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CONJUGATED AND SYMMETRIC POLYNOMIAL EQUATIONS

I: Continuous-Time Systems

JAN JEŽEK

The paper is divided into two separate parts. In Part I properties of conjugated and symmetric equations are investigated. The equations occur in the synthesis of continuous quadratically optimal controllers. In Part II a similar study is done for the equation from the corresponding discrete problems.

INTRODUCTION

In recent years, the polynomial equation approach was successfully used for formulation and solving of many problems in control theory [1]. For single-input, single-output systems, the equation

$$a(\lambda)\, x(\lambda) + b(\lambda)\, y(\lambda) = c(\lambda) \tag{1}$$

is relevant. The indeterminate λ stands for either the derivative operator (for continuous-time systems) or the delay operator (for discrete-time ones). In essence, this equation is equivalent to the problem of partial fraction decomposition:

$$\frac{c(\lambda)}{a(\lambda)\, b(\lambda)} = \frac{x(\lambda)}{b(\lambda)} + \frac{y(\lambda)}{a(\lambda)}$$

The theory of (1) is well developed along with practical solution methods.

In linear-quadratic control problems where an integral

$$I = \int_{-\infty}^{\infty} e^2(t)\, \mathrm{d}t = \frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} E(s)\, E(-s)\, \mathrm{d}s$$

plays a central role, polynomials $x(-s)$ appear along with $x(s)$. The polynomial approach to such problems, as used until now, removes the "conjugated" polynomials $x(-s)$ in early stages of problem formulation and yields at the end the equation (1).

An alternative method is possible: to treat the conjugation as a basic operation in polynomial algebra and to develop a theory of conjugated and symmetric polynomial equations. The latter approach seems to be more natural and going more deeply into the mathematical essence of the problem. This is done in the presented paper, namely the equations

$$a(s)\,x(-s) + b(-s)\,y(s) = c(s) \tag{1a}$$

$$a(s)\,x(-s) + a(-s)\,x(s) = 2\,b(s) \tag{1b}$$

are investigated in Part I and their discrete-time counterparts

$$a(\zeta)\,x(\zeta^{-1}) + b(\zeta^{-1})\,y(\zeta) = c(\zeta) + d(\zeta^{-1}) \tag{1c}$$

$$a(\zeta)\,x(\zeta^{-1}) + a(\zeta^{-1})\,x(\zeta) = b(\zeta) + b(\zeta^{-1}) \tag{1d}$$

in Part II. Some properties of (1a)–(1d) are similar to those of (1), other ones are strikingly different. For better understanding, the explanation begins with some algebraic preliminaries.

The equations (1a)–(1d) occur in other places of control theory and signal theory, too. The corresponding partial fraction problems, e.g.

$$\frac{b(s)}{a(-s)\,a(s)} = \frac{x(s)}{a(s)} + \frac{x(-s)}{a(-s)}$$

arise in manipulation with rational spectral densities. The most important application of (1b) is in an iterative method for solution of the quadratic polynomial equation

$$x(-s)\,x(s) = 2\,b(s)$$

where a stable polynomial $x(s)$ is looked for – [2]. This problem is known as "spectral factorization" and plays a key role in linear-quadratic control problems. For multiple-input, multiple-output control, (1a)–(1d) represent basic tools for investigating similar matrix equations.

PRELIMINARIES

We shall use the ring of polynomials over the field of reals, ∂f will mean degree of $f(s)$. The theory of polynomial equations [1] also will be used. We define an operation of "conjugation" $f^*(s) = f(-s)$. The following properties are evident:

$$[f(s) + g(s)]^* = f^*(s) + g^*(s)\,, \quad [f(s)\,g(s)]^* = f^*(s)\,g^*(s)\,, \tag{2}$$
$$f^{**}(s) = f(s)\,, \quad \partial f^* = \partial f\,.$$

If $f(-s) = f(s)$ then the polynomial is called even, it can be expressed $f(s) = \hat{f}(s^2)$. If $f(-s) = -f(s)$ then f is called odd, it can be expressed $f(s) = s\,\hat{f}(s^2)$. Every polynomial has a unique decomposition into its even part and odd part:

(3) $$f(s) = f_e(s^2) + s f_o(s^2)$$

It is $f(-s) = f_e(s^2) - s f_o(s^2)$. The parts are

(4) $$f_e(s^2) = \frac{f(s) + f(-s)}{2}, \quad f_o(s^2) = \frac{f(s) - f(-s)}{2s}$$

For ∂f even:

$$\partial f_e = \frac{\partial f}{2}, \quad \partial f_o \leqq \frac{\partial f}{2} - 1, \quad \partial f_o \leqq \partial f_e - 1$$

For ∂f odd:

$$\partial f_o = \frac{\partial f - 1}{2}, \quad \partial f_e \leqq \frac{\partial f - 1}{2}, \quad \partial f_e \leqq \partial f_o$$

In the next chapters, we shall use two following lemmas concerning common divisors.

Lemma 1. $g(s) = \gcd(a(s), a(-s))$ satisfies one of the following conditions:

$$\text{a)}\ g(-s) = g(s) \qquad \text{b)}\ g(-s) = -g(s)$$

Proof. First, we shall prove that $g(-s) = \gcd(a(-s), b(-s))$ follows from $g(s) = \gcd(a(s), b(s))$. Actually, from

$$a(s) = a_0(s)\, g(s), \quad b(s) = b_0(s)\, g(s)$$

by taking the conjugates

$$a(-s) = a_0(-s)\, g(-s), \quad b(-s) = b_0(-s)\, g(-s)$$

we see that $g(-s)$ is a common divisor of $a(-s)$ and $b(-s)$. To prove that it is the greatest one, suppose that $h(s)$ is another common divisor of $a(-s)$, $b(-s)$. By taking the conjugates we see that $h(-s)$ is a common divisor of $a(s)$, $b(s)$. So $h(-s)$ divides $g(s)$, $h(s)$ divides $g(-s)$, $g(-s)$ is the greatest one.

Second, applying this for $a(s)$, $a(-s)$ we see that $g(-s)$ is $\gcd(a(s), a(-s))$ as well as $g(s)$ is. Therefore $g(s)$, $g(-s)$ are associated: $g(s) = e\, g(-s)$ where e is a nonzero number. Substituting here its own conjugate, we have $g(s) = e^2\, g(s)$, $e = \pm 1$. □

Lemma 2. $g(s) = \gcd(a(s), a(-s))$ can be expressed:

$$\text{a) for}\ \ g(-s) = g(s) \qquad g(s) = w(-s)\, a(s) + w(s)\, a(-s)$$

$$\text{b) for}\ \ g(-s) = -g(s) \qquad g(s) = -w(-s)\, a(s) + w(s)\, a(-s)$$

where $w(s)$ is a polynomial.

Proof. As the ring of polynomials is a principal ideal ring, $g(s)$ can be expressed as

$$g(s) = u(-s)\,a(s) + v(s)\,a(-s)$$

where $u(s)$, $v(s)$ are polynomials. We have:

$$\text{a)} \qquad g(-s) = g(s)\,, \quad g(s) = \frac{g(s) + g(-s)}{2} =$$

$$= \frac{v(-s) + u(-s)}{2}\,a(s) + \frac{v(s) + u(s)}{2}\,a(-s)$$

$$\text{b)} \qquad g(-s) = -g(s)\,, \quad g(s) = \frac{g(s) - g(-s)}{2} =$$

$$= -\frac{v(-s) - u(-s)}{2}\,a(s) + \frac{v(s) - u(s)}{2}\,a(-s)$$

□

THE SYMMETRIC EQUATION THEORY

It is evident that the equation (1a) can be transformed to a common polynomial equation (1) by the substitution

$$x(s) = x'(-s)\,, \quad b'(s) = b(-s)\,.$$

That is why it is no more interesting. Let us note in anticipation of Part II that the situation is quite different in the discrete case.

In this chapter, the general solution of (1b) and some "minimal" one will be derived.

Theorem 1. The general solution of the homogeneous equation

$$a(s)\,x(-s) + a(-s)\,x(s) = 0 \tag{6}$$

where $g(s) = \gcd\,(a(s), a(-s))$ and $a_0(s) = a(s)/g(s)$ has one of the two forms:

$$\begin{aligned} &\text{a)}\ g(-s) = g(s) &\quad x(s) = a_0(s)\,s\,t(s^2) \\ &\text{b)}\ g(-s) = -g(s) &\quad x(s) = a_0(s)\,t(s^2) \end{aligned} \tag{7}$$

where t is an arbitrary polynomial.

Proof.

a) For $g(-s) = g(s)$ the equation turns by cancellation by $g(s)$ into an equivalent equation

$$a_0(s)\,x(-s) + a_0(-s)\,x(s) = 0 \tag{8}$$

with $a_0(s)$, $a_0(-s)$ coprime. Let us consider a polynomial equation with two unknown polynomials

$$a_0(s)\,y(s) + a_0(-s)\,x(s) = 0 \tag{9}$$

with the general solution

$$x(s) = a_0(s)\,q(s)\,, \quad y(s) = -a_0(-s)\,q(s) \tag{10}$$

where $q(s)$ is an arbitrary polynomial. Let us select only such solution which satisfy $y(s) = x(-s)$. By substituting the general solution into this condition we obtain $q(-s) = -q(s)$, i.e. $q(s)$ is an arbitrary odd polynomial, $q(s) = s\,t(s^2)$ where t is an arbitrary polynomial.

b) For $g(-s) = -g(s)$ the equation obtained by cancellation is

$$a_0(s)\,x(-s) - a_0(-s)\,x(s) = 0$$

The polynomial equation

$$a_0(s)\,y(s) - a_0(-s)\,x(s) = 0$$

with the general solution

$$x(s) = a_0(s)\,q(s)\,, \quad y(s) = a_0(-s)\,q(s)$$

and with the condition $y(s) = x(-s)$ leads to $q(-s) = q(s)$, i.e. $q(s)$ is an arbitrary even polynomial, $q(s) = t(s^2)$. □

Theorem 2. The equation (1b) is solvable iff $b(-s) = b(s)$, i.e. $b(s) = \hat{b}(s^2)$ and $g(s) = \gcd(a(s), a(-s))$ divides $b(s)$. Further,

a) for $g(-s) = g(s)$, i.e. $g(s) = \hat{g}(s^2)$:

a1) for $\partial\hat{b} < \partial a - \partial\hat{g}$ it exists a unique "minimal" solution satisfying $\partial x < \partial a - \partial g$

a2) for $\partial\hat{b} = \partial a - \partial\hat{g}$ it exists a unique "minimal" solution satisfying $\partial x = \partial a - \partial g$ and no solution of lower degree

a3) for $\partial\hat{b} > \partial a - \partial\hat{g}$ all solutions have $\partial x \geqq \partial a - \partial g + 2$. No ∂x exists for which the solution is unique.

b) for $g(-s) = -g(s)$, i.e. $g(s) = s\,\hat{g}(s^2)$:

b1) for $\partial\hat{b} < \partial a - \partial\hat{g}$ it exists a unique "minimal" solution satisfying $\partial x < \partial a - \partial g$

b2) for $\partial\hat{b} \geqq \partial a - \partial\hat{g}$ all solutions have $\partial x \geqq \partial a - \partial g + 1$. No ∂x exists for which the solution is unique.

Proof. Both conditions are necessary: let $x(s)$ be a solution, the left-hand side of (1) is an even polynomial, so must be the right-hand one:

$$b(s) = \hat{b}(s^2)\,, \quad \partial b = 2\,\partial\hat{b} \tag{11}$$

The left-hand side is divisible by $g(s)$, so must be the right-hand one. Conversely, let (11) hold and

$$b(s) = b_0(s)\,g(s)\,, \quad \partial b_0 = \partial b - \partial g$$

Let us express $g(s)$ according to Lemma 2:

a) $$g(-s) = g(s)\,, \quad g(s) = \hat{g}(s^2)\,, \quad \partial g = 2\,\partial\hat{g}\,,$$
$$b_0(-s) = b_0(s)\,, \quad \partial b_0 \text{ even}\,,$$
$$g(s) = w(-s)\,a(s) + w(s)\,a(-s)$$

b) $$g(-s) = -g(s)\,, \quad g(s) = s\,\hat{g}(s^2)\,, \quad \partial g = 2\,\partial\hat{g} + 1\,,$$
$$b_0(-s) = -b_0(s)\,, \quad \partial b_0 \quad \text{odd}\,,$$
$$g(s) = -w(-s)\,a(s) + w(s)\,a(-s)$$

From here we obtain in both cases

$$b(s) = b_0(s)\,g(s) = a(s)\,b_0(-s)\,w(-s) + a(-s)\,b_0(s)\,w(s)$$

i.e. $x(s) = 2\,b_0(s)\,w(s)$ is a solution.

Let us investigate the minimal solution. We denote

$$a(s) = a_0(s)\,g(s)\,, \quad \partial a_0 = \partial a - \partial g$$

a1) $\partial\hat{b} < \partial a - \partial\hat{g}\,, \quad \partial b < 2\,\partial a - \partial g\,, \quad \partial b_0 < 2\,\partial a_0$.
The equation turns by cancellation by $g(s)$ into

(12) $$\tfrac{1}{2}[a_0(s)\,x(-s) + a_0(-s)\,x(s)] = b_0(s)$$

Let us consider the polynomial equation with two unknowns

(13) $$\tfrac{1}{2}[a_0(s)\,y(s) + a_0(-s)\,x(s)] = b_0(s)$$

For $\partial b_0 < 2\,\partial a_0$ the minimal solution exists [3], i.e. $\partial x < \partial a_0$, $\partial y < \partial a_0$. We shall prove that this solution satisfies $y(s) = x(-s)$. The right-hand side of (13) is even, so is the left-hand one:

$$a_0(s)\,y(s) + a_0(-s)\,x(s) = a_0(-s)\,y(-s) + a_0(s)\,x(-s)$$
$$a_0(s)\,[x(-s) - y(s)] - a_0(-s)\,[x(s) - y(-s)] = 0$$

We denote $z(s) = x(s) - y(-s)$, $\partial z < \partial a_0$. According to Theorem 1, the general solution of

(14) $$a_0(s)\,z(-s) - a_0(-s)\,z(s) = 0$$

is $z(s) = a_0(s)\,t(s^2)$ where t is an arbitrary polynomial. For nonzero z it must be $\partial z \geqq \partial a_0$. But we have $\partial z < \partial a_0$, hence it is $z = 0$, $y(s) = x(-s)$. We have found a solution of (12) satisfying $\partial x < \partial a_0 = \partial a - \partial g$. It is unique, otherwise we would have two different minimal solutions of (13) which is not possible.

a2) $\partial\hat{b} = \partial a - \partial\hat{g}\,, \quad \partial b = 2\,\partial a - \partial g\,, \quad \partial b \quad \text{even}\,, \quad \partial g \quad \text{even}$.
We see in (1b) that no solution exists with $\partial x < \partial b - \partial a = \partial a - \partial g$. Let us look

for a solution $\partial x = \partial b - \partial a$. By matching the highest terms in (1b) we obtain

$$x_{\partial b - \partial a} = (-1)^{\partial a} \frac{b_{\partial b}}{a_{\partial a}} \tag{15}$$

It is possible to express

$$x(s) = x'(s) + (-1)^{\partial a} \frac{b_{\partial b}}{a_{\partial a}} s^{\partial b - \partial a} \tag{16}$$

By substituting into (1b) we have

$$\tfrac{1}{2}[a(s)\, x'(-s) + a(-s)\, x'(s)] = b'(s) \tag{17}$$

where

$$b'(s) = b(s) - \frac{b_{\partial b}}{a_{\partial a}} s^{\partial b - \partial a} \frac{a(s) + (-1)^{\partial a} a(-s)}{2} \tag{18}$$

$$b'(-s) = b(s), \quad \partial b' < \partial b$$

For $x'(s)$ we have an equation of the same form but with lower degree of the right-hand side: $\partial \hat{b}' < \partial a - \partial \hat{g}$. According to what was proved in a1), it exists the unique solution $x'(s)$ satisfying $\partial x' < \partial a - \partial g$. Then (16) yields the unique solution of (1b) satisfying $\partial x = \partial a - \partial g$.

a3) $\partial \hat{b} > \partial a - \partial \hat{g}, \quad \partial b > 2\, \partial a - \partial g$.

In the latter inequality, both sides are even numbers, therefore $\partial b \geqq 2\, \partial a - \partial g + 2$. We see in (1b) that $\partial x \geqq \partial a - \partial g + 2$ for every solution. According to Theorem 1, case a), infinitely many solutions of the homogeneous equation (6) exist with $\partial x = \partial a - \partial g + 1$. We shall obtain them by taking arbitrary number in place of $t(s^2)$. All these solutions can be added to $x(s)$ giving new solutions of the same degree.

b1) $\partial \hat{b} < \partial a - \partial \hat{g}, \quad \partial b < 2\, \partial a - \partial g + 1, \quad \partial b_0 < 2\, \partial a_0 + 1$.

In the latter inequality both sides are odd numbers, therefore $\partial b_0 < 2\, \partial a_0$ holds as well. The equation obtained by cancellation by $g(s)$ is

$$\tfrac{1}{2}[a_0(s)\, x(-s) - a_0(-s)\, x(s)] = b_0(s) \tag{19}$$

Let us consider the polynomial equation

$$\tfrac{1}{2}[a_0(s)\, y(s) - a_0(-s)\, x(s)] = b_0(s) \tag{20}$$

For $\partial b_0 < 2\, \partial a_0$ the minimal solution exists, i.e. $\partial x < \partial a_0$, $\partial y < \partial a_0$. We shall prove that it satisfies $y(s) = x(-s)$. The right-hand side is odd, so is the left-hand one:

$$a_0(s)\, y(s) - a_0(-s)\, x(s) = -a_0(-s)\, y(-s) + a_0(s)\, x(-s)$$

$$a_0(s)\, [x(-s) - y(s)] + a_0(-s)\, [x(s) - y(-s)] = 0$$

We denote $z(s) = x(s) - y(-s)$, $\partial z < \partial a_0$. The general solution of

$$a_0(s)\, z(-s) + a_0(-s)\, z(s) = 0$$

is $z(s) = a_0(s)\, s\, t(s^2)$. For nonzero z it must be $\partial z \geqq \partial a_0 + 1$. But we have $\partial z < \partial a_0$, hence $z = 0$ and $y(s) = x(-s)$. We have found a solution of (19) satisfying $\partial x < \partial a_0 = \partial a - \partial g$. It is unique as in a1).

b2) $\partial \hat{b} \geqq \partial a - \partial \hat{g}$, $\partial b \geqq 2\, \partial a - \partial g + 1$.

We see in (1b) $\partial x \geqq \partial a - \partial g + 1$. As in a3), solution of (6) can be added. According to Theorem 1, case b), there are infinitely many of them with $\partial x = \partial a - \partial g$. □

THE SYMMETRIC EQUATION SOLUTION METHOD

The way how Theorem 2 was proved can serve as a numerical solution method for finding the minimal solution. The method consists of replacing the symmetric equation by non-symmetric one, solving it [3] and by restoring the symmetry.

The most important case is that of stable polynomial $a(s)$ and $\partial \hat{b} < \partial a$ or $\partial \hat{b} = \partial a$. Stability means $a(s) \neq 0$ for $\mathrm{Re}\, s \geqq 0$. For that case, $g(s) = 1$ and the minimal solution is $\partial x < \partial a$ resp. $\partial x = \partial a$. The properties of stable polynomials make another algorithm possible which utilizes the symmetry and needs approximately four times less numerical operations. It was described in [1] in terms of Hurwitz matrices. Let us formulate it here in polynomial terms.

Consider the equation

$$\tfrac{1}{2}[a(s)\, x(-s) + a(-s)\, x(s)] = \hat{b}(s^2)\,, \quad a(s) \text{ stable}\,, \quad \partial \hat{b} \leqq \partial a \tag{21}$$

Let us express $a(s)$ by its even and odd parts (3) and let us do the same with the conjugate of $x(s)$:

$$\begin{aligned} a(s) &= a_e(s^2) + s\, a_o(s^2) \qquad & x(s) &= x_e(s^2) - s\, x_o(s^2) \\ a(-s) &= a_e(s^2) - s\, a_o(s^2) \qquad & x(-s) &= x_e(s^2) + s\, x_o(s^2) \end{aligned} \tag{22}$$

By substituting (22) into (21) the equation turns into a non-symmetric polynomial equation which is of approximately half degree:

$$a_e(s^2)\, x_e(s^2) + s^2\, a_o(s^2)\, x_o(s^2) = \hat{b}(s^2) \tag{23}$$

Methods for polynomial equations can be used. The equation is usualy attacked "from the left": For a stable $a(s)$, $a_0 \neq 0$, $a_1 \neq 0$ holds; by matching the absolute terms we have $x_0 = b_0/a_0$ and we can express

$$\begin{aligned} x_e(s^2) &= \frac{b_0}{a_0} + s^2\, x_o'(s^2) \\ x_o(s^2) &= x_e'(s^2) - \frac{a_0}{a_1}\, x_o'(s^2) \end{aligned} \tag{24}$$

By substituting it into (23) we have

(25) $$a'_e(s^2)\, x'_e(s^2) + s^2\, a'_o(s^2)\, x'_o(s^2) = \hat{b}'(s^2)$$

where

$$a'_e(s^2) = a_o(s^2)$$

(26) $$a'_o(s^2) = \frac{1}{s^2}\left[a_e(s^2) - \frac{a_0}{a_1}\, a_o(s^2)\right]$$

$$\hat{b}'(s^2) = \frac{1}{s^2}\left[\hat{b}(s^2) - \frac{b_0}{a_0}\, a_e(s^2)\right]$$

It is an equation of the same form as (23), hence it is equivalent to an equation

(27) $$\tfrac{1}{2}[a'(s)\, x'(-s) + a'(-s)\, x'(s)] = \hat{b}'(s^2)$$

where $a'(s)$, $x'(s)$ are defined by their even and odd parts as in (22). Let us investigate the degrees. From (26):

$$\partial a'_e = \partial a_o\,, \quad \partial \hat{b}' < \max(\partial \hat{b}, \partial a_e) \leqq \max(\partial \hat{b}, \partial a)$$

a) $$\partial a \ \text{ even}\,, \quad \partial a = 2\,\partial a_e\,, \quad \partial a_o \leqq \partial a_e - 1\,, \quad \text{from (26)} \quad \partial a'_o = \partial a_e - 1\,,$$

$$\partial a'_e \leqq \partial a'_o\,, \quad \partial a' \ \text{ odd}\,, \quad \partial a' = 2\,\partial a'_o + 1 = \partial a - 1$$

b) $$\partial a \ \text{ odd}\,, \quad \partial a = 2\,\partial a_o + 1\,, \quad \partial a_e \leqq \partial a_o\,,$$

$$\partial a'_o < \partial a'_e\,, \quad \partial a' \ \text{ even}\,, \quad \partial a' = 2\,\partial a'_e = \partial a - 1$$

The degrees got lowered. It follows from Routh-Hurwitz theory that $a'(s)$ is again stable, $a'_0 \neq 0$, $a'_1 \neq 0$. By repeating this process we shall come to the case $\partial a = 0$, $\partial a_e = 0$, $a_o(s) \equiv 0$ which can be solved directly. By performing all substitutions backwards we obtain $x(s)$. The forward part of the algorithm is the same as in Routh stability test. The procedure is easily mechanized for a computer or a calculator.

EXAMPLES

Very simple examples can be presented to illustrate Theorem 2. First of all, cases without common factor, e.g. $a(s) = 1 + 2s$, $g(s) = 1$:

a1) $b(s) = 1$, minimal solution $x(s) = 1$
a2) $b(s) = 1 - 6s^2$, minimal solution $x(s) = 1 + 3s$
a3) $b(s) = 1 - s^2 - 2s^4$, every solution must have $\partial x \geqq 3$, e.g. $x(s) = 1 + s + s^2 + s^3 + \xi(s + 2s^2)$ where ξ is an arbitrary number.

Cases with $g(-s) = g(s)$ e.g. $a(s) = (1 + 2s)(1 + s^2) = 1 + 2s + s^2 + 2s^3$, $g(s) = 1 + s^2$:

a1) $b(s) = 1 + s^2$, $x(s) = 1$

a2) $b(s) = (1 - 6s^2)(1 + s^2) = 1 - 5s^2 - 6s^4$, $x(s) = 1 + 3s$

a3) $b(s) = (1 - s^2 - 2s^4)(1 + s^2) = 1 - 3s^4 - 2s^6$,
$x(s) = 1 + s + s^2 + s^3 + \xi(s + 2s^2)$

Cases with $g(-s) = -g(s)$ e.g. $a(s) = (1 + 2s)\,s = s + 2s^2$, $g(s) = s$:

b1) $b(s) = 2s^2$, $x(s) = 1$

b2) $b(s) = s^2 + 2s^4$, every solution must have $\partial x \geqq 2$, e.g. $x(s) = 1 + s + s^2 + \xi(1 + 2s)$.

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Ing. Jan Ježek CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.