : x_3 = 8\}.
\end{aligned}$$

Let us suppose that the unacceptability of consequences is analogous to Example 1. It means that a consequence $c \in C$ is not acceptable if

$$u_1(c) < 2 \text{ or } u_2(c) < 2 \text{ or } u_3(c) < 6$$

or if

$$u_1(c) + u_2(c) < 4 \text{ or } u_2(c) + u_3(c) < 9 \text{ or } u_1(c) + u_3(c) < 6.$$

Then

$$\begin{aligned}
U(I) &= W(I), \\
U(\{1, 2\}) &= \{x \in \mathbb{R}^3 : x_1 + x_2 \leq 4\} \cap W(\{1, 2\}) = \\
&= \{x \in \mathbb{R}^3 : x_1 + x_2 \leq 4, x_1 \leq 6, x_2 \leq 6\}, \\
U(\{2, 3\}) &= \{x \in \mathbb{R}^3 : x_2 + x_3 \leq 9\} \cap W(\{2, 3\}) = \\
&= \{x \in \mathbb{R}^3 : (x_2 + x_3 \leq 9, x_2 \leq 2, x_3 \leq 8) \text{ or} \\
&\quad (x_2 + x_3 \leq 9, x_2 \leq 6, x_3 \leq 4) \text{ or } (x_2 \leq 3, x_3 \leq 5)\}, \\
U(\{1, 3\}) &= \{x \in \mathbb{R}^3 : x_1 + x_3 \leq 6\} \cap W(\{1, 3\}) = \\
&= \{x \in \mathbb{R}^3 : x_1 + x_3 \leq 6, x_1 \leq 6, x_3 \leq 8\}, \\
U(\{1\}) &= \{x \in \mathbb{R}^3 : x_1 \leq 2\}, \quad U(\{2\}) = \{x \in \mathbb{R}^3 : x_2 \leq 2\}, \\
U(\{3\}) &= \{x \in \mathbb{R}^3 : x_3 \leq 6\},
\end{aligned}$$

(see also Figure 4), and

$$\begin{aligned}
U^*(I) &= W^*(I), \\
U^*(\{1, 2\}) &= (\mathbb{R}^3 - U(\{1, 2\})) \cup \{x \in \mathbb{R}^3 : x_1 + x_2 = 4\} = \\
&= \{x \in \mathbb{R}^3 : (x_1 + x_2 \geq 4) \text{ or } (x_1 > 6) \text{ or } (x_2 > 6)\},
\end{aligned}$$

$$\begin{aligned}
U^*(\{2, 3\}) &= (\mathbb{R}^3 - U(\{2, 3\})) \cup \{x \in \mathbb{R}^3 : x_2 + x_3 = 9\} \cup \\
&\cup \{x \in \mathbb{R}^3 : x_2 = 3, x_3 = 5\} = \\
&= \{x \in \mathbb{R}^3 : (x_2 + x_3 \geq 9) \text{ or } (x_2 > 2, x_2 > 5) \text{ or } \\
&\quad (x_2 > 3, x_3 > 4) \text{ or } (x_2 = 3, x_3 = 5) \text{ or } (x_2 > 6) \\
&\quad \text{or } (x_3 > 8)\}, \\
U^*(\{1, 3\}) &= (\mathbb{R}^3 - U(\{1, 3\})) \cup \{x \in \mathbb{R}^3 : x_1 + x_3 = 6\} = \\
&= \{x \in \mathbb{R}^3 : (x_1 + x_3 \geq 6) \text{ or } (x_1 > 6) \text{ or } (x_3 > 8)\}, \\
U^*(\{1\}) &= \{x \in \mathbb{R}^3 : x_1 \geq 2\}, \quad U^*(\{2\}) = \{x \in \mathbb{R}^3 : x_2 \geq 2\}, \\
U^*(\{3\}) &= \{x \in \mathbb{R}^3 : x_3 \geq 6\}.
\end{aligned}$$

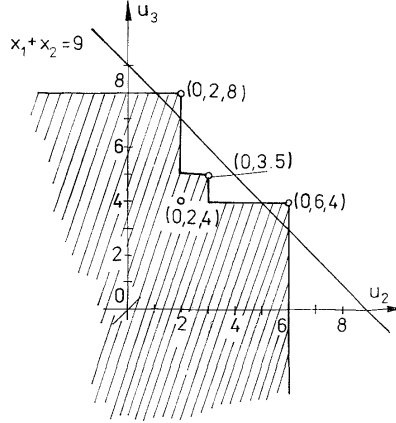


Fig. 4. Intersection of the sets $W(\{2, 3\})$ (hatched) and $U(\{2, 3\})$ with the plane $\{x \in \mathbb{R}^3 : x_1 = 0\}$.

Then the set of all acceptable utility vectors is the following one:

$$\begin{aligned}
&W(I) \cap U^*(I) \cap U^*(\{1, 2\}) \cap U^*(\{2, 3\}) \cap U^*(\{1, 3\}) \cap U^*(\{1\}) \cap \\
&\cap U^*(\{2\}) \cap U^*(\{3\}) = W(I) \cap U^*(I) \cap U^*(\{2, 3\}) \cap U^*(\{3\}) = \\
&= \{x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}\} \cap U^*(\{2, 3\}) \cap U^*(\{3\}) = \{x^{(3)}\}.
\end{aligned}$$

It means that there is exactly one acceptable utility vector $x^{(3)}$ corresponding to the decision $\delta^{(3)}$ and its consequence $c^{(3)}$.

8. CONCLUSIVE REMARKS

It was shown above that some features of the multicriterial decision-making can be modelled by the general coalition-games. The game theoretical tools enable us to reduce the set of possible decisions and their consequences, and to eliminate the evidently unacceptable ones. The final set of acceptable utility vectors (connected with acceptable decisions and their consequences) is formed by the utility vectors which are really achievable and are not dominated by any unacceptable utility vector. These utility vectors and the corresponding decisions can be treated by other, more sophisticated, methods in order to find the optimal decision. The application of that detailed methods is not influenced by the properties of unacceptable decisions.

The game theoretical methods can also lead to elimination of some criteria which do not appear to be essential for the considered problem. A few results of that type are presented above. However, the effective methods of elimination of less important criteria represent an open problem which can be solved by the model given in this paper.

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