

**BANG-BANG CONTROL
OF A SECOND-ORDER NON-LINEAR STABLE PLANT
WITH FOURTH-ORDER NONLINEARITY**

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In this paper the design of a controller for a relay controlled second order non-linear stable plant with fourth order nonlinearity is considered. The task of the controller is the simultaneous reduction of plant's output and output derivative to zero with the input to the feed back system being at zero. It will be shown that for all initial values of output and output derivative it would be possible to bring them to zero, simultaneously and in the shortest possible time with almost one switching reversal of the relay. Through simple transformation, it would be shown that the equation of switching curve can be made independent of constant gain of the plant as well as the coefficient of the non-linear term.

1. INTRODUCTION

Bang-bang control of a number of linear stable plants has been thoroughly discussed by Athans and Falb [1]. Second order linear unstable relay controlled plants have been studied by Vakilzadeh [2, 3]. In [1], time optimal control of a class of second order non-linear systems is investigated and it is shown that for these non-linear systems, like linear ones, bang-bang control is indeed time optimal. A number of examples are considered, but no analytical treatment for obtaining the equation of switching curve is presented. Recently Vakilzadeh and Keshavarz [4, 5] considered bang-bang control of second order non-linear stable plant with second-order nonlinearity and unstable plant with third-order nonlinearity. In this paper we shall consider a second-order non-linear stable plant with fourth-order nonlinearity.

2. ANALYSIS

Consider a plant whose input-output is governed by the second-order non-linear differential equation

$$(1) \quad \ddot{e} + a|\dot{e}|e^3 = Ku \quad a > 0, \quad u = \pm 1$$

Since $a > 0$, the above system is stable. The plant represents a moving body subject to non-linear drag force which is proportional to the fourth power of velocity. Here the object is the design of a controller for simultaneous reduction of $c(t)$ and $\dot{c}(t)$ to zero when the input $r(t) = 0$. Therefore the system's error differential equation is given by

$$(2) \quad \ddot{e} + a|\dot{e}| \dot{e}^3 = -Ku$$

The reduction of error and error derivative to zero will mean the reduction of output $c(t)$ and output derivative $\dot{c}(t)$ to zero. Now

$$(3) \quad \ddot{e} = \frac{d\dot{e}}{dt} = \dot{e} \frac{d\dot{e}}{de}$$

From equations (2) and (3), we have

$$(4) \quad \dot{e} \frac{d\dot{e}}{de} + a|\dot{e}| \dot{e}^3 = -Ku$$

For the solution of the non-linear differential equation (4), two cases for \dot{e} are considered.

Case (i) $\dot{e} > 0$

In this case, equation (4) can be written as

$$(5) \quad \dot{e} \frac{d\dot{e}}{de} + a\dot{e}^4 = -Ku$$

or

$$(6) \quad \frac{d\dot{e}}{de} = -\frac{a\dot{e}^4 + Ku}{\dot{e}}$$

Let

$$(7) \quad y_1 = 4\sqrt[4]{(aK)} e$$

$$(8) \quad y_2 = \left(\frac{a}{K}\right)^{1/4} \dot{e}$$

Substituting (7) and (8) in (6), we get

$$(9) \quad dy_2 = -\frac{y_2^4 + u}{4y_2} dy_1$$

Integrating (9), we have for $u \neq +1$,

$$(10) \quad y_1 - \beta_1 = 2[\arctg \beta_2^2 - \arctg y_2^2]$$

and for $u = -1$,

$$(11) \quad y_1 - \beta_1 = \ln \left[\left(\frac{1 + y_2^2}{1 - y_2^2} \right) \left(\frac{1 - \beta_2^2}{1 + \beta_2^2} \right) \right]$$

where $\beta_1 = y_1(0)$ and $\beta_2 = y_2(0)$. From equation (10) we have for control $u = +1$

$$(12) \quad y_2 = + \left[\frac{\beta_2^2 - \tan \left(\frac{y_1 - \beta_1}{2} \right)}{1 + \beta_2^2 \tan \left(\frac{y_1 - \beta_1}{2} \right)} \right]^{1/2}$$

and from (11), we have for $u = -1$

$$(13) \quad y_2 = + \left[\frac{(1 + \beta_2^2) - (1 - \beta_2^2) \exp(-y_1 + \beta_1)}{(1 + \beta_2^2) + (1 - \beta_2^2) \exp(-y_1 + \beta_1)} \right]^{1/2}$$

Case (ii) $\dot{e} < 0$.

In this case, equation (4) can be written as

$$(14) \quad \dot{e} \frac{d\dot{e}}{de} - a\dot{e}^4 = -Ku$$

or, as in equations (6) to (8), we have

$$(15) \quad dy_2 = \frac{y_2^4 - u}{4y_2} dy_1$$

Integrating (15), we have for $u = +1$,

$$(16) \quad y_1 - \beta_1 = \ln \left[\left(\frac{1 - y_2^2}{1 + y_2^2} \right) \left(\frac{1 + \beta_2^2}{1 - \beta_2^2} \right) \right]$$

and for $u = -1$,

$$(17) \quad y_1 - \beta_1 = 2[\arctg y_2^2 - \arctg \beta_2^2]$$

where $\beta_1 = y_1(0)$ and $\beta_2 = y_2(0)$.

From equation (16), we have for $u = +1$,

$$(18) \quad y_2 = - \left[\frac{(1 + \beta_2^2) - (1 - \beta_2^2) \exp(y_1 - \beta_1)}{(1 + \beta_2^2) + (1 - \beta_2^2) \exp(y_1 - \beta_1)} \right]^{1/2}$$

and from (17), we have for $u = -1$,

$$(19) \quad y_2 = - \left[\frac{\beta_2^2 + \tan \left(\frac{y_1 - \beta_1}{2} \right)}{1 - \beta_2^2 \tan \left(\frac{y_1 - \beta_1}{2} \right)} \right]^{1/2}$$

3. CONSTRUCTION OF SWITCHING CURVE

Figure 1 shows the phase plane trajectories. The solid curves are for control $u = +1$ and the dashed ones for $u = -1$, the arrows show the direction of increasing time. These trajectories originate from the initial state (β_1, β_2) and terminate to the final state $(-\infty, -1)$ for control $u = +1$ and $(+\infty, 1)$ for control $u = -1$. The two trajectories which pass through the origin $(0, 0)$ are marked AOA' or γ_+ which is for control $u = +1$ and BOB' or γ_- which is for control $u = -1$.

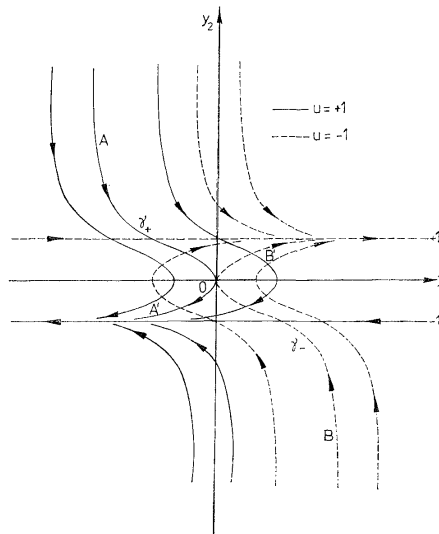


Fig. 1. The $u = +1$ and $u = -1$ forced trajectories in the $y_1 - y_2$ plane.

From equations (12), (13) and (18), (19) we have

Equation of AO:

$$(20) \quad y_2 = + \left[-\tan \frac{y_1}{2} \right]^{1/2}$$

Equation of AO':

$$(21) \quad y_2 = - \left[\frac{1 - \exp(y_1)}{1 + \exp(y_1)} \right]^{1/2}$$

Equation of BO:

$$(22) \quad y_2 = - \left[\tan \frac{y_1}{2} \right]^{1/2}$$

Equation of OB':

$$(23) \quad y_2 = + \left[\frac{1 - \exp(-y_1)}{1 + \exp(-y_1)} \right]^{1/2}$$

The switching curve is the union of segments AO and BO. From equations (20) and (22), the equation of switching curve is given by

$$(24) \quad y_2 = - \frac{y_1}{|y_1|} \left[\tan \frac{|y_1|}{2} \right]^{1/2}$$

As seen from Figure 1, for all states initially above the switching curve we must first apply control $u = +1$ and when the state reaches the switching curve then the control $u = -1$. Conversely for all states initially below the switching curve we must first apply control $u = -1$ and when the state reaches the switching curve then the control

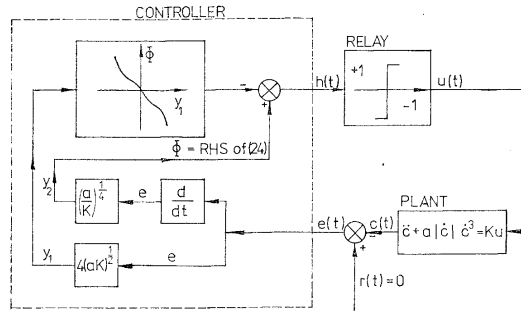


Fig. 2. Block diagram of bang-bang control of the nonlinear plant with designed controller.

$u = +1$. Hence, all the initial states (β_1, β_2) can be brought to the origin $(0, 0)$ simultaneously and in the shortest possible time by the application of one of the four possible control sequences

$$[+1], [-1], [+1, -1], [-1, +1].$$

Figure 2 shows the relay controlled plant with the designed controller.

4. CONCLUSION

In this paper the bang-bang control of a second order non-linear stable plant with fourth-order nonlinearity was discussed. Through simple transformation an equation for the switching curve was obtained which is independent of the values of constant gain (K) and the coefficient (a) of the non-linear term. Hence if $r(t) = 0$, then the designed controller will bring any output $c(t)$ and output derivative $\dot{c}(t)$ to zero simultaneously and in the shortest possible time with maximum one switching reversal of the relay.

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