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MEASURES OF VECTOR INFORMATION WITH THE BRANCHING PROPERTY

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It is known that, for a large class of monoids $(S, *)$, all solutions of the functional equation

$$\Delta(s, t) + \Delta(s * t, u) = \Delta(s, t * u) + \Delta(t, u)$$

for $\Delta : S^2 \to G$ (with $(G, +)$ any divisible abelian group) have representations of the form

$$\Delta(s, t) = f(s) + f(t) - f(s * t) + \psi(s, t),$$

where ψ is antisymmetric and bi-additive. We show that this class is closed under the formation of direct products. This result is then used to characterize branching measures of vector information on strings of monoid elements.

1. INTRODUCTION

An entropy in the "classical" (probabilistic) sense is a sequence (I_n) of mappings from the set of all n-ary complete probability distributions into the real numbers. More generally, I_n can be a function of several probability distributions, as is the case for directed divergence or inaccuracy. Discussions, properties, and characterizations of such information measures can be found in the book [2] by Aczél and Daróczy.

Measures of information in a different sense have been proposed and studied by the author [3], [4]. In [3], a measure of information is a function of a string of elements from a monoid. In [4], the information measure is a function of pairs of strings (quantities and "attractions") and appears as a utility function. In the present setting, a measure of information will be a function of m-vectors of strings.

The main results are contained in Sections 2 and 4. In Section 3, a fundamental functional equation is derived. Its general solution is found in Section 4 and used in Section 5 to prove the main result of Section 2.

2. BRANCHING MEASURES OF VECTOR INFORMATION

A *measure of m-vector information* ($m = 1, 2, \ldots$) is a sequence $\mu_n: \mathsf{X}_{j=1}^{m} S_j^n \to G$ ($n = 3, 4, \ldots$), where $(G, +)$ is a divisible abelian group and each S_j is a commutative monoid (with identity e_j)from a certain class **S** defined below. For notational convenience, we write the argument of μ_n as if it belonged to $(\mathsf{X}_{j=1}^{m} S_j)^n$ instead of $\mathsf{X}_{j=1}^{m} S_j^n$.

The essential property for our measures is that of branching. A measure μ_n of m-vector information is said to be *branching* if there are maps $\Delta_{ni} : \mathsf{X}_{j=1}^{m} S_j^2 \to G$, for all $i = 1, 2, \ldots, n - 1$ ($n = 3, 4, \ldots$), such that

$$(2.1) \qquad \mu_n(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \\ = \mu_n(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_i \circ \mathbf{v}_{i+1}, \mathbf{e}, \mathbf{v}_{i+2}, \ldots, \mathbf{v}_n) + \Delta_{ni}(\mathbf{v}_i, \mathbf{v}_{i+1}),$$

for all $\mathbf{v}_k \in \mathsf{X}_{j=1}^{m} S_j$ ($k = 1, 2, \ldots, n$), where $\mathbf{e} = (e_1, e_2, \ldots, e_m)$ and $\mathbf{u} \circ \mathbf{w} = (u_1 w_1, u_2 w_2, \ldots, u_m w_m)$ for all $(\mathbf{u}, \mathbf{w}) \in \mathsf{X}_{j=1}^{m} S_j^2$. For the sake of readability, the operations of all S_j are designated simply by juxtaposition, as this leads to no confusion here.

All monoids S_j are from the class **S** defined as follows.

Definition 2.1. A commutative monoid $(S, *)$ is said to belong to class **S** if all solutions $\Delta : S^2 \to G$ ($(G, +)$ any divisible abelian group) of the functional equation

$$(2.2) \qquad \Delta(s, t) + \Delta(s * t, u) = \Delta(s, t * u) + \Delta(t, u),$$

for all $(s, t, u) \in S^3$, have a representation

$$(2.3) \qquad \Delta(s, t) = \delta(s) + \delta(t) - \delta(s * t) + \psi(s, t)$$

for some map $\delta : S \to G$ and a map $\psi : S^2 \to G$ which is *antisymmetric*

$$(2.4) \qquad \psi(s, t) = -\psi(t, s), \quad \forall (s, t) \in S^2,$$

and *bi-additive*. (Additivity in the first variable, for example, means that

$$\psi(st, u) = \psi(s, u) + \psi(t, u), \quad \forall (s, t, u) \in S^3.)$$

Equations (2.2) and (2.1) have been studied extensively and on many different domains. On branching measures of information, see [3] and [4] for results on strings of ($m =$) 1- and 2-vectors. On equation (2.2), see [8], [5], [6], [1], [7], [3], [4].

The following summarizes the above results on (2.2).

Theorem 2.2. A commutative monoid $(S, \cdot)$ belongs to class **S** if $(S, \cdot)$ is any of the following: idempotent, a monoid with zero, a thread, a group, the set of positive elements of an ordered group, a cancellative w-thread, a near-thread.

Definition 2.3. A w-*thread* (or *thread in the wider sense*) is a connected, totally ordered topological semigroup. A *thread* is a w-thread with a greatest and a least element, both of which are idempotent. Finally, a *near-thread* is a semigroup obtained by removing the zero from a w-thread $(S^*, \cdot)$ which has a zero as least element and the property $S^* . S^* = S^*$ (global idempotence).

The main result, which is proved in Section 5, is the following.

Theorem 2.4. The measure $\mu_n : \mathop{\times}\limits_{j=1}^{m} S_j^n \to G$ $(n = 3, 4, \ldots)$ of m-vector information has the (2.1) branching property, if and only if it admits a representation

$$(2.5)\qquad \mu_n(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \varphi_{n0}(\mathbf{v}_1 \circ \ldots \circ \mathbf{v}_n) + \sum_{i=1}^{n} \varphi_{ni}(\mathbf{v}_i) + \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} \psi_n(\mathbf{v}_i, \mathbf{v}_k),$$

for all $(\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \mathop{\times}\limits_{j=1}^{m} S_j^n$, for some maps $\varphi_{ni} : \mathop{\times}\limits_{j=1}^{m} S_j \to G$ $(i = 0, 1, \ldots, n)$ and $\psi_n : \mathop{\times}\limits_{j=1}^{m} S_j^2 \to G$, where ψ_n is antisymmetric and bi-additive in the m-tuples $\mathbf{v}_i, \mathbf{v}_k$. That is, ψ_n satisfies

$$(2.6)\qquad \psi_n(\mathbf{u}, \mathbf{w}) = -\psi_n(\mathbf{w}, \mathbf{u}), \quad \forall(\mathbf{u}, \mathbf{w}) \in \mathop{\times}\limits_{j=1}^{m} S_j^2$$

$$\psi_n(\mathbf{u} \circ \mathbf{v}, \mathbf{w}) = \psi_n(\mathbf{u}, \mathbf{w}) + \psi_n(\mathbf{v}, \mathbf{w}), \quad \forall(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathop{\times}\limits_{j=1}^{m} S_j^3.$$

We can easily obtain the following consequence.

Corollary 2.5. A measure μ_n of m-vector information is (2.1) branching and symmetric in its arguments $\mathbf{v}_i$ $(i = 1, \ldots, n)$, if and only if there exist maps $\tilde{\varphi}_n, \varphi_n : \mathop{\times}\limits_{j=1}^{m} S_j \to G$ such that

$$(2.7)\qquad \mu_n(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \tilde{\varphi}_n(\mathbf{v}_1 \circ \ldots \circ \mathbf{v}_n) + \sum_{i=1}^{n} \varphi_n(\mathbf{v}_i).$$

Proof. Assume that μ_n has the (2.1) branching property. μ_n has representation (2.5) by Theorem 2.4. Interchanging $\mathbf{v}_k$ with $\mathbf{v}_{k+1}$ $(1 \leqq k \leqq n-1)$, by the symmetry hypothesis, we get

$$(2.8)\qquad \varphi_{nk}(\mathbf{v}_k) + \varphi_{n,k+1}(\mathbf{v}_{k+1}) + \psi_n(\mathbf{v}_k, \mathbf{v}_{k+1}) = $$
$$= \varphi_{nk}(\mathbf{v}_{k+1}) + \varphi_{n,k+1}(\mathbf{v}_k) + \psi_n(\mathbf{v}_{k+1}, \mathbf{v}_k),$$

for all $(\mathbf{v}_k, \mathbf{v}_{k+1}) \in \mathop{\times}\limits_{j=1}^{m} S_j^2$, from (2.5). Letting $\mathbf{v}_{k+1} = \mathbf{e}$ and using $\psi_n(\mathbf{e}, \mathbf{v}_k) = \psi_n(\mathbf{v}_k, \mathbf{e}) =$

$= 0$ (which follow from bi-additivity), we have

$$\varphi_{n,k+1}(\mathbf{u}) = \varphi_{nk}(\mathbf{u}) + c_{nk}, \quad \forall \mathbf{u} \in \mathop{\mathsf{X}}_{j=1}^{m} S_j . \tag{2.9}$$

Now (2.8) yields

$$\psi_n(\mathbf{v}_k, \mathbf{v}_{k+1}) = \psi_n(\mathbf{v}_{k+1}, \mathbf{v}_k), \quad \forall(\mathbf{v}_k, \mathbf{v}_{k+1}) \in \mathop{\mathsf{X}}_{j=1}^{m} S_j^2 .$$

By the (2.6) antisymmetry of ψ_n, therefore,

$$\psi_n = 0 . \tag{2.10}$$

Thus, by (2.9) and (2.10), (2.5) becomes (2.7), where φ_n and $\tilde{\varphi}_n$ are defined by $\varphi_n := \varphi_{n1}$, $\tilde{\varphi}_n := \varphi_{n0} + \sum_{k=1}^{n-1} (n-k)\, c_{nk}$.

The converse is easy to check. □

3. DERIVATION OF THE FUNDAMENTAL EQUATION

We begin by using equation (2.1) in two different ways for a given string $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \in \mathop{\mathsf{X}}_{j=1}^{m} S_j^n$. On one hand, apply (2.1) for $i = k + 1$ $(1 \leqq k \leqq n - 2)$, then for $i = k$; on the other hand, apply (2.1) first for $i = k$, then for $i = k + 1$, then for $i = k$ again. Comparing the two results, we find that

$$\begin{aligned} &\Delta_{n,k+1}(\mathbf{v}_{k+1}, \mathbf{v}_{k+2}) + \Delta_{nk}(\mathbf{v}_k, \mathbf{v}_{k+1} \circ \mathbf{v}_{k+2}) = \\ &= \Delta_{nk}(\mathbf{v}_k, \mathbf{v}_{k+1}) + \Delta_{n,k+1}(\mathbf{e}, \mathbf{v}_{k+2}) + \Delta_{nk}(\mathbf{v}_k \circ \mathbf{v}_{k+1}, \mathbf{v}_{k+2}), \end{aligned} \tag{3.1}$$

for any $(\mathbf{v}_k, \mathbf{v}_{k+1}, \mathbf{v}_{k+2}) \in \mathop{\mathsf{X}}_{j=1}^{m} S_j^3$. With $\mathbf{v}_k = \mathbf{e}$, (3.1) becomes

$$\begin{aligned} \Delta_{n,k+1}(\mathbf{v}_{k+1}, \mathbf{v}_{k+2}) &= \Delta_{nk}(\mathbf{v}_{k+1}, \mathbf{v}_{k+2}) + \Delta_{nk}(\mathbf{e}, \mathbf{v}_{k+1}) + \\ &+ \Delta_{n,k+1}(\mathbf{e}, \mathbf{v}_{k+2}) - \Delta_{nk}(\mathbf{e}, \mathbf{v}_{k+1} \circ \mathbf{v}_{k+2}) . \end{aligned} \tag{3.2}$$

From (3.1), using (3.2), we now get (with $\mathbf{u} = \mathbf{v}_k$, $\mathbf{v} = \mathbf{v}_{k+1}$, $\mathbf{w} = \mathbf{v}_{k+2}$)

$$\begin{aligned} &\Delta_{nk}(\mathbf{v}, \mathbf{w}) - \Delta_{nk}(\mathbf{e}, \mathbf{v} \circ \mathbf{w}) + \Delta_{nk}(\mathbf{u}, \mathbf{v} \circ \mathbf{w}) = \\ &= \Delta_{nk}(\mathbf{u}, \mathbf{v}) - \Delta_{nk}(\mathbf{e}, \mathbf{v}) + \Delta_{nk}(\mathbf{u} \circ \mathbf{v}, \mathbf{w}), \end{aligned} \tag{3.3}$$

for all $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathop{\mathsf{X}}_{j=1}^{m} S_j^3$ and all $k = 1, 2, \ldots, n-1$. (To get (3.3) for $k = n - 1$, use (3.1) and (3.2) for $k = n - 2$ and solve (3.2) for $\Delta_{n,n-2}(\mathbf{v}_{n-1}, \mathbf{v}_n)$ instead of $\Delta_{n,n-1}(\mathbf{v}_{n-1}, \mathbf{v}_n)$; then substitute for $\Delta_{n,n-2}$ terms in (3.1).)

Fix $n \geqq 2$ and $k = 1$ (temporarily) and define $F : \mathop{\mathsf{X}}_{j=1}^{m} S_j^2 \to G$ by

$$F(\mathbf{u}, \mathbf{v}) := \Delta_{n1}(\mathbf{u}, \mathbf{v}) - \Delta_{n1}(\mathbf{e}, \mathbf{v}), \quad \forall(\mathbf{u}, \mathbf{v}) \in \mathop{\mathsf{X}}_{j=1}^{m} S_j^2 .$$

Then, by (3.3), F satisfies

$$F(\mathbf{u}, \mathbf{v}) + F(\mathbf{u} \circ \mathbf{v}, \mathbf{w}) = F(\mathbf{u}, \mathbf{v} \circ \mathbf{w}) + F(\mathbf{v}, \mathbf{w}) \tag{3.4}$$

for all $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \bigtimes_{j=1}^{m} S_j^3$, and Δ_{n1} is given by

$$\Delta_{n1}(\mathbf{u}, \mathbf{v}) = F(\mathbf{u}, \mathbf{v}) + \Delta_{n1}(\mathbf{e}, \mathbf{v}), \quad \forall (\mathbf{u}, \mathbf{v}) \in \bigtimes_{j=1}^{m} S_j^2, \tag{3.5}$$

for an arbitrary map $\Delta_{n1}(\mathbf{e}, \cdot) : \bigtimes_{j=1}^{m} S_j \to G$.

Thus we have proved the following.

Lemma 3.1. If μ_n $(n = 3, 4, \ldots)$ is a (2.1) branching measure of m-vector information, then the branching functions Δ_{ni} $(i = 2, 3, \ldots, n-1)$ can be obtained recursively from Δ_{n1} and arbitrary one-place functions $\Delta_{ni}(\mathbf{e}, \cdot)$ through (3.2). Moreover, Δ_{n1} is given by (3.5) for an arbitrary solution, $F : \bigtimes_{j=1}^{m} S_j^2 \to G$, of (3.4).

Our immediate goal is, therefore, to solve (3.4).

4. SOLUTION OF THE FUNDAMENTAL EQUATION

The principal tool to be used in solving equation (3.4) is the following.

Lemma 4.1. If $(X, \oplus)$ is a commutative monoid, and if $(S, *) \in \mathbf{S}$, then a map $F : (X \times S)^2 \to G$ (with $(G, +)$ a divisible abelian group) satisfies

$$\begin{aligned} F(x, r; y, s) + F(x \oplus y, r * s; z, t) &= \\ = F(x, r; y \oplus z, s * t) &+ F(y, s; z, t), \end{aligned} \tag{4.1}$$

for all $(x, y, z) \in X^3$ and all $(r, s, t) \in S^3$, if and only if there exist maps $\varphi : X \times S \to G$, $\psi : (X \times S)^2 \to G$, $\tilde{F} : X^2 \to G$ such that

$$\begin{aligned} F(x, r; y, s) = \tilde{F}(x, y) + \varphi(x, r) + \varphi(y, s) - \varphi(x \oplus y, r * s) + \\ + \psi(x, r; y, s), \end{aligned} \tag{4.2}$$

for all $(x, y) \in X^2$, $(r, s) \in S^2$, where ψ is (2.6) antisymmetric and bi-additive, and $\tilde{F}$ satisfies

$$\tilde{F}(x, y) + \tilde{F}(x \oplus y, z) = \tilde{F}(x, y \oplus z) + \tilde{F}(y, z), \quad \forall (x, y, z) \in X^3. \tag{4.3}$$

Proof. Lemma 4.1 in [3]. □

With Lemma 4.1, we obtain the following result.

Theorem 4.2. Let $S_j \in \mathbf{S}$, $j = 1, 2, \ldots, m$; let $(G, +)$ be a divisible abelian group.

A map $F : \mathop{\times}_{j=1}^{m} S_j^2 \to G$ satisfies (3.4) for all $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathop{\times}_{j=1}^{m} S_j^3$, if and only if there exist a map $\varphi : \mathop{\times}_{j=1}^{m} S_j \to G$ and an (2.6) antisymmetric bi-additive map $\psi : \mathop{\times}_{j=1}^{m} S_j^2 \to G$ for which

$$(4.4) \qquad F(\mathbf{u}, \mathbf{v}) = \varphi(\mathbf{u}) + \varphi(\mathbf{v}) - \varphi(\mathbf{u} \circ \mathbf{v}) + \psi(\mathbf{u}, \mathbf{v}),$$

for all $(\mathbf{u}, \mathbf{v}) \in \mathop{\times}_{j=1}^{m} S_j^2$.

Proof. Let F satisfy (3.4). In this direction, the proof is by induction on m. For $m = 1$, it is simply a restatement of Definition 2.1.

Assume the truth of the theorem for $m = n$. Let $\tilde{\mathbf{v}}_i$ denote $(v_1, v_2, \ldots, v_i)$ = the first i coordinates of $\mathbf{v} \in \mathop{\times}_{j=1}^{m} S_j$, for $i = 1, 2, \ldots, m - 1$. Also, let $\tilde{\mathbf{u}}_i \oplus \tilde{\mathbf{v}}_i$ denote $(u_1 v_1, u_2 v_2, \ldots, u_i v_i)$ = the restriction of $\mathbf{u} \circ \mathbf{v}$ to its first i coordinates ($i = 1, 2, \ldots$ $\ldots, m - 1$), for any $(\mathbf{u}, \mathbf{v}) \in \mathop{\times}_{j=1}^{m} S_j^2$. Then, for $m = n + 1$, (3.4) can be written

$$(3.4) \qquad F(\tilde{\mathbf{u}}_n, u_{n+1}; \tilde{\mathbf{v}}_n, v_{n+1}) + F(\tilde{\mathbf{u}}_n \oplus \tilde{\mathbf{v}}_n, u_{n+1} v_{n+1}; \tilde{\mathbf{w}}_n, w_{n+1}) =$$
$$= F(\tilde{\mathbf{u}}_n, u_{n+1}; \tilde{\mathbf{v}}_n \oplus \tilde{\mathbf{w}}_n, v_{n+1} w_{n+1}) + F(\tilde{\mathbf{v}}_n, v_{n+1}; \tilde{\mathbf{w}}_n, w_{n+1}).$$

Now, applying Lemma 4.1 with $X = \mathop{\times}_{j=1}^{n} S_j$ and $S = S_{n+1}$, we get

$$(4.5) \qquad F(\tilde{\mathbf{u}}_n, u_{n+1}; \tilde{\mathbf{v}}_n, v_{n+1}) = \tilde{F}(\tilde{\mathbf{u}}_n, \tilde{\mathbf{v}}_n) + \hat{\varphi}(\mathbf{u}) + \hat{\varphi}(\mathbf{v}) - \hat{\varphi}(\mathbf{u} \circ \mathbf{v}) + \hat{\psi}(\mathbf{u}, \mathbf{v}),$$

where $\tilde{F}$ satisfies (4.3) and $\hat{\psi}$ is (2.6) antisymmetric and bi-additive. But, by the induction hypothesis, $\tilde{F}$ has a representation of the form

$$(4.6) \qquad \tilde{F}(\tilde{\mathbf{u}}_n, \tilde{\mathbf{v}}_n) = \tilde{\varphi}(\tilde{\mathbf{u}}_n) + \tilde{\varphi}(\tilde{\mathbf{v}}_n) - \tilde{\varphi}(\tilde{\mathbf{u}}_n \oplus \tilde{\mathbf{v}}_n) + \tilde{\psi}(\tilde{\mathbf{u}}_n, \tilde{\mathbf{v}}_n),$$

where $\tilde{\psi}$ is (2.6) antisymmetric and bi-additive. Combining (4.6) with (4.5) and defining $\varphi(\mathbf{u}) := \hat{\varphi}(\mathbf{u}) + \tilde{\varphi}(\tilde{\mathbf{u}}_n)$, $\psi(\mathbf{u}, \mathbf{v}) := \hat{\psi}(\mathbf{u}, \mathbf{v}) + \tilde{\psi}(\tilde{\mathbf{u}}_n, \tilde{\mathbf{v}}_n)$ for all $(\mathbf{u}, \mathbf{v}) \in \mathop{\times}_{j=1}^{n+1} S_j^2$, we have (4.4) for $m = n + 1$ (with ψ (2.6) antisymmetric and bi-additive), as required.

The converse is easy to check. □

Remark. Theorem 4.2 says that $\mathbf{S}$ is closed under the formation of direct products.

5. PROOF OF THEOREM 2.4.

Let μ_n ($n = 3, 4, \ldots$) be a (2.1) branching measure of m-vector information. By Lemma 3.1 and Theorem 4.2, Δ_{n1} has a representation in the form

$$\Delta_{n1}(\mathbf{u}, \mathbf{v}) = \varphi_{n1}(\mathbf{u}) + \varphi_{n2}(\mathbf{v}) - \varphi_{n1}(\mathbf{u} \circ \mathbf{v}) + \psi_n(\mathbf{u}, \mathbf{v})$$

for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathop{\times}\limits_{j=1}^{m} S_j^2$, where $\varphi_{n2}(\boldsymbol{v}) := \varphi_{n1}(\boldsymbol{v}) + \Delta_{n1}(\mathbf{e}, \boldsymbol{v})$ and ψ_n is (2.6) antisymmetric and bi-additive. Furthermore, $\Delta_{n2}, \Delta_{n3}, \ldots, \Delta_{n,n-1}$ are defined recursively through (3.2), giving

$$\Delta_{ni}(\boldsymbol{u}, \boldsymbol{v}) = \varphi_{ni}(\boldsymbol{u}) + \varphi_{n,i+1}(\boldsymbol{v}) - \varphi_{ni}(\boldsymbol{u} \circ \boldsymbol{v}) + \psi_n(\boldsymbol{u}, \boldsymbol{v}) \tag{5.1}$$

for all $(\boldsymbol{u}, \boldsymbol{v}) \in \mathop{\times}\limits_{j=1}^{m} S_j^2$ and all $i = 1, 2, \ldots, n-1$, where φ_{ni} is defined recursively by $\varphi_{n,i+1}(\boldsymbol{u}) := \varphi_{ni}(\boldsymbol{u}) + \Delta_{ni}(\mathbf{e}, \boldsymbol{u})$. We have also used the fact that $\psi_n(\mathbf{e}, \boldsymbol{u}) = 0$ for any $\boldsymbol{u} \in \mathop{\times}\limits_{j=1}^{m} S_j$, which follows from additivity in the first variable.

Now, by (2.1) and (5.1), we have

$$\begin{aligned}\mu_n(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) &= \mu_n(\mathbf{v}_1, \ldots, \mathbf{v}_{n-2}, \mathbf{v}_{n-1} \circ \mathbf{v}_n, \mathbf{e}) + \varphi_{n,n-1}(\mathbf{v}_{n-1}) + \\ &+ \varphi_{nn}(\mathbf{v}_n) - \varphi_{n,n-1}(\mathbf{v}_{n-1} \circ \mathbf{v}_n) + \psi_n(\mathbf{v}_{n-1}, \mathbf{v}_n) = \ldots \\ &\ldots = \mu_n(\mathbf{v}_1 \circ \mathbf{v}_2 \circ \ldots \circ \mathbf{v}_n, \mathbf{e}, \mathbf{e}, \ldots, \mathbf{e}) + \sum_{i=1}^{n} \varphi_{ni}(\mathbf{v}_i) - \\ &- \varphi_{n1}(\mathbf{v}_1 \circ \mathbf{v}_2 \circ \ldots \circ \mathbf{v}_n) + \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} \psi_n(\mathbf{v}_i, \mathbf{v}_k),\end{aligned}$$

where we have also used the additivity of ψ_n in the second variable. Defining $\varphi_{n0}(\boldsymbol{u}) := \mu_n(\boldsymbol{u}, \mathbf{e}, \mathbf{e}, \ldots, \mathbf{e}) - \varphi_{n1}(\boldsymbol{u})$, we obtain the asserted form (2.5).

Again, the converse is easy to check, and Theorem 2.4 is established. □

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