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ON A MULTIPLACE FUNCTIONAL EQUATION RELATED TO INFORMATION MEASURES AND FUNCTIONS

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This paper deals with a functional equation which is related to measures of information such as Shannon's entropy, inaccuracy, divergence, entropy of degree β etc. and information functions. The general solutions of the functional equation are determined, without the assumption of any regularity condition on the functions involved.

1. INTRODUCTION

Real valued functions satisfying the functional equation

$$f(x) + (1 - x)^\beta f\left(\frac{y}{1 - x}\right) = f(y) + (1 - y)^\beta f\left(\frac{x}{1 - y}\right) \tag{1.1}$$

are called information functions of degree β, which is related to the Shannon's entropy when $\beta = 1$, otherwise to the entropy of degree β $(\neq 0)$ [2, 4]. In [5, 7, 2] a generalization of (1.1),

$$F(x) + (1 - x)^\beta G\left(\frac{y}{1 - x}\right) = H(y) + (1 - y)^\beta K\left(\frac{x}{1 - y}\right) \tag{1.2}$$

for $x, y \in [0, 1[$ with $x + y \in [0, 1]$ was considered, giving details of its connection with directed divergence of type β, inaccuracy of type β etc.

The purpose of this paper is to consider in generalization of these and other results, a multiplace functional equation

(1.3)

$$f(x, y) + (1 - x)^\beta g\left(\frac{u}{1 - x}, \frac{v}{1 - y}\right) = h(u, v) + (1 - u)^\beta k\left(\frac{x}{1 - u}, \frac{y}{1 - v}\right)$$

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for $x, u \in [0, 1[$ with $x + u \in [0, 1]$, $y, v, y + v \in]0, 1[^n$ and to find the general solution of (1.3) by simple methods using familiar functional equations such as Pexider equations etc., without any further assumptions (regularity conditions) on the functions, by relating (1.3) to (1.2).

2. SOLUTION OF (1.3)

Theorem. Let $f, h\colon [0, 1[\times]0, 1[^n \to \mathbb{R}$ reals, $g, k\colon [0, 1] \times]0, 1[^n \to \mathbb{R}$ satisfy the functional equation (1.3) for $x, u \in [0, 1[$ with $x + u \in [0, 1]$, $y, v, y + v \in]0, 1[^n$, $\beta \in \mathbb{R}$. Then the general solution is given by

$$\begin{aligned}
(2.1)\qquad f(x, y) &= x\, l(x) + (1 - x)\, l(1 - x) + x\, M_1(y) + (1 - x)\, M_1(1 - y) + \\
&\quad + C_1 x + C_2\,, \\
g(u, y) &= u\, l(u) + (1 - u)\, l(1 - u) + u\, M_1(y) + (1 - u)\, M_1(1 - y) + \\
&\quad + C_3 u + C_5 - C_2 + C_4 - C_3\,, \\
h(x, y) &= x\, l(x) + (1 - x)\, l(1 - x) + x\, M_1(y) + (1 - x)\, M_1(1 - y) + \\
&\quad + C_4 x + C_5\,, \\
k(u, y) &= u\, l(u) + (1 - u)\, l(1 - u) + u\, M_1(y) + (1 - u)\, M_1(1 - y) + \\
&\quad + (C_1 - C_5 + C_2 - C_4 + C_3)\, u + C_4 - C_3\,,
\end{aligned}$$

for $x \in [0, 1[$, $u \in [0, 1]$, $y \in]0, 1[^n$, if $\beta = 1$;

(2.2)

$$f(x, y) = \begin{cases} M_1(1 - y) + M_2(y) + a_1\,, & x \in]0, 1[\\ M_1(1 - y) + M_2(y) + a_2\,, & x = 0 \end{cases}$$

$$g(x, y) = \begin{cases} M_1(1 - y) + M_3(y) - M_3(1 - y) + b_1 - a_1 + b_2\,, & x \in]0, 1[\\ M_1(1 - y) + M_3(y) - M_3(1 - y) + a_3\,, & x = 0 \\ M_1(1 - y) + M_3(y) - M_3(1 - y) + a_5\,, & x = 1 \end{cases}$$

$$h(x, y) = \begin{cases} M_1(1 - y) + M_2(1 - y) + M_3(y) - M_3(1 - y) + b_1\,, & x \in]0, 1[\\ M_1(1 - y) + M_2(1 - y) + M_3(y) - M_3(1 - y) + a_1 + a_3 - b_2\,, & x = 0 \end{cases}$$

$$k(x, y) = \begin{cases} M_1(1 - y) - M_3(1 - y) + M_2(y) + b_2\,, & x \in]0, 1[\\ M_1(1 - y) - M_3(1 - y) + M_2(y) + a_2 - a_1 - b_2\,, & x = 0 \\ M_1(1 - y) - M_3(1 - y) + M_2(y) + a_1 + a_4 - b_1\,, & x = 1 \end{cases}$$

for $y \in]0, 1[^n$, if $\beta = 0$;

$$\begin{aligned}
(2.3)\qquad f(x, y) &= -d(x) + a_1 x^2 + a_2 (1 - x)^2 + a_3\,, & x \in [0, 1[\\
g(x, y) &= d(x) + a_4 x^2 + a_5 (1 - x)^2 - a_2\,, & x \in [0, 1] \\
h(x, y) &= -d(x) + a_4 x^2 + a_6 (1 - x)^2 + a_3\,, & x \in [0, 1[\\
k(x, y) &= -d(x) + a_1 x^2 + a_5 (1 - x)^2 - a_6\,, & x \in [0, 1]
\end{aligned}$$

for $y \in]0, 1[^n$, if $\beta = 2$; and

$$f(x, y) = a_1 x^\beta + a_2(1 - x)^\beta + a_3,$$
$$g(x, y) = b_1 x^\beta + b_2(1 - x)^\beta - a_2,$$
$$h(x, y) = b_1 x^\beta + b_3(1 - x)^\beta + a_3,$$
$$k(x, y) = a_1 x^\beta + b_2(1 - x)^\beta - b_3, \tag{2.4}$$

for any $y \in]0, 1[^n$, if $\beta \neq 0, 1, 2$, where l, M_i and d are arbitrary solutions of

$$l(uv) = l(u) + l(v)\,(u, v \in]0, \infty[) \tag{2.5}$$

with

$$0 \,.\, l(0) := 0, \tag{2.6}$$

$$M_i(uv) = M_i(u) + M_i(v)\,(u, v \in]0, 1[^n), \tag{2.7}$$

$$d(x + y) = d(x) + d(y)$$
$$d(xy) \quad = x\,d(y) + y\,d(x) \quad x, y \in \mathbb{R} \tag{2.8}$$

(that is d is a real derivation) and C_i, a_i, b_i are arbitrary constants.

Remark 1. The convention $0 \,.\, l(0) := 0$ is used only to represent f, g, h, k as given in (2.1) and is not used in the proof at all. We also follow the convention $0^\beta = 0$ $(\beta \neq 0)$, $1^\beta = 1$.

The following lemma will be useful in the sequel and elsewhere.

Lemma. If $M_1(u) = M_2(1 - u) + C$, where M_1 and M_2 are solutions of (2.7), then $M_1 = M_2 \equiv 0$ and $C = 0$. That is, $1, M_1(u)$ and $M_2(1 - u)$ are linearly independent.

Proof. By (2.7),

$$M_1(uv) = M_1(u) + M_1(v) = M_2(1 - u) + M_2(1 - v) + 2C =$$
$$= M_2(1 - u - v + uv) + 2C = M_1(u + v - uv) + C$$

That is, $M_1(uv/(u + v - uv)) = C$ for $u, v \in]0, 1[^n$.

For fixed $u \in]0, 1[^n$, as v varies over $]0, 1[^n$, $uv/(u + v - uv)$ varies over $]0, u[$. As u is arbitrary, it follows that $M_1(t) = C$, for $t \in]0, 1[^n$. Thus $M_1 = 0$, $C = 0$. Hence $M_2 = 0$. This proves the lemma. □

Proof of the Theorem.

Case 1. Let us first treat the case when $\beta = 1$. For fixed $v, y \in]0, 1[^n$, (1.3) becomes a special case of (1.2) $(\beta = 1)$ treated in [5, 2], with

$$F(x) = f(x, y), \quad G(x) = \left(x, \frac{y}{1 - v}\right), \quad H(x) = h(x, v), \quad K(x) = k\left(x, \frac{v}{1 - y}\right)$$

so that F, G, H, K are given by

(2.9)
$$\begin{aligned} F(x) &= m(x) + d_1 x + d_2 \\ G(x) &= m(x) + d_3 x + d_5 - d_2 + d_4 - d_3, \quad x \in]0,1[\\ H(x) &= m(x) + d_4 x + d_5 \\ K(x) &= m(x) + (d_1 - d_5 + d_2 - d_4 + d_3)\,x + d_4 - d_3 \end{aligned}$$

where m is a symmetric $(m(x) = m(1-x))$ solution of

$$m(x) + (1-x)\,m\left(\frac{y}{1-x}\right) =$$
$$= m(y) + (1-y)\,m\left(\frac{x}{1-y}\right), \quad \text{for} \quad x, y, x+y \in]0,1[.$$

Then from [9, 10], it follows that

$$m(x) = x\,l(x) + (1-x)\,l(x), \quad x \in]0,1[$$

where l is an arbitrary solution of (2.5). Either by [2] or by the examination of F, G, H, K at the boundary points, the solution (2.9) can be extended to 0, 1, that is valid at the appropriate boundary points too. Thus from (2.9) and the form of $m(x)$, with the convention (2.6) we have

(2.10)
$$\begin{aligned} f(x, y) &= x\,l(x) + (1-x)\,l(1-x) + d_1(y, v)\,x + d_2(y, v), \\ g\left(u, \frac{v}{1-y}\right) &= u\,l(u) + (1-u)\,l(1-u) + d_3(y, v)\,u + \\ &\quad + (d_5 - d_2 + d_4 - d_3)(y, v), \\ h(x, v) &= x\,l(x) + (1-x)\,l(1-x) + d_4(y, v)\,x + d_5(y, v) \\ k\left(u, \frac{y}{1-v}\right) &= u\,l(u) + (1-u)\,l(1-u) + \\ &\quad + (d_1 - d_5 + d_2 - d_4 + d_3)(y, v)\,u + (d_4 - d_3)(y, v) \end{aligned}$$

for $x \in [0, 1[$, $u \in [0, 1]$, $y, v, y + v \in]0, 1[$". Here l would be a function of v and y.

Now we determine the d_i and l. From the forms of f and h in (2.10) we can conclude that l is independent of y and v, and that

(2.11) $d_1(y, v) =$ a function of y alone $= d_1(y)$ say; similarly

$d_2(y, v) = d_2(y)$ say; $d_4(y, v) = d_4(v)$ say; $d_5(y, v) = d_5(v)$ say;

and from the forms of g and k in (2.10) that,

(2.12)
$$d_3(y, v) = d_3\left(\frac{v}{1-y}\right),$$

$$(2.13)\qquad (d_5 - d_2 + d_4 - d_3)(y, v) = A\left(\frac{v}{1-y}\right),$$

$$(2.14)\qquad (d_1 - d_5 + d_2 - d_4 + d_3)(y, v) = B\left(\frac{y}{1-v}\right),$$

$$(2.15)\qquad (d_4 - d_3)(y, v) = C\left(\frac{y}{1-v}\right), \quad \text{for} \quad y, v, y + v \in]0, 1[^n .$$

Now (2.11), (2.14) and (2.15) give, $(d_1 + d_2)(y) - d_5(v) = (B + C)(y/(1 - v))$, which is the Pexider equation, $(d_1 + d_2)(rs) = d_5(1 - s) + (B + C)(r)$, $(r = y/(1-v) \in]0, 1[^n, s = 1 - v \in]0, 1[^n)$. The general solution is given by [1] as,

$$(2.16)\qquad (d_1 + d_2)(y) = L_1(y) + a_1 + a_2,$$

$$d_5(y) = L_1(1 - y) + a_1, (B + C)(y) = L_1(y) + a_2, \quad y \in]0, 1[^n,$$

with arbitrary constants a_1, a_2 and L_1 an arbitrary solution of (2.7). Similarly, (2.11), (2.12) and (2.13) imply the Pexider equation $(d_4 + d_5)(v) = d_2(y) + (d_3 + A) \cdot (v/(1 - y))$ so that

$$(2.17)\qquad (d_4 + d_5(y) = L_2(y) + b_1 + b_2, d_2(y) = L_2(1 - y) + b_2,$$

$$(d_3 + A)(y) = L_2(y) + b_2, \quad \text{for} \quad y \in]0, 1[^n,$$

where b_1, b_2 are arbitrary constants and L_2 is a solution of (2.7).

From (2.16), (2.17) and (2.15), we get,

$$(2.18)\qquad d_1(y) = L_1(y) - L_2(1 - y) + a_1 + a_2 - b_1,$$

$$(2.19)\qquad d_4(y) = L_2(y) - L_1(1 - y) + b_1 + b_2 - a_1.$$

Now, (2.15) can be written as,

$$(2.20)\qquad d_4(v) = d_3\left(\frac{v}{1-y}\right) + C\left(\frac{y}{1-v}\right)$$

which by (2.19) and by the substitution $v/(1 - y) = t$, $y/(1 - v) = r \in]0, 1[^n$, thus $v = t(1 - r)/(1 - rt)$, $1 - v = 1 - t/(1 - rt)$, reduces to the Pexider equation

$$[d_3(t) - L_2(t) + L_1(1 - t)] + (C(r) - L_2(1 - r) - b_1 - b_2 + a_1) =$$
$$= L_1(1 - tr) - L_2(1 - tr)$$

Now, the use of the lemma gives $L_1 = L_2$ and

$$(2.21)\qquad d_3(y) = L_1(y) - L_1(1 - y) + C_3,$$

where C_3 is a constant.

Now the functions d_1, d_2, d_3, d_4, d_5, are given by (2.18), (2.16), (2.21), (2.19) and

(2.16) respectively. Thus the d_i's are of the form

$$(2.22)\qquad \begin{aligned} &d_1(y) = L_1(y) - L_1(1-y) + C_1,\ d_2(y) = L_1(1-y) + C_2\,,\\ &d_3(y) = L_1(y) - L_2(1-y) + C_3\,, \qquad\qquad y \in\,]0,1[^n\\ &d_4(y) = L_1(y) - L_1(1-y) + C_4,\ d_5(y) - L_1(1-y) + C_5\,, \end{aligned}$$

where L_1 is a solution of (2.7) and C_i's are constants. Indeed f, g, h, k given by (2.10) with d_i's given by (2.22) satisfy the functional equation (1.3) when $\beta = 1$. Thus f, g, h, k are of the form given in (2.1).

Case 2. Let us now consider the case when $\beta = 0$. As in Case 1, for fixed $v, y, \in\,]0,1[^n$ (1.3) can be reduced to a special case of (1.2) ($\beta = 0$) [5, 3], so that F, G, H, K are given by

$$F(x) = m(x) + a\,,\quad G(x) = m(x) + b\,,\quad H(x) = m(x) + c\,,\quad K(x) = m(x) + d\,,$$

for $x \in\,]0,1[$, where $F(x) = f(x, y)$, $G(x) = g(x, v/(1-y))$, $H(x) = h(x, v)$, $K(x) = k(x, y/(1-v))$ and m, a symmetric solution of

$$m(x) + m\left(\frac{u}{1-x}\right) = m(u) + m\left(\frac{x}{1-u}\right),\quad x, u, x + u \in\,]0,1[\,.$$

By letting $T(p,q) = m(p) + 2m(q)$, it is easy to show (as in [6]) that T is symmetric, that is, m is a constant, so that

$$F(x) = C_1, G(x) = C_2\,,\quad H(x) = C_3 \quad\text{and}\quad K(x) = C_4 \quad\text{on}\quad]0,1[\,.$$

The use of [3] or the examination of F, G, H, K satisfying (1.3) at the boundary points reveals that

(2.23)

$$f(x,y) = \begin{cases} C_3 + C - C_2\,, & x = 0\\ C_1 & x \in\,]0,1[\end{cases}\qquad h(x,v) = \begin{cases} C_3 + a - C_2\,, & x = 0\\ C_3\,, & x \in\,]0,1[\end{cases}$$

$$g\left(x, \frac{v}{1-y}\right) = \begin{cases} a\,, & x = 0\\ C_2\,, & x \in\,]0,1[\\ b\,, & x = 1\end{cases}\qquad k\left(x, \frac{y}{1-v}\right) = \begin{cases} C, & x = 0\\ C_1 + C_2 - C_3\,, & x \in\,]0,1[\\ C_1 + b - C_3\,, & x = 1\end{cases}$$

where a, b, C, C_i are functions of $y, v \in\,]0,1[^n$.

From the forms of f, g, h, k it is easy to see that C_1 is a function of y only, say $C_1(y)$, similarly C_3, a, C_2, b and C are of the form $C_3(v)$, $a(v/(1-y))$, $C_2(v/(1-y))$, $b(v/(1-y))$ and $C(y/(1-v))$ and

$$(2.24)\quad \begin{aligned} &(C_3 + C - C_2)(y, v) = A(y)\,(\text{say});\ (C_3 + a - C_2)(y, v) = B(v)\,(\text{say})\\ &(C_1 + C_2 - C_3)(y, v) = E\left(\frac{y}{1-v}\right)(\text{say}) \quad\text{and}\quad (C_1 + b - C_3)(y, v) =\\ &\qquad = D\left(\frac{y}{1-v}\right)(\text{say})\,. \end{aligned}$$

From the third equation in (2.24) and the forms of C_1, C_2, C_3 given above we get

$$(2.25)\qquad C_1(y) + C_2\left(\frac{v}{1-y}\right) = C_3(v) + E\left(\frac{y}{1-v}\right), \quad \text{for} \quad y, v, y+v \in]0,1[^n .$$

Now we prove by induction on n, that

(2.26)

$$C_1(y) = M_1(1-y) + M_2(y) + a_1$$

$$C_2(y) = M_1(1-y) + M_3(y) - M_3(1-y) + b_1 - a_1 + b_2$$

$$C_3(y) = M_1(1-y) + M_2(1-y) + M_3(y) - M_3(1-y) + b_1 , \quad \text{for} \quad y \in]0,1[^n ,$$

where M_i are solutions (2.7), a_1, b_1, b_2 arbitrary constants.

For $n = 1$, the result follows by [10]. Let the result (2.26) hold for $(n-1)$. Now to prove (2.26) for n.

Let $y = (t, s)$, $v = (r, w)$ with $t, r, t+r \in]0,1[^{n-1}$, $s, w, s+w \in]0,1[$, so that (2.25) can be rewritten as

$$C_1(t,s) + C_2\left(\frac{r}{1-t}, \frac{w}{1-s}\right) = C_3(v, w) + E\left(\frac{t}{1-r}, \frac{s}{1-w}\right).$$

For fixed $s, w \in]0,1[$, by induction hypothesis, from the above operation we obtain

$$(2.27)\qquad C_1(t,s) = m_1(1-t) + m_2(t) + a_1$$

$$C_2\left(t, \frac{w}{1-s}\right) = m_1(1-t) - m_3(1-t) + m_3(t) + b_1 - a_1 + c_4$$

$$C_3(t,w) = m_1(1-t) + m_2(1-t) - m_3(1-t) + m_3(t) + b_1$$

$$E\left(t, \frac{s}{1-w}\right) = m_1(1-t) - m_3(1-t) + m_2(t) + C_4 ,$$

for $t \in]0,1[^{n-1}$, $s, w, s+w \in]0,1[$ where m_i, solutions of $\alpha(tr) = \alpha(t) + \alpha(r)$ for $t, r \in]0,1[^{n-1}$ and constants a_1, b_1, C_4 are functions of s and w. From the forms of C_1, C_2, C_3 and E given above and the lemma, we conclude that,

$$(2.28)\qquad m_1(1-t,s,w) = n_1(1-t,s)\,(\text{say}), \quad m_2(t,s,w) = n_2(t,s),$$

$$a_1(s,w) = a_1(s).$$

$$(2.29)\quad (m_1 - m_3)(1-t,s,w) = n\left(1-t, \frac{w}{1-s}\right), \quad m_3(t,s,w) = n_3\left(t, \frac{w}{1-s}\right),$$

$$(b_1 - a_1 + c_4)(s,w) = e\left(\frac{w}{1-s}\right)$$

$$(2.30) \qquad m_3(t, s, w) = v_3(t, w), \quad b_1(s, w) = b_1(w)$$

$$(2.31) \qquad m_2(t, s, w) = v_2\left(t, \frac{s}{1-w}\right), \quad c_4(s, w) = c_4\left(\frac{s}{1-w}\right).$$

From the forms m_2 given by (2.28) and (2.31) it is easy to see that $m_2(t, s, w)$ is independent of s and w, that is, m_2 is a function of t only say $m_2(t)$. Similarly $m_3(t, s, w)$ occurring in (2.29) and (2.30) is a function of t only say $m_3(t)$. From this it follows that $m_1(1 - t, s, w)$ occurring in (2.28) and (2.29) is a function of t only say $m_1(1 - t)$. Further, from (2.28) to (2.31), we get

$$b_1(w) - a_1(s) + c_4\left(\frac{s}{1-w}\right) = e\left(\frac{w}{1-s}\right),$$

for $w, s, w + s \in]0, 1[$. Then by [10], it follows that

$$(2.32) \qquad \begin{aligned} b_1(w) &= l_1(1-w) + l_2(w) + a_2 \\ c_4(w) &= l_1(1-w) + l_3(w) - l_3(1-w) + b_2 - a_2 + b_3 \\ a_1(w) &= l_1(1-w) + l_2(1-w) - l_3(1-w) + l_3(w) + b_2, \end{aligned}$$

for $w \in]0, 1[$, where l_i are solutions of (2.5) and a_2, b_2, b_3 are constants, From (2.27) and (2.32), it follows that C_1, C_2, C_3 indeed are of the form given by (2.26).

Similarly, the first, the second and the fourth equations in (2.24) and the forms of C_i's given by (2.26) yield

$$C_3(v) + C\left(\frac{y}{1-v}\right) = A(y) + C_2\left(\frac{v}{1-y}\right) \text{etc.},$$

so that

$$\begin{aligned} C(y) &= M_1(1-y) - M_3(1-y) + M_2(y) + a_5, \\ a(y) &= M_1(1-y) + M_3(y) - M_3(1-y) + a_3, \\ b(y) &= M_1(1-y) + M_3(y) - M_3(1-y) + a_4, \quad y \in]0, 1[^n. \end{aligned}$$

With these values of C_1, C_2, C_3, a, b, C the functions f, g, h, k given in (2.23) satisfy (1.3) for $\beta = 0$, provided they are of the form given by (2.2).

Case 3. Let us treat the case $\beta = 2$.

For fixed $v, y \in]0, 1[^n$, (1.3) can be rewritten as a special case of (1.2) ($\beta = 2$) with $F(x) = f(x, y)$, $G(x) = g(x, v/(1 - y))$, $H(x) = h(x, v)$, $K(x) = k(x, y/(1 - u))$, so that from [5], we see that

$$\begin{aligned} F(x) &= n(1-x) + a_1 x^2 + a_2(1-x)^2 + a_3, \\ G(x) &= n(x) + b_1 x^2 + b_2(1-x)^2 + b_3, \\ H(x) &= n(x) + C_1 x^2 + C_3(1-x)^2 + C_3, \\ K(x) &= n(1-x) + d_1 x^2 + d_2(1-x)^2 + d_3, \quad x \in]0, 1[\end{aligned}$$

where a_i, b_i, c_i, d_i are constants and n is a solution of

$$n(1-x) + (1-x)^2 n\left(\frac{t}{1-x}\right) = n(t) + (1-t)^2 n\left(1 - \frac{x}{1-t}\right).$$

From [2, 11], it follows that

$$n(x) = d(x) + a(x^2 + (1-x)^2 - 1), \quad \text{for} \quad x \in]0, 1[$$

where d is a derivation, that is a solution of (2.8).

These F, G, H, K with n given above satisfy (1.2) for $\beta = 2$, provided

$$f(x, y) = F(x) = -d(x) + a_1 x^2 + a_2(1-x)^2 + a_3,$$

$$g\left(u, \frac{y}{1-v}\right) = G(u) = d(u) + C_1 u^2 + b_1(1-u)^2 - a_2,$$

$$h(x, v) = H(x) = d(x) + C_1 x^2 + C_2(1-x)^2 + a_3,$$

$$k\left(u, \frac{v}{1-y}\right) = K(u) = -d(u) + a_1 u^2 + b_1(1-u)^2 - C_2, \quad x, u \in]0, 1[$$

where d satisfying (2.8), a_i, b_1, C_i constants are functions of y and v. Examination at the boundary points with the help of the equation (1.3) shows that (also refer to [2]), the above forms hold for $x \in [0, 1[$, $u \in [0, 1]$.

From the forms of f and h it is easy to see that d is independent of y and v. Noting that $1, x^2, (1-x)^2$ are linearly independent, a_1, occurring in f and k shows that $a_1(y, v)$ is a function of y and $v/(1-y)$, that is a_1 is a constant. Similarly, we can show that a_2, a_3, C_1, C_2 and b_1 are constants. That is f, g, h, k are of the form (2.3) for $x \in [0, 1[$, $u \in [0, 1]$, $y \in]0, 1[^n$.

Case 4. Finally, let us consider β any real number, other than 0, 1 and 2.

As before, for fixed $y, v \in]0, 1[^n$, (1.3) takes the form (1.2) with $F(x) = f(x, y)$, $G(x) = g(x, v/(1-y))$, $H(x) = h(x, v)$, $K(x) = k(x, y/(1-v))$, so that, from [5], it follows that

$$f(x, y) = F(x) = a_1 x^\beta + a_2(1-x)^\beta + a_3,$$

$$g\left(u, \frac{v}{1-y}\right) = G(u) = a_1 u^\beta + b_2(1-u)^\beta - a_2,$$

$$h(x, v) = H(x) = b_1 x^\beta + C_1(1-x)^\beta + a_3,$$

$$k\left(u, \frac{y}{1-v}\right) = K(u) = a_1 u^\beta + b_2(1-u)^\beta - C_1,$$

for $x \in [0, 1[$, $u \in [0, 1]$, $y, v, y + v \in]0, 1[^n$ where the constants a_i, b_i, C_1 are functions of y and v.

The argument showing that $a_1, a_2, a_3, b_1, b_2, C_3$ are constants is analogous to that given in Case 3 (notice the fact that $1, x^\beta, (1-x)^\beta$ ($\beta \neq 0, 1, 2$) are linearly independent). Thus, in this case f, g, h, k are of the from given by (2.4).

This completes the proof of the theorem. □

Remark. 2. From [8], it follows that M satisfying (2.7) is of the form

$$M(u) = \sum_{i=1}^{n} m_i(u_i), \quad u = (u_1, u_2, \ldots, u_n) \in]0, 1[^n$$

where m_i is a solution of (2.5). Further, if f, g, h, k are measurable in each variable, then l, M_1, M_2, M_3, d occuring in (2.1), (2.2) and (2.3) are given by $l(x) = a \log x$, $M_j(u) = \sum_{i=1}^{n} a_{ij} \log u, j = 1, 2, 3$ and $d = 0$. The solutions (2.4) are regular even though no regularity condition was supposed.

Remark 3. Some remarks on the solutions (2.2), (2.3) and (2.4) are in order. The solution (2.2) is independent of x because of the domain of definition of x. Whereas the domain of definition of x is $[0, 1]$ which includes the end points, the domain of definition of y is $]0, 1[^n$, which excludes the end points. If the domain of definitions of x and y were $]0, 1[$ and $]0, 1[^n$, then indeed the solutions f, g, h and k would depend upon x and y (refer (2.6)). The solutions (2.3) and (2.4) are independent of y, this is simply because the multiplication factors of the second and fourth terms in (1.3) depend only on the variables in the first place. In order to obtain the dependency of the solution in the second variable y also, the equation to be considered would be

$$f(x, y) + (1-x)^\beta (1-y)^\gamma g\left(\frac{u}{1-x}, \frac{v}{1-y}\right) =$$

$$= h(u, v) + (1-u)^\beta (1-v)^\gamma k\left(\frac{x}{1-u}, \frac{y}{1-v}\right)$$

for $x, y \in [0, 1[$ with $x + u \in [0, 1]$, $y, v, y + v \in]0, 1[^n$ where $(1-y)^\gamma$ means $(1-y_1)^{\gamma_1}(1-y_2)^{\gamma_2} \ldots (1-y_n)^{\gamma_n}$, $y_i \in]0, 1[$, $\gamma_i \in R$ which would be dealt elsewhere. For example, if $f = g = h = k$ in the above equation, a solution would be $f(x, y) = ax^\beta y^\gamma + b(1-x)^\beta (1-y)^\gamma - b$, for suitable β, γ (refer [5, 7]).

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