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ADDITIONALS SIGNALS IN LINEAR DISCRETE-TIME CONTROL SYSTEMS II
Additional Feedback Signal

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Using the algebraic approach a linear discrete-time control system interconnected by the additional feedback is investigated. The results are presented and evaluated for a closed-loop system stability and time optimal as well as least squares optimal control synthesis.

The second part of the paper is concerned with the application of additional feedback signal (AFS) in linear discrete-time (sampled-data) control systems. The principles and fundamental applications of AFS (or additional controlled variable) have been discussed, e.g., in [3]–[6].

This part of the paper represents the independent continuation of Part I [7], but the reader is assumed to be familiar with the fundamental symbols and operations of algebraic (polynomial) theory of linear discrete-time systems ([1], [2]) summarized in Sections I and II of [7].

At first closed-loop stability of the presented system structure is investigated. Then time optimal as well as least squares optimal control is solved. Optimal reference tracking and optimal disturbance compensation are distinguished in the both cases. The conditions under which the control performance index is improved by AFS are specified and verified.

1. ADDITIONAL FEEDBACK SIGNAL IN LINEAR DISCRETE-TIME SYSTEM

The block diagram of a discrete-time (sampled-data) control system using AFS is shown in Fig. 1. Auxiliary output of the selected first part $\mathscr{S}_1$ of a controlled system is assumed to be sampled and fed back through the additional controller R_2. For simplicity, synchronous samplers preceding both digital controllers R_1 and R_2 are not pictured in Fig. 1 and a continuously operating subsystem is limited by the dashed line. All the signals outside this line are considered to be in the discrete-time forms. Possible disturbances $\mathscr{V}_1$ and $\mathscr{V}_2$ affect the system continuously. Let the resulting sampled-data effect of $\mathscr{V}_1$ after passing the corresponding part of $\mathscr{S}_1$ is denoted by V_A and a similar output effect of both $\mathscr{V}_1$ and $\mathscr{V}_2$ is represented by V_B. Then Fig. 1 can be, for discrete instants of time, replaced by an equivalent block diagram in Fig. 2 where the transfer sequences

$$(1) \qquad G = \frac{b}{a}, \quad (a, b) \sim 1, \quad \text{and} \quad G_1 = \frac{b_1}{a_1}, \quad (a_1, b_1) \sim 1,$$

represent the discrete-time mathematical model of the overall controlled system (including a data reconstructor $\mathscr{H}$, subsystems $\mathscr{S}_1$ and $\mathscr{S}_2$) and the model of the first part of the controlled system (including $\mathscr{H}$ and $\mathscr{S}_1$), respectively,

$$(2) \qquad R_1 = \frac{m_1}{n_1}, \quad (n_1, m_1) \sim 1, \quad \text{and} \quad R_2 = \frac{m_2}{n_2}, \quad (n_2, m_2) \sim 1,$$

are transfer sequences of the controllers.

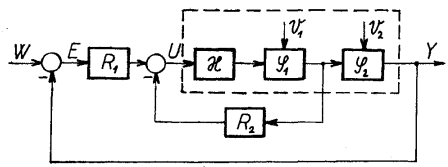

Fig. 1.

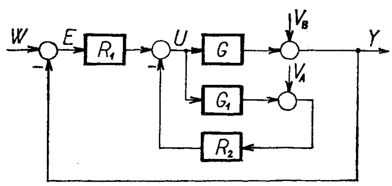

Fig. 2.

Note that controlled systems with the properties $z^{-1}|b$ and $z^{-1}|b_1$ are assumed and $a_1|a$ but generally b_1 does not divide b.

The following equations are valid for the system in Fig. 2:

$$(3) \quad Y = (1 + GR_1 + G_1R_2)^{-1} [GR_1W - GR_2V_A + (1 + G_1R_2) V_B],$$

$$(4) \quad E = (1 + GR_1 + G_1R_2)^{-1} [(1 + G_1R_2) W + GR_2V_A - (1 + G_1R_2) V_B]$$

and

$$(5) \quad U = (1 + GR_1 + G_1R_2)^{-1} (R_1W - R_2V_A - R_1V_B).$$

Using the denotations (1), (2) and

$$a_2 = \frac{a}{a_1} \tag{6}$$

the transfer matrices

$$\boldsymbol{G} = [G \ G_1] = a^{-1}[b \ a_2 b_1] \tag{7}$$

and

$$\boldsymbol{R} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} m_1 n_{20} \\ m_2 n_{10} \end{bmatrix} n_0^{-1} \tag{8}$$

can be written (in the coprime factorization forms, [1], [2]) where

$$n_{10} = \frac{n_1}{(n_1, n_2)}, \quad n_{20} = \frac{n_2}{(n_1, n_2)} \quad \text{and} \quad n_0 = (n_1, n_2)\, n_{10} n_{20}\,.$$

Then the equations (3)–(5) can be arranged into the vector-matrix form

$$\begin{bmatrix} Y \\ E \\ U \end{bmatrix} = \begin{bmatrix} K_{W/Y} & K_{V_A/Y} & K_{V_B/Y} \\ K_{W/E} & K_{V_A/E} & K_{V_B/E} \\ K_{W/U} & K_{V_A/U} & K_{V_B/U} \end{bmatrix} \begin{bmatrix} W \\ V_A \\ V_B \end{bmatrix} = \tag{9}$$

$$= (an_0 + bm_1n_{20} + a_2b_1m_2n_{10})^{-1} \times$$

$$\times \begin{bmatrix} bm_1n_{20} & -bm_2n_{10} & an_0 + a_2b_1m_2n_{10} \\ an_0 + a_2b_1m_2n_{10} & bm_2n_{10} & -(an_0 + a_2b_1m_2n_{10}) \\ am_1n_{20} & -am_2n_{10} & -am_1n_{20} \end{bmatrix} \begin{bmatrix} W \\ V_A \\ V_B \end{bmatrix}$$

and the closed-loop pseudocharacteristic polynomial

$$l = an_0 + bm_1n_{20} + a_2b_1m_2n_{10}\,. \tag{10}$$

2. CLOSED-LOOP SYSTEM STABILITY AND CAUSALITY

Theorem 1. A closed-loop system with AFS pictured in Fig. 2 and described by the relations (1)–(6) is stable and causal (physically realizable) if and only if

$$R_1 = M_1 N^{-1} \quad \text{and} \quad R_2 = M_2 N^{-1} \tag{11}$$

where M_1, M_2 and N are stable sequences which satisfy closed-loop stability (CLS) equation

$$aN + bM_1 + a_2b_1M_2 = 1 \tag{12}$$

and N^{-1} is causal.

Proof.

1. It will be proved at first that the closed-loop system is stable if and only if the closed-loop transfer sequences $K_{W/Y}$ and $K_{W/E}$ in (9) have the form

$$K_{W/Y} = bM_1 \quad \text{and} \quad K_{W/E} = aN + a_2 b_1 M_2 , \tag{13}$$

where M_1, M_2 and N are stable sequences.

a) Only if:

According to (9) and (10)

$$K_{W/Y} = bm_1 n_{20} l^{-1} \quad \text{and} \quad K_{W/E} = (an_0 + a_2 b_1 m_2 n_{10})\, l^{-1} .$$

Denoting

$$M_1 = m_1 n_{20} l^{-1} , \quad M_2 = m_2 n_{10} l^{-1} \quad \text{and} \quad N = n_0 l^{-1} \tag{14}$$

then $K_{W/Y}$ and $K_{W/E}$ stand in (13). Since a stable system pseudocharacteristic polynomial l is a stable polynomial then M_1, M_2 and N are stable sequences.

b) If:

Let us assume that M_1, M_2 and N in (13) are stable but pseudocharacteristic polynomial $l = l^+ l^-$ with $l^- \nsim 1$. Then analyzing (14) $l^- | m_1 n_{20}$, $l^- | m_2 n_{10}$ and $l^- | n_0$ must be valid at the same time. But $(m_1 n_{20}, m_2 n_{10}, n_0) \sim 1$ and hence the only $l^- \sim 1$ is allowed. Therefore stable M_1, M_2 and N ensure stability of the pseudocharacteristic polynomial which is equivalent to stability of the system.

Note that the remaining closed-loop transfer sequences

$$K_{V_A/Y} = -K_{V_A/E} = -bM_2 , \quad K_{V_B/Y} = -K_{V_B/E} = K_{W/E} ,$$
$$K_{W/U} = -K_{V_B/U} = aM_1 \quad \text{and} \quad K_{V_A/U} = -aM_2$$

are then stable too.

2. Since substituting (14)

$$aN + bM_1 + a_2 b_1 M_2 = (an_0 + bm_1 n_{20} + a_2 b_1 m_2 n_{10})\, l^{-1} = ll^{-1} = 1$$

the relation (12) is proved.

It follows from the comparison of (3)–(5) and (9) that

$$K_{W/E} R_1 = K_{W/U}(1 + G_1 R_2) \tag{15}$$

and

$$K_{W/E} R_2 = -K_{V_A/U}(1 + G_1 R_2) . \tag{16}$$

If $K_{W/E}$, $K_{W/U}$ and $K_{V_A/U}$ are substituted into (15) and (16) then

$$(aN + a_2 b_1 M_2)\, R_1 = aM_1 + a_2 b_1 M_1 R_2$$

and

$$(aN + a_2 b_1 M_2) R_2 = aM_2 + a_2 b_1 M_2 R_2 .$$

Hence R_1 and R_2 result in (11).

3. Causality of the controllers (11) needs a causal sequence N^{-1}. □

3. TIME OPTIMAL CONTROL

In the case of time optimal control (TOC) causal controllers (11) must satisfy CLS equation (12) and moreover ensure the error sequence E to be finite and as short as possible. The control sequence U is required to be either stable (stable TOC) or finite (finite TOC).

3.1. Time optimal reference tracking

Theorem 2. Given a discrete-time system with AFS pictured in Fig. 2, described by the relations (1) – (6) and subjected to the reference $W = f/h$, $(h, f) \sim 1$, $V_A = V_B = 0$, then

a) stable TOC is assured by the controllers (11) with

$$(17) \qquad N = \frac{h_{21}x}{a_{20}^+ a_{12}^+ f^+} = \frac{h_0 x}{a_0^+ f^+}, \quad M_1 = \frac{y}{b^+ f^+} \quad \text{and} \quad M_2 = \frac{h_{22} v}{a_{20}^+ b_{12}^+ f^+}$$

where

$$(18) \qquad a_{20} = \frac{a_2}{(a_2, h)}, \quad h_2 = \frac{h}{(a_2, h)},$$

$$(19) \qquad a_{12} = \frac{a_1}{(a_1, h_2)}, \quad h_{21} = \frac{h_2}{(a_1, h_2)}, \quad b_{12} = \frac{b_1}{(b_1, h_2)} \quad \text{and} \quad h_{22} = \frac{h_2}{(b_1, h_2)}.$$

The polynomials x, y, v satisfy the equation

$$(20) \qquad h\, a_{20}^-(a_{12}^- x + b_{12}^- v) + b^- y = f^+$$

in such a way that x is causal and $\deg (a_{12}^- x + b_{12}^- v)$ attains its minimum.
The error sequence (polynomial)

$$(21) \qquad E = e = a_{20}^- f^-(a_{12}^- x + b_{12}^- v) = f^-(a_0^- x + a_{20}^- b_{12}^- v)$$

with

$$(22) \qquad \deg e < \deg a_{20}^- + \deg f^- + \deg b^- ,$$

the control sequence

$$(23) \qquad U = \frac{a_{12} a_{20} f^- y}{h_{21} b^+} = \frac{a_0 f^- y}{h_0 b^+}$$

where

$$a_0 = \frac{a}{(a, h)} \quad \text{and} \quad h_0 = \frac{h}{(a, h)} . \tag{24}$$

The optimal solution exists if and only if $h_0 = h_{21}$ is stable. Optimal controllers R_1 and R_2 are not unique while the resulting optimal error (21) is given unambiguously.

b) finite TOC is attained by the controllers (11) with

$$N = \frac{h_{21}x}{a_{20}^+ a_{12}^+ f^+} = \frac{h_0 x}{a_0^+ f^+}, \quad M_1 = \frac{y}{f^+} \quad \text{and} \quad M_2 = \frac{h_{22}v}{a_{20}^+ b_{12}^+ f^+} \tag{25}$$

where a_{20}, h_2, a_{12}, h_{21}, b_{12} and h_{22} are given by (18) and (19) and the polynomials x, y, v represent the solution of the equation

$$h\, a_{20}^-(a_{12}^- x + b_{12}^- v) + by = f^+ \tag{26}$$

with x causal and min deg $(a_{12}^- x + b_{12}^- v)$.

The error polynomial is given by (21) with

$$\deg e < \deg a_{20}^- + \deg f^- + \deg b \tag{27}$$

and the control sequence

$$U = \frac{a_{20}a_{12}f^- y}{h_{21}} = \frac{a_0 f^- y}{h_0} \tag{28}$$

where a_0 and h_0 stand in (24).

The optimal solution exists if and only if $h_0 \sim 1$. Optimal controllers are not unique while the resulting optimal error is given unambiguously.

Proof. Writing CLS equation (12) in the form

$$a_2(a_1 N + b_1 M_2) = 1 - bM_1$$

and multiplying both its sides by $W = f/h$ then

$$a_2 L \frac{f}{h} = \frac{f}{h} - bM_1 \frac{f}{h} = E \tag{29}$$

where

$$L = a_1 N + b_1 M_2 . \tag{30}$$

Since N and M_2 are stable sequences then L in (30) must be stable too. The error sequence is required to be polynomial $E = e$ and therefore the optimal choice of L considering (18) is

$$L = \frac{h_2 s}{a_{20}^+ f^+} \tag{31}$$

where polynomial s is undetermined till now.

Hence the resulting error

$$e = a_{20}^{-} f^{-} s \tag{32}$$

From equation (29)

$$f - he = bM_1 f \tag{33}$$

must be a polynomial, too.

a) Therefore optimal stable M_1 stands in (17). Substituting L and M_1 into (29) then

$$h\, a_{20}^{-} s + b^{-} y = f^{+} . \tag{34}$$

Equation (34) is solvable provided that $(h, b^{-}) \sim 1$. Among all solutions of (34) the unique min deg s solution s, y is the optimal one, cf. (32).

Since the sequence L must be realized in the system structure through N and M_2 equation (30) with L standing in (31)

$$\frac{a_{20}^{+} a_1 f^{+}}{h_2} N + \frac{a_{20}^{+} b_1 f^{+}}{h_2} M_2 = s \tag{35}$$

must be always solvable for any s resulting from (34). The choice (17) of N and M_2 satisfies this condition since the resulting equation

$$a_{12}^{-} x + b_{12}^{-} v = s \tag{36}$$

is always solvable due to $(a_{12}^{-}, b_{12}^{-}) \sim 1$. All solutions x, v of (36) are allowed, therefore N and M_2 as well as R_1 and R_2 are not unique.

The control sequence resulting from (5) stands in (23). Since

$$a_2 = a_{20} h_1 \quad \text{and} \quad h = h_2 h_1 \quad \text{where} \quad h_1 = (a_2, h)$$

then

$$h_0 = \frac{h}{(a_1 a_2, h)} = \frac{h_2 h_1}{(a_1 a_{20} h_1, h_2 h_1)} = \frac{h_2}{(a_1, h_2)} = h_{21} \tag{37}$$

and

$$a_0 = \frac{a_1 a_2}{(a_1 a_2, h)} = \frac{a_1 a_{20} h_1}{(a_1 a_{20} h_1, h_2 h_1)} = a_{20} \frac{a_1}{(a_1, h_2)} = a_{20} a_{12} . \tag{38}$$

Then both the forms of U in (23) as well as of N in (17) and e in (21) are identical.

Analyzing (23) stable TOC is solvable if and only if $h_0 = h_{21}$ is stable. Then the equation (34) is always solvable since $(a_{20}^{-} h, b^{-}) \sim (h_0, b^{-}) \sim 1$. Its min deg s solution s, y possesses the property $\deg s < \deg b^{-}$ and hence (22) is valid.

Equations (34) and (36) can be combined into the only equation (20).

b) The choice (17) of M_1 does not ensure a finite control sequence (23). It follows from (23) and (33) that M_1 must be according to (25) and equation (34) changes into

(39) $$h a_{20}^{-} s + b y = f^{+} .$$

The stable sequences L, N and M_2 as well as equation (36) stay unchanged.

The error polynomial has the form (21) or (32) where $s = a_{12}^{-} x + b_{12}^{-} v$ belongs to min deg s solution s, y of (39). The resulting control sequence U standing in (28) is finite if and only if $h_0 = h_{21} \sim 1$. In this case the equation (39) is always solvable and deg $s <$ deg b; hence (27) is valid. Combining equations (39) and (36) the only equation (26) can be written and solved. □

3.2. Time optimal disturbance compensation

If disturbances $\mathscr{V}_1$ and $\mathscr{V}_2$ affect the system according to Fig. 1 then generally $V_A \neq 0$ and $V_B \neq 0$ in Fig. 2 and this block diagram can be transformed into equivalent Fig. 3.

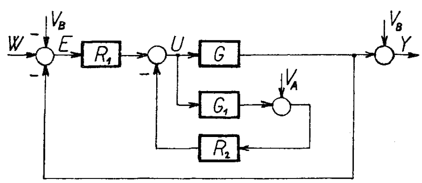

Fig. 3.

Provided $\mathscr{V}_1 = 0$ then $V_A = 0$ and the only modified reference signal $W_1 = W - V_B$ may be considered for the optimal control design. Putting $W_1 = f/h$ all the relations and results of Theorem 2 are valid unchanged.

The problem becomes rather complicated if $\mathscr{V}_1 \neq 0$ and consequently $V_A \neq 0$. The following theorem describes the solution of this case.

Theorem 3. Given a discrete-time system with AFS pictured in Fig. 2, described by the relations (1)–(6) and subjected to the inputs W, V_A and V_B where

$$W - V_B = W_1 = \frac{f}{h}, \quad (h, f) \sim 1, \quad V_A = \frac{p}{q}, \quad (q, p) \sim 1,$$

then

a) stable TOC is satisfied by the controllers (11) with

(40) $$N = \frac{h_0 x}{a_0^+ f^+}, \quad M_1 = \frac{y}{b^+ f^+ d_h^+} \quad \text{and} \quad M_2 = \frac{kv}{g^+(f, p)^+}.$$

Here a_0 and h_0 are given by (24), k and g follow from the relation

(41) $$\frac{a_2 b_1 f_p}{h} + \frac{b p_f}{q} = \frac{g}{k}, \quad (k, g) \sim 1; g \neq 0 \quad \text{is assumed,}$$

where

(42) $$f_p = \frac{f}{(f, p)} \quad \text{and} \quad p_f = \frac{p}{(f, p)},$$

(43) $$d_h = \frac{d}{(d, h)} \quad \text{and} \quad h_d = \frac{h}{(d, h)}$$

if

(44) $$\frac{bkp_f}{g^+ q} = \frac{c}{d}; \quad (d, c) \sim 1 .$$

The triplet of polynomials x, y, v is the solution of the equation

(45) $$hd_h a_0^- f_p^- x + b^- d_h^- f_p^- y + h_d(dg^- - c) v = d_h f^+ f_p^-$$

with x causal and min deg $(a_0^- f_p^- x + g^- v)$.

The error polynomial

(46) $$e = (f, p)^- (a_0^- f_p^- x + g^- v)$$

and

(47) $$\deg e = \deg (f, p)^- + \deg (a_0^- f_p^- x + g^- v) .$$

The control sequence

(48) $$U = \frac{a_0 f^- y}{b^+ d_h^+ h_0} - \frac{akp_f^+ p^- v}{g^+ q} .$$

The optimal solution exists if and only if $h_0 \sim h_0^+$ and $q_{ak} = q/(ak, q) \sim q_{ak}^+$ and it is not generally unique. If the unique solution exists optimal error as well as optimal controllers are unique.

b) finite TOC is satisfied by the controllers (11) with

(49) $$N = \frac{h_0 x}{a_0^+ f^+}, \quad M_1 = \frac{y}{f^+} \quad \text{and} \quad M_2 = \frac{kv}{(f, p)^+}$$

where h_0 and a_0 stand in (24) and $g \neq 0$ and k follow from (41). The triplet of polynomials x, y, v is the solution of the equation

(50) $$hd_h a_0^- f_p^- x + bd_h f_p^- y + h_d(dg - c) v = d_h f^+ f_p^-$$

with x causal and min deg $(a_0^- f_p^- x + gv)$ where

(51) $$\frac{bkp_f}{q} = \frac{c}{d}, \quad (d, c) \sim 1 ,$$

and d_h, h_d are according to (43).

The error polynomial

$$e = (f, p)^- (a_0^- f_p^- x + gv) \tag{52}$$

with

$$\deg e = \deg (f, p)^- + \deg (a_0^- f_p^- x + gv) . \tag{53}$$

The control sequence (polynomial)

$$U = u = \frac{a_0 f^- y}{h_0} - \frac{akp_f^+ p^- v}{q} . \tag{54}$$

The optimal solution exists if and only if $h_0 \sim 1$ and $q_{ak} = q/(ak, q) \sim 1$ and it is not generally unique. The unique solution results in unique error and optimal controllers.

Proof. Starting with equation (4) and using the substitution (41) the error signal is given by

$$E = \frac{a_0 f}{h_0} N + (f, p) \frac{g}{k} M_2 = (f, p)(a_0 f_p, g) L \tag{55}$$

where

$$L = \frac{a_0 f_p}{(a_0 f_p, g) h_0} N + \frac{g}{(a_0 f_p, g) k} M_2 = \frac{s}{l_0} . \tag{56}$$

Writing (56) in the other form

$$h_0 k L = \frac{a_0 f_p k}{(a_0 f_p, g)} N + \frac{g h_0}{(a_0 f_p, g)} M_2$$

both sides of this equation must be stable sequences as N and M_2 are required to be stable.

At the same time the error must be polynomial $E = e$ and therefore from (55) the property $l_0 | (f, p)(a_0 f_p, g)$ is necessary. Since $(h_0, (f, p)(a_0 f_p, g)) \sim 1$ as well as $(k, (f, p)(a_0 f_p, g)) \sim 1$ (cf. (41) $k|h$, $k|q$) the optimal choice of L is

$$L = \frac{s}{(f, p)^+ (a_0 f_p, g)^+} \tag{57}$$

where polynomial s is undetermined till now.
Then

$$e = (f, p)^- (a_0 f_p, g)^- s . \tag{58}$$

But any s and L must be realized through stable N and M_2 in such a way that the equation

$$\frac{a_0 f_p (f, p)^+}{h_0} N + \frac{g (f, p)^+}{k} M_2 = (a_0 f_p, g)^- s \tag{59}$$

resulting directly from (56) and (57) is always solvable for any s.

a) The choice (40) of N and M_2 satisfies this requirement since it results in the equation

$$a_0^- f_p^- x + g^- v = (a_0 f_p, g)^- s \tag{60}$$

which is always solvable. Hence comparing (60) and (58) e stands in (46).

Applying CLS equation (12) the error signal can be also expressed in the form

$$e = W_1 - bM_1 W_1 + bM_2 V_A = \frac{f}{h} - bM_1 \frac{f}{h} + \frac{c(f, p)^- v}{d} \tag{61}$$

where M_2 and the denotation (44) have been substituted. Hence

$$f - he = bM_1 f - \frac{ch_d(f, p)^- v}{d_h}$$

with h_d and d_h given by (43) and therefore

$$d_h f - hd_h e + ch_d(f, p)^- v = bd_h f M_1 \tag{62}$$

must be a polynomial too. Consequently optimal M_1 stands in (40); substituting M_1 and e given by (46) into equation (62) we obtain the form (45).

The control sequence follows from (5) and results in (48). Obviously it is stable if and only if $h_0 \sim h_0^+$ and $q_{ak} = q/(ak, q) \sim q_{ak}^+$.

In this case $(h, b^-) \sim 1$ and equation (45) is always solvable as

$$(hd_h a_0^- f_p^-, b^- d_h^- f_p^-, h_d(dg^- - c)) \sim (d_h^- f_p^-, h_d(dg^- - c)) \sim$$
$$\sim (f_p^-, dg^- - c) \quad \text{and} \quad (f_p^-, dg^- - c) | d_h f^+ f_p^- .$$

General solution of (45) must be usually analyzed to find the solution with min deg $(a_0^- f_p^- x + g^- v)$. Writing the general solution in the form

$$x = x_0 + b^- t_1 + \frac{dg^- - c}{(a_0^- f_p^-, dg^- - c)} t_2 , \tag{63}$$

$$y = y_0 - hd_h^+ a_0^- t_1 \quad \text{and} \quad v = v_0 - d \frac{a_0^- f_p^-}{(a_0^- f_p^-, dg^- - c)} t_2$$

where x_0, y_0, v_0 is a particular solution and t_1 and t_2 any arbitrary polynomials then

$$a_0^- f_p^- x + g^- v = a_0^- f_p^- x_0 + g^- v_0 + a_0^- f_p^- b^- t_1 - c \frac{a_0^- f_p^-}{(a_0^- f_p^-, dg^- - c)} t_2 . \tag{64}$$

The suitable choice of t_1 and t_2 gives optimal solution with min deg $(a_0^- f_p^- x + g^- v)$. If this choice is unique then x, y, v and consequently e, R_1 and R_2 are given unambiguously.

b) The choice (40) of M_1 and M_2 does not ensure finite control sequence U. Analyzing (48) and (59) M_2 must be according to (49) and equation (60) changes into the form

$$a_0^- f_p^- x + gv = (a_0 f_p, g)^- s \tag{65}$$

which is always solvable for any s.

Applying CLS equation (12)

$$e = \frac{f}{h} - bM_1 \frac{f}{h} + \frac{c(f, p)^- v}{d}$$

where M_2 and the denotation (51) have been substituted. Hence (62) can be written but stable M_1 must be chosen according to (49). Then substituting into (62) equation (50) is obtained.

The resulting control sequence U stands in (54). It has a polynomial form if and only if $h_0 \sim 1$ and $q_{ak} = q/(ak, q) \sim 1$.
In this case $(h, b) \sim 1$ and equation (50) is always solvable. The special min deg $(a_0^- f_p^- x + gv)$ solution of (50) can be usually determined through the general solution

$$x = x_0 + bt_1 + \frac{dg - c}{(a_0^- f_p^-, dg - c)} t_2, \quad y = y_0 - ha_0^- t_1, \tag{66}$$

$$v = v_0 - d \frac{a_0^- f_p^-}{(a_0^- f_p^-, dg - c)} t_2$$

only and

$$a_0^- f_p^- x + gv = a_0^- f_p^- x_0 + gv_0 + a_0^- f_p^- bt_1 - c \frac{a_0^- f_p^-}{(a_0^- f_p^-, dg - c)} t_2$$

where x_0, y_0, v_0 is a particular solution of (50). The choice of the polynomials t_1 and t_2 must assure min deg $(a_0^- f_p^- x + gv)$; if this choice is unique then x, y, v and consequently e, R_1 and R_2 are given unambiguously. □

Note. The special case $g = 0$ in (41) has been excluded in Theorem 3. If this case occurs then

$$E = aN \frac{f}{h} = \frac{f}{h} - bM_1 \frac{f}{h} + bM_2 \frac{p}{q}. \tag{67}$$

Optimal stable N stands in (40) or (49) and

$$e = a_0^- f^- x. \tag{68}$$

If $g = 0$ then $bp/q = -a_2 b_1 f/h$ and

$$f^+ - ha_0^- x = (b, b_1) f^+ L \tag{69}$$

where

$$L = \frac{b}{(b, b_1)} M_1 + \frac{a_2 b_1}{(b, b_1)} M_2 . \tag{70}$$

a) Solving stable TOC optimal stable

$$L = \frac{r}{(b, b_1)^+ f^+} \tag{71}$$

and equation (69) has the form

$$h a_0^- x + (b, b_1)^- r = f^+ . \tag{72}$$

Provided $(h, b^-, b_1^-) \sim 1$ equation (72) is solvable and its min deg x solution $x, \boldsymbol{r}$ with $\deg x < \deg (b, b_1)^-$, x causal, is the optimal one. Substituting (71) into equation (70) and choosing

$$M_1 = \frac{y}{b^+ f^+} \quad \text{and} \quad M_2 = \frac{v}{a_2^+ b_1^+ f^+} \tag{73}$$

the equation

$$b^- y + a_2^- b_1^- v = (b, b_1)^- r \tag{74}$$

is obtained and always solvable for any r.

The control sequence

$$U = \frac{a_0 f^- y}{b^+ h_0} - \frac{a p v}{a_2^+ b_1^+ f^+ q} = \frac{a_0 f^- y}{b^+ h_0} + \frac{a_2^- b_1^- a_0 f^- v}{b h_0} = \tag{75}$$

$$= \frac{a_0 f^- (b, b_1)^- r}{b h_0} = \frac{a_0 f^- r}{b^+ h_0 b_0^-}$$

where

$$b_0 = \frac{b}{(b, b_1)} \tag{76}$$

and the problem is solvable if $h_0 \sim h_0^+$ and $b_0^- | f^-$.

b) In the case of finite TOC optimal stable $L = r/f^+$ must be chosen in (69) and hence the equation

$$h a_0^- x + (b, b_1) r = f^+ \tag{77}$$

is solved for min deg x; x causal, $\deg x < \deg (b, b_1)$.

If

$$M_1 = \frac{y}{f^+} \quad \text{and} \quad M_2 = \frac{v}{a_2^+ f^+} \tag{78}$$

the resulting equation

$$(79) \qquad by + a_2^- b_1 v = (b, b_1)\, r$$

is always solvable.

The resulting control sequence

$$(80) \qquad U = \frac{a_0 f^- y}{h_0} - \frac{apv}{a_2^+ f^+ q} = \frac{a_0 f^- y}{h_C} + \frac{a_0 a_2^- b_1 f^- v}{b h_0} = \frac{a_0 f^- r}{b_0 h_0}$$

and the problem is solvable if $h_0 \sim 1$ and $b_0 | f^-$.

4. LEAST SQUARES CONTROL

In the case of least squares control (LSC) the minimum value of $\sigma_E = \|E\|^2$ must be attained by optimal causal controllers (11) which satisfy CLS equation (12) at the same time. The control sequence U is required to be stable.

4.1. Least squares reference tracking

Theorem 4. Given a discrete-time system with AFS pictured in Fig. 2, described by the relations (1) – (6) and subjected to the reference $W = f/h$, $(h, f) \sim 1$, $V_A = V_B = 0$, then LSC is ensured by the controllers (11) where

$$(81) \qquad N = \frac{h_{21} x}{a_{12}^+ a_{20}^* f^* b^{-\sim}}, \quad M_1 = \frac{y}{b^* f^* a_{20}^{-\sim}} \quad \text{and} \quad M_2 = \frac{h_{22} v}{b_{12}^+ a_{20}^* f^* b^{-\sim}}$$

with a_{20}, h_{21}, a_{12}, b_{12} and h_{22} given by (18) and (19).

The polynomials x, y, v represent the solution of the equation

$$(82) \qquad h\, a_{20}^-(a_{12}^- x + b_{12}^- v) + b^- y = b^{-\sim} f^* a_{20}^{-\sim}$$

with $\deg(a_{12}^- x + b_{12}^- v) < \deg b^-$, x causal.

The optimal error sequence

$$(83) \qquad E = \frac{a_{20}^- f^-}{a_{20}^{-\sim} f^{-\sim} b^{-\sim}} (a_{12}^- x + b_{12}^- v),$$

the control sequence

$$(84) \qquad U = \frac{a_0 f^- y}{b^* f^{-\sim} a_{20}^{-\sim} h_0}$$

and the optimal control performance index

$$(85) \qquad \sigma_{E_{\min}} = \left\langle \frac{\bar{s}}{\bar{b}^-} \frac{s}{b^-} \right\rangle$$

where $s = a_{12}^- x + b_{12}^- v$.

The optimal solution exists if and only if $h_0 \sim h_0^+$. Optimal controllers are not unique while the resulting optimal error sequence is given unambiguously.

Proof. Provided $V_A = V_B = 0$ any stable error sequence $E = W - bM_1W$. Denoting $E^* = W^* - bM_1W^*$ where $W^* = f^*/h$ then

$$E = E^* \frac{f^-}{f^{-\sim}}, \tag{86}$$

$\bar{E}E = \overline{E^*}E^*$ and hence $\sigma_E = \langle \overline{E^*}E^* \rangle$.
Using (86) we can write

(87)

$$\overline{E^*}E^* = (\overline{W^*} - \overline{bM_1W^*})(W^* - bM_1W^*) = (\bar{Z} - \overline{b^*M_1W^*})(Z - b^*M_1W^*)$$

and the identities

$$b^*\bar{Z} = b\overline{W^*}, \quad \overline{b^*}Z = \bar{b}W^* \quad \text{and} \quad \bar{Z}Z = \overline{W^*}W^*$$

follow from the comparison of the multiplied terms in (87) seeing that $c\bar{c} = c^*\overline{c^*} = c^\sim\overline{c^\sim}$ for any polynomial c. Hence (87) can be rewritten into the form

$$\overline{E^*}E^* = \left(\frac{b}{b^*}\overline{W^*} - \overline{b^*M_1W^*}\right)\frac{\overline{c^\sim}}{\bar{c}}\left(\frac{\bar{b}}{\overline{b^*}}W^* - b^*M_1W^*\right)\frac{c^\sim}{c} = \bar{E}_0E_0$$

where

$$E_0 = \frac{\bar{b}c^\sim}{\overline{b^*}c}W^* - b^*\frac{c^\sim}{c}M_1W^* = \frac{b^{-\sim}c^\sim f^*}{b^-ch} - b^*\frac{c^\sim f^*}{ch}M_1 \tag{88}$$

with a polynomial c undetermined till now.

Obviously $\sigma_E = \langle \bar{E}_0E_0 \rangle$ too. If the decomposition

$$\frac{b^{-\sim}c^\sim f^*}{b^-ch} = \frac{s}{b^-} + \frac{y}{ch} \tag{89}$$

and the denotation

$$X = \frac{y}{ch} - b^*\frac{c^\sim f^*}{ch}M_1\,. \tag{90}$$

are used in (88) then

$$E_0 = \frac{s}{b^-} + X \tag{91}$$

and

$$\sigma_E = \left\langle\left(\frac{\bar{s}}{\overline{b^-}} + \bar{X}\right)\left(\frac{s}{b^-} + X\right)\right\rangle. \tag{92}$$

The relation (89) results in the polynomial equation

$$hcs + b^- y = b^{-\sim} c^\sim f^* \tag{93}$$

the general solution of which can be written as

$$s = s_2 - \frac{b^-}{(hc, b^-)}\, t\,, \quad y = y_2 + \frac{hc}{(hc, b^-)}\, t \tag{94}$$

where s_2, y_2 is the particular solution with $\deg s_2 < \deg b^-$ and t is an arbitrary polynomial.

Since $\bar{m} = m^\sim z^{\deg m}$ and $m = \overline{m^\sim} z^{-\deg m}$ for any polynomial m then

$$\frac{\bar{s}_2}{\bar{b}^-} = \frac{s_2^\sim}{b^{-\sim}}\, z^{-v} \quad \text{and} \quad \frac{s_2}{b^-} = \frac{\overline{s_2^\sim}}{\overline{b^{-\sim}}}\, z^v \tag{95}$$

where

$$v = \deg b^- - \deg s_2 > 0\,.$$

Therefore substituting s given by (94) into (92) this expression is reduced into

$$\sigma_E = \left\langle \frac{\bar{s}_2 s_2}{\bar{b}^- b^-} \right\rangle + \left\langle \left(\bar{X} - \frac{\bar{t}}{\overline{(hc, b^-)}} \right) \left(X - \frac{t}{(hc, b^-)} \right) \right\rangle \tag{96}$$

and $\sigma_{E_{\min}}$ given by (85) is attained if $s = s_2$ and $X = t/(hc, b^-)$. Then the equation (90) can be rewritten into the form

$$\frac{y_2}{hc} + \frac{t}{(hc, b^-)} - b^* \frac{c^\sim f^*}{hc} M_1 - \frac{t}{(hc, b^-)} = 0$$

and the optimal stable sequence

$$M_1 = \frac{y}{b^* f^* c^\sim} \quad \text{with} \quad y = y_2\,. \tag{97}$$

Substituting M_1 into (86) and using equation (93)

$$E^* = \frac{f^*}{h} - \frac{by}{b^* c^\sim h} = \frac{cs}{c^\sim b^{-\sim}}\,. \tag{98}$$

According to (9) we can write

$$E^* = aN \frac{f^*}{h} + a_2 b_1 M_2 \frac{f^*}{h} = \frac{a_{20} f^*}{h_2} L \tag{99}$$

where a_{20} and h_2 stand in (18) and

$$L = a_1 N + b_1 M_2\,. \tag{100}$$

Comparing (99) and (98) the optimal sequence

$$L = \frac{h_2 cs}{c^\sim b^{-\sim} a_{20} f^*}\,.$$

Since L must be stable the choice $c = a_{20}^-$ is necessary. Then

$$L = \frac{h_2 s}{a_{20}^* f^* b^{-\sim}} \tag{101}$$

and equation (93) takes the form

$$h\, a_{20}^- s + b^- y = b^{-\sim} f^* a_{20}^{-\sim} \,. \tag{102}$$

The sequence L must be realized by N and M_2 according to (100). Substituting L into equation (100) we obtain the form

$$a_1 \frac{a_{20}^* f^* b^{-\sim}}{h_2} N + b_1 \frac{a_{20}^* f^* b^{-\sim}}{h_2} M_2 = s \,.$$

The choice (81) of N and M_2 results in the equation

$$a_{12}^- x + b_{12}^- v = s \tag{103}$$

which is always solvable for any s. General solution of (103) is allowed and therefore N and M_2 as well as R_1 and R_2 are not unique.

Considering (103) the resulting error sequence E stands in (83) and the control sequence U in (84). Obviously $h_0 \sim h_0^+$ is only allowed; in this case equation (102) is always solvable and its particular solution s, y with $\deg s < \deg b^-$ is unique and identical with min deg s solution. Therefore the resulting error (83) is given unambiguously.

Equations (103) and (102) can be combined into the only equation (82).

4.2. LEAST SQUARES DISTURBANCE COMPENSATION

In accordance with the consideration given in Section 3.2 the case $V_A \neq 0\ (\mathscr{V}_1 \neq 0)$ will be teated separately.

Theorem 5. Given a discrete-time system with AFS pictured in Fig. 2, described by the relations (1)–(6) and subject to the inputs W, V_A and V_B where $W - V_B = W_1 = f/h$, $(h, f) \sim 1$, $V_A = p/q$, $(q, p) \sim 1$, then LSC is attained by the controllers (11) with

$$N = \frac{h_0 x}{a_0^+ f_p^+ (bf_p, c)^{-\sim} (f, p)^* (a_0 f_p, g)^{-\sim}}\,, \tag{104}$$

$$M_1 = \frac{y}{b^+ f_p^+ d_h^+ (bf_p, c)^{-\sim} (f, p)^* (a_0 f_p, g)^{-\sim}}$$

and

$$M_2 = \frac{kv}{g^+(bf_p, c)^{-\sim} (f, p)^* (a_0 f_p, g)^{-\sim}} .$$

There are a_0, h_0 given by (24), f_p and p_f by (42), $g \neq 0$ and k by (41) and c, d_h and h_d follow from (43) and (44). The triplet of the polynomials x, y, v is the solution of the equation

(105) $$hd_h a_0^- f_p^- x + b^- f_p^- d_h^- y + h_d(dg^- - c) v = \\ = d_h(bf_p, c)^{-\sim} (a_0 f_p, g)^{-\sim} (f, p)^* f_p$$

with x causal and

(106) $$\deg (a_0^- f_p^- x + g^- v) < \deg (bf_p, c)^- + \deg (a_0 f_p, g)^- - \deg (f_p, g, c)^- .$$

Optimal error sequence

(107) $$E = \frac{(f, p)^- (a_0^- f_p^- x + g^- v)}{(f, p)^{-\sim} (a_0 f_p, g)^{-\sim} (bf_p, c)^{-\sim}} ,$$

control sequence

(108) $$U = \frac{1}{(bf_p, c)^{-\sim} (f, p)^{-\sim} (a_0 f_p, g)^{-\sim}} \left(\frac{a_0 f^- y}{b^+ d_h^+ h_0} - \frac{akp_f^+ p^- v}{g^+ q} \right)$$

and the optimal control performance index

(109) $$\sigma_{E_{\min}} = \left\langle \frac{\bar{s}}{\overline{(bf_p, c)^-}} \frac{s}{(bf_p, c)^-} \right\rangle$$

where

(110) $$s = \frac{a_0^- f_p^- x + g^- v}{(a_0 f_p, g)^-} .$$

The optimal solution exists if and only if $h_0 \sim h_0^+$ and $q_{ak} = q/(ak, q) \sim q_{ak}^+$. It is not generally unique. In the case of the unique solution all the optimal error E and controllers R_1 and R_2 are given unambiguously.

Proof. According to (9)–(12)

$$E = W_1 - bM_1 W_1 + bM_2 V_A = \frac{(f, p)}{h} \left(f_p - bf_p M_1 + \frac{g^+ ch_d}{kd_h} M_2 \right) = \\ = \frac{(f, p)}{h} (f_p - mF)$$

where the relations (44) and (43) and the denotations

(111) $$m = (bf_p, g^+ch_d)$$

and

(112) $$F = \frac{bf_p}{m} M_1 - \frac{g^+ch_d}{kd_h m} M_2$$

have been used.

Let us define $W_p^* = (f, p)^*/h$ and

(113) $$E^* = W_p^*(f_p - mF) = E \frac{(f, p)^{-\sim}}{(f, p)^-} .$$

Hence

$$\overline{E}E = \overline{E^*}E^* = \overline{W_p^*}(\overline{f}_p - \overline{m}\overline{F})\, W_p^*(f_p - mF) = (\overline{Z} - \overline{m^*}\,\overline{W_p^*}\overline{F})\,(Z - m^*W_p^*F)$$

where a sequence Z satisfies the identities

$$\overline{Z}Z = \overline{f}_p f_p \overline{W_p^*} W_p^* , \quad m^*\overline{Z} = m\overline{f}_p \overline{W_p^*} \quad \text{and} \quad \overline{m^*}Z = \overline{m} f_p W_p^* .$$

Then

$$\overline{E^*}E^* = \left(\frac{m}{m^*}\overline{f}_p \overline{W_p^*} - \overline{m^*}\,\overline{W_p^*}\overline{F}\right)\frac{\overline{n^\sim}}{\overline{n}}\left(\frac{\overline{m}}{\overline{m^*}} f_p W_p^* - m^* W_p^* F\right)\frac{n^\sim}{n} = \overline{E_0}E_0$$

where

(114) $$E_0 = \frac{m^{-\sim} n^\sim (f, p)^* f_p}{m^- nh} - m^* \frac{(f, p)^* n^\sim}{nh} F$$

an n is a polynomial undetermined till now. The decomposition

$$\frac{m^{-\sim} n^\sim (f, p)^* f_p}{m^- nh} = \frac{s}{m^-} + \frac{r}{nh}$$

results in the equation

(115) $$hns + m^- r = m^{-\sim} n^\sim (f, p)^* f_p .$$

If we denote

(116) $$X = \frac{r}{nh} - m^* \frac{(f, p)^* n^\sim}{nh} F$$

then

$$\sigma_E = \langle \overline{E_0}E_0 \rangle = \left\langle \left(\frac{\overline{s}}{\overline{m^-}} + \overline{X}\right)\left(\frac{s}{m^-} + X\right)\right\rangle .$$

Applying the results proved in Theorem 4 we can determine

$$\sigma_{E_{\min}} = \left\langle \frac{\bar{s}}{\overline{m^-}} \frac{s}{m^-} \right\rangle$$

and

(117) $$F = \frac{r}{m^*(f, p)^* n^\sim}$$

where s, r is the solution of the equation (115) with

(118) $$\deg s < \deg m^- = \deg (bf_p, ch_d)^- .$$

Then using (113)

(119) $$E^* = \frac{(f, p)^*}{h}\left(f_p - \frac{mr}{m^*(f, p)^* n^\sim}\right) = \frac{ns}{n^\sim m^{-\sim}} .$$

The relation

(120) $$E^* = (f, p)^* \left(\frac{a_0 f_p}{h_0} N + \frac{g}{k} M_2\right) = (f, p)^* (a_0 f_p, g) L$$

is also valid according to (12). Comparison of (120) and (119) gives

$$L = \frac{ns}{n^\sim m^{-\sim} (f, p)^* (a_0 f_p, g)}$$

and stability of L needs the choice $n = (a_0 f_p, g)^-$. Then

$$L = \frac{s}{(a_0 f_p, g)^* (f, p)^* m^{-\sim}} , \quad F = \frac{r}{(a_0 f_p, g)^{-\sim} (f, p)^* m^*}$$

and equation (115) obtains the form

(121) $$h(a_0 f_p, g)^- s + m^- r = m^{-\sim}(a_0 f_p, g)^{-\sim}(f, p)^* f_p .$$

But the sequences F and L can be realized through M_1, M_2 and N only. Therefore s and r in (121) must satisfy the additional requirements.
Substituting the resulting L into (120) and choosing stable N and M_2 according to (104) if $(b, h_d)^- \sim 1$ is assumed in advance, then the equation

(122) $$(a_0 f_p, g)^- s = a_0^- f_p^- x + g^- v$$

is obtained.
To determine remaining M_1 we substitute M_2 and F into equation (112). Resulting M_1 stands in (104) and the equation

(123) $$d_h m^- r = b^- f_p^- d_h^- y - ch_d v$$

is valid.

All the equations (122), (123) and (121) must be satisfied for the polynomials s, r, x, y and v. If (122) and (123) are substituted into equation (121) then the equation

(124)
$$hd_h(a_0^- f_p^- x + g^- v) + b^- f_p^- d_h^- y - ch_d v = d_h m^{-\sim}(a_0 f_p, g)^{-\sim} (f, p)^* f_p$$

is obtained.

The condition (118) is transformed into

(125) $$\deg (a_0^- f_p^- x + g^- v) < \deg m^- + \deg (a_0 f_p, g)^-$$

and therefore the solution x, y, v of (124) with the property (125) must be found assuming (124) is solvable.

Then the resulting error stands in (107), the control sequence results in (108) and it is stable if and only if $h_0 \sim h_0^+$ and $q_{ak} = q/(ak, q) \sim q_{ak}^+$. Then, really, $(b, h_d)^- \sim 1$, $m^- = (bf_p, c)^-$, equation (124) obtains the final form (105) and is always solvable seeing that $(hd_h a_0^- f_p^-, b^- f_p^- d_h^-, h_d(dg^- - c)) \sim (f_p, g, c)^-$ and $(f_p, g, c)^- | d_h(bf_p, c)^{-\sim} (a_0 f_p, g)^{-\sim} (f, p)^* f_p$.

Since generally $(f_p, g, c)^- \nsim 1$ the condition (125) can be written in the final form (106).

The special solution of (105) with the property (106) must be usually determined by means of general solution which has also the form (63). If the choice of t_1 and t_2 is unique then error sequence as well as controllers are given unambiguously. In the other case the solution of the problem is not unique. □

Note. The special case $g = 0$ excluded in Theorem 5 must be solved separately with the following results:

$$N = \frac{h_0 x}{a_0^* f^*(b, b_1)^{-\sim}}, \quad M_1 = \frac{y}{b^+ a_0^{-\sim} f^*(b, b_1)^{-\sim}} \quad \text{and}$$

$$M_2 = \frac{v}{a_2^+ b_1^+ a_0^{-\sim} f^*(b, b_1)^{-\sim}},$$

$$E = \frac{a_0^- f^- x}{a_0^{-\sim} f^{-\sim} (b, b_1)^{-\sim}}, \quad U = \frac{a_0 f^- y}{b^+ h_0 (b, b_1)^{-\sim}} + \frac{a_0 a_2^- b_1^- f^- v}{bh_0 (b, b_1)^{-\sim} a_0^{-\sim} f^{-\sim}}$$

and

$$\sigma_{E_{\min}} = \left\langle \frac{\bar{x}}{\overline{(b, b_1)^-}} \, \frac{x}{(b, b_1)^-} \right\rangle$$

where the triplet of polynomials x, y, v is the solution of the equation

$$ha_0^- x + b^- y + a_2^- b_1^- v = (b, b_1)^{-\sim} f^* a_0^{-\sim}$$

with x causal and $\deg x < \deg (b, b_1)^-$.

Optimal solution exists if and only if $h_0 \sim h_0^+$ and $b_0^- | f^-$ where $b_0^- = b^- / (b, b_1)^-$ with unique resulting error; optimal controllers are not unique.

The results are presented without proof which can be simply executed by the reader using the previous approach.

5. CONCLUSIONS

The relations derived above will be evaluated by the comparison with the well-known results which are valid in simple control systems ($R_2 = 0$) ([1], [7]).

1. With regard to solvability all the optimal problems treated above are not solvable using AFS unless being solvable in a simple control system.

There is no difference in solvability of reference tracking problems between both the system structures. The same condition, i.e., $h_0 \sim h_0^+$ for stable TOC and LSC and $h_0 \sim 1$ for finite TOC must be valid.

The additional condition $q_{ak} \sim q_{ak}^+$ or $q_{ak} \sim 1$ must be fulfilled in disturbance compensation problems ($\mathscr{V}_1 \neq 0$) if they are solved by AFS. This condition is necessary for the required stable or finite control sequence as it can be seen from the relations (48) and (108) or (54), respectively. Usually (but not always) $q | h$; in this case the additional condition is redundant.

2. The application of AFS brings an effect in optimality if $\lambda_2 < \lambda_1$ where λ_1 and λ_2 denote a control performance index in simple and AFS system structure, respectively ([7]); $\lambda_2 \leqq \lambda_1$.

a) Analyzing the relations (20), (21), (26) and (82), (83) and comparing them with the case $v = 0$ (simple system) then obviously AFS can improve a reference tracking process for unstable controlled systems provided that $a_{20}^- \nsim a_0^-$, i.e., $a_{12}^- \nsim 1$. Therefore the additional feedback ought to be chosen to enclose the possible unstable part of a system.

b) Optimal results of disturbance ($\mathscr{V}_1 \neq 0$) compensation problems are given by the special solutions of equations (45) (50) and (105) for stable TOC, finite TOC and LSC, respectively. But the special solution requirements are referred to the polynomial s and the equations

$$a_0^- f_p^- x + g^- v = (a_0 f_p, g)^- s$$

and

$$b^- d_h^- f_p^- y - c h_d v = (b f_p, c)^- d_h r \quad \text{for stable TOC and LSC}$$

or

$$a_0^- f_p^- x + g v = (a_0 f_p, g)^- s$$

and

$$b d_h f_p^- y - c h_d v = (b f_p^-, c) d_h r \quad \text{for finite TOC}$$

must be satisfied at the same time.

Then considering a simple control system ($R_2 = v = 0$) as the special case of AFS structure the solution with $v = 0$ can be the optimal one only if

$$(a_0 f_p, g)^- \sim a_0^- f_p^-$$

and

$$(bf_p, c)^- \sim b^- f_p^- \quad \text{for stable TOC and LSC}$$

or

$$(bf_p^-, c) \sim bf_p^- \quad \text{for finite TOC.}$$

Optimal solution with $v \neq 0$ (with AFS) must be expected in the other cases provided the problem is solvable. Thus, the application of AFS is not restricted to unstable controlled systems only provided a disturbance $\mathscr{V}_1 \neq 0$ is compensated. It can be recommended if at least one of the conditions

$$(a_0 f_p, g)^- \nsim a_0^- \text{ and } (bf_p, c)^- \nsim b^- f_p^- \quad \text{for stable TOC or LSC}$$

and

$$(a_0 f_p, g)^- \nsim a_0^- \text{ and } (bf_p^-, c) \nsim bf_p^- \quad \text{for finite TOC}$$

is valid.

3. Analyzing the technical requirements of AFS the only additional sampler preceding R_2 is needed for application provided both controllers sequences R_1 and R_2 are realized by computer programs.

Examples

1. Let us consider the system shown in Fig. 4. The continuous time controlled subsystems are described in the block diagram by their transfer functions (in Laplace transform) and sampling period $\tau = 1$ sec.

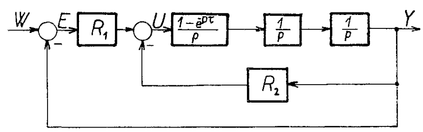

Fig. 4.

Let us solve TOC problem if

$$W = \frac{f}{h} = \frac{1}{1 - 0{\cdot}3679\, z^{-1}}$$

(reference tracking). The controlled system discrete-time transfer sequence is determined to be

$$G = \frac{b}{a} = \frac{0{\cdot}5 z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}\,.$$

Then $a_0^- = a$ and $h_0 = h = h^+ \sim 1$. Therefore stable TOC problem is only solvable and optimal additional feedback is the output feedback in Fig. 4 such that $G_1 = G$, i.e., $a_1 = a$, $b_1 = b$, $a_2 = 1$. Then $a_{12}^- = a$, $b_{12}^- = b$, $h_{22} = h_2 = h$, $f^+ = f^- = 1$.

The solution can start with equation (20)

$$(1 - 0{\cdot}3679z^{-1})\left[(1 - z^{-1})^2 x + 0{\cdot}5z^{-1}(1 + z^{-1}) v\right] + 0{\cdot}5z^{-1}(1 + z^{-1}) y = 1$$

which is decomposed into equation (34)

$$(1 - 0{\cdot}3679z^{-1}) s + 0{\cdot}5z^{-1}(1 + z^{-1}) y = 1$$

with min deg s solution $s = 1 + 0{\cdot}2690z^{-1}$, $y = 0{\cdot}1979$ and equation (36)

$$(1 - z^{-1})^2 x + 0{\cdot}5z^{-1}(1 + z^{-1}) v = 1 + 0{\cdot}2690z^{-1}$$

the general solution of which is

$$x = 1 + 0{\cdot}8173z^{-1} - 0{\cdot}5z^{-1}(1 + z^{-1}) t\,,$$

$$v = 2{\cdot}9034 - 1{\cdot}6345z^{-1} + (1 - z^{-1})^2 t\,.$$

Then according to (21) unique optimal error $e = s = 1 + 0{\cdot}2690z^{-1}$ with deg $e =$ $= 1$. Controllers R_1 and R_2 given by (11) and (17) are not unique with respect to an arbitrary t in x and v. Choosing $t = 0$ the simplest pair of controllers is

$$R_1 = \frac{0{\cdot}1979}{(1 - 0{\cdot}3679z^{-1})(1 + 0{\cdot}8173z^{-1})} \quad \text{and} \quad R_2 = \frac{2{\cdot}9034 - 1{\cdot}6345z^{-1}}{1 + 0{\cdot}8173z^{-1}}$$

and

$$U = \frac{0{\cdot}1979(1 - z^{-1})^2}{1 - 0{\cdot}3679z^{-1}}\,.$$

The given problem solved in simple control system ($R_2 = 0$) results in

$$e = 1 - 1{\cdot}1827z^{-1} - 0{\cdot}6345z^{-2} + 0{\cdot}8173z^{-3},$$

$$R = \frac{3{\cdot}1012(1 - 0{\cdot}8716z^{-1} + 0{\cdot}1939z^{-2})}{(1 - 0{\cdot}3679z^{-1})(1 + 0{\cdot}8173z^{-1})}$$

and

$$U = \frac{3{\cdot}1012(1 - 0{\cdot}8716z^{-1} + 0{\cdot}1939z^{-2})(1 - z^{-1})^2}{1 - 0{\cdot}3679z^{-1}}\,.$$

Hence deg $e = 3$ and $\lambda_1 - \lambda_2 = 2$.

2. The control system is according to Fig. 5. Let us solve LSC problem if the sampling period $\tau = 1$ sec, reference sequence $W = 0{\cdot}5/(1 - z^{-1})$ and continuous-time disturbance with Laplace transform $\mathscr{V}_1(p) = 1/(p + 1)$ affects the system.

At first discrete-time transfer sequences

$$G = \frac{b}{a} = \frac{0{\cdot}3679z^{-1}(1 + 0{\cdot}7181z^{-1})}{(1 - z^{-1})(1 - 0{\cdot}3679z^{-1})} \quad \text{and} \quad G_1 = \frac{b_1}{a_1} = \frac{0{\cdot}6321z^{-1}}{1 - 0{\cdot}3679z^{-1}}$$

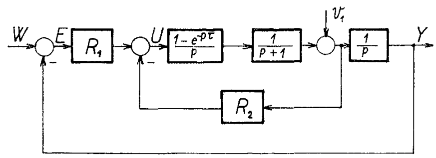

Fig. 5.

and input sequences

$$W_1 = W - V_B = \frac{f}{h} = \frac{0{\cdot}5(1 - 1{\cdot}6321z^{-1})}{(1 - z^{-1})(1 - 0{\cdot}3679z^{-1})} \quad \text{and} \quad V_A = \frac{p}{q} = \frac{1}{1 - 0{\cdot}3679z^{-1}}$$

have been determined.

Since $a_0 = h_0 = 1$ and $q|h$ the problem is solvable. Putting $b^+ = 0{\cdot}3679(1 + 0{\cdot}7181z^{-1})$, $b^- = z^{-1}$, $b_1^+ = 0{\cdot}6321$, $b_1^- = z^{-1}$, $(f, p) = (f, p)^* = 1$, $f_p^+ = 0.5$, $f_p^- = 1 - 1{\cdot}6321z^{-1}$, $p_f^+ = p_f^- = 1$ and considering $a_2 = 1 - z^{-1}$ then according to (41), (43) and (44)

$$g^+ = 0{\cdot}6840,\ g^- = z^{-1},\ k = 1,\ c^+ = b^+,\ c^- = b^-,$$

$$d = 0{\cdot}6840(1 - 0{\cdot}3679z^{-1}),\quad (d, h) = 1 - 0{\cdot}3679z^{-1},\quad d_h = d_h^+ = 0{\cdot}6840,$$

$$d_h^- = 1,\quad h_d = 1 - z^{-1},\quad (f_p, g, c)^- = 1,$$

$$(a_0 f_p, g)^- = (a_0 f_p, g)^{-\sim} = 1,\quad (bf_p, c)^- = z^{-1},\quad (bf_p, c)^{-\sim} = 1.$$

Hence $(a_0 f_p, g)^- \sim a_0^- f_p^-$ as well as $(bf_p, c)^- \sim b^- f_p^-$. Equation (105) has the form

$$0{\cdot}6840(1 - z^{-1})(1 - 0{\cdot}3679z^{-1})(1 - 1{\cdot}6321z^{-1})x + z^{-1}(1 - 1{\cdot}6321z^{-1})y +$$
$$+ 0{\cdot}3161z^{-1}(1 - z^{-1})(1 - 1{\cdot}6321z^{-1})v = 0{\cdot}3420(1 - 1{\cdot}6321z^{-1})$$

and its general solution can be written as follows:

$$x = 0{\cdot}5 + z^{-1}t_1 + 0{\cdot}3161z^{-1}t_2,$$

$$y = 0{\cdot}4678 - 0{\cdot}1258z^{-1} - 0{\cdot}6840(1 - z^{-1})(1 - 0{\cdot}3679z^{-1})t_1$$

and

$$v = -0{\cdot}6840(1 - 0{\cdot}3679z^{-1})t_2$$

with any arbitrary t_1 and t_2.

The optimal solution with the property (106)

$$\deg\left[(1 - 1{\cdot}6321z^{-1})\,x + z^{-1}v\right] < 1 \quad \text{must be found.}$$

This solution

$$x = 0{\cdot}5 - 0{\cdot}2375z^{-1}\,, \quad y = 0{\cdot}2972 + 0{\cdot}1076z^{-1} - 0{\cdot}0628z^{-2}$$

and

$$v = 1{\cdot}0535(1 - 0{\cdot}3679z^{-1})$$

corresponds to the choice $t_1 = 0{\cdot}2494$ and $t_2 = -1{\cdot}5404$ and is unique. Hence

$$E = 0{\cdot}5\,, \quad \sigma_E = 0{\cdot}25\,, \quad R_1 = \frac{0{\cdot}2972 + 0{\cdot}1076z^{-1} - 0{\cdot}0628z^{-2}}{0{\cdot}2516(1 + 0{\cdot}7181z^{-1})(0{\cdot}5 - 0{\cdot}2375z^{-1})}\,,$$

$$R_2 = \frac{0{\cdot}7701(1 - 0{\cdot}3679z^{-1})}{0{\cdot}5 - 0{\cdot}2375z^{-1}} \quad \text{and} \quad U = -\frac{0{\cdot}3593(1 + 1{\cdot}3894z^{-1})}{1 + 0{\cdot}7181z^{-1}}$$

are given unambiguously.

Solving the given problem in a simple control system the results are

$$E = \frac{0{\cdot}8161(1 - 1{\cdot}6321z^{-1})}{1{\cdot}6321 - z^{-1}} = 0{\cdot}5 - 0{\cdot}5097z^{-1} - 0{\cdot}3123z^{-2} -$$

$$- 0{\cdot}1914z^{-3} - 0{\cdot}1172z^{-4} - \ldots\,; \quad \sigma_E = 0{\cdot}6655\,,$$

$$R = \frac{2{\cdot}0528(1 - 0{\cdot}4872z^{-1})}{1 + 0{,}7181z^{-1}} \quad \text{and} \quad U = \frac{1{\cdot}6752(1 - 1{\cdot}6321z^{-1})(1 - 0{\cdot}4872z^{-1})}{(1{\cdot}6321 - z^{-1})(1 + 0{\cdot}7181z^{-1})}\,.$$

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