KYBERNETIKA — VOLUME *18* (1982), NUMBER 5

ADDITIONAL SIGNALS IN LINEAR DISCRETE-TIME CONTROL SYSTEMS I

Additional Control Signal

VÁCLAV SOUKUP

The interconnected discrete-time linear systems are gradually investigated using the algebraic (polynomial) approach. Time optimal and least squares optimal control with additional control (feedforward) signal is treated in the first part of this paper.

INTRODUCTION

The additional loops and signals are gradually studied in the paper provided they are applied in linear, discrete-time (sampled-data), single input-single output control systems. The algebraic (polynomial) approach established and developed by Kučera ([1], [2]) is chosen for the investigation. At the same time the fundamental polynomial operations used in a single variable system analysis are only needed. Some problems presented in this work have been solved under supervision of the author in [11] and [12].

Let us note that the multiloop structures are well known and frequently used in automatic control practice for a long time ([3]–[10]) but their applications are often based upon the designer's experience only. The results derived in this paper answer the question when an additional signal can improve the control process provided usual discrete-time synthesis approach is applied.

Following an introductory survey of polynomial theory fundamentals, double controller system structure using the additional control signal (ACS) is treated in this first part. The closed-loop stability condition is formulated and then time optimal as well as least squares control are solved. It is shown that ACS can improve the given control performance index in the case of controlled systems which are described by non-minimum phase discrete-time models especially.

1. POLYNOMIALS, SEQUENCES AND LINEAR DISCRETE-TIME SYSTEM DESCRIPTION

The necessary notions, symbols and operations concerning the algebraic theory of linear discrete-time systems will be briefly mentioned here. More thorough details can be found in [1] or [2].

Given the real field R, constants $\alpha_i \in R$, $i \in [0, n]$, and an indeterminate z^{-1} over R a polynomial

$$a = \alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots + \alpha_n z^{-n} \tag{1}$$

is defined and there is

a) degree of $a = \deg a = n$ if $\alpha_n \neq 0$; $\deg 0 = -\infty$;
   α_n is called the leading coefficient of a;

b) causal a if and only if $\alpha_0 \neq 0$;

c) stable a if and only if a satisfies the stability test ([1], [2]); (if z^{-1} were regarded in (1) as a complex variable then all zeros z_j^{-1} of a stable a would posses the known property $|z_j^{-1}| > 1$, $j = 1, \ldots, n$);

d) factorization $a = a^+ a^-$ where a^+ is the stable polynomial of the greatest degree which is contained in a;

e) $\bar{a} = \alpha_0 + \alpha_1 z + \ldots + \alpha_n z^n$;

f) $a^\sim = z^{-n}\bar{a} = \alpha_n + \alpha_{n-1} z^{-1} + \ldots + \alpha_0 z^{-n}$;

g) $a^* = a^+(a^-)^\sim = a^+ a^{-\sim}$.

Considering two polynomials a, b of the type (1) the following properties and operations are defined and algorithmized in [1] and [2]:

a) division

$$a = bu + v \quad \text{for} \quad b \neq 0 \tag{2}$$

where u and v are given uniquely with $\deg v < \deg b$; b divides a, $b \mid a$, if $v = 0$;

b) $a \sim b$ if and only if $a = ib$ where $\deg i = 0$; obviously $i \sim 1$ and $a \mid b$, $b \mid a$ if $a \sim b$;

c) the greatest common divisor (GCD) $d = (a, b)$;

d) polynomial fraction

$$\frac{b}{a} = \frac{\beta_0 + \beta_1 z^{-1} + \ldots + \beta_m z^{-m}}{\alpha_0 + \alpha_1 z^{-1} + \ldots + \alpha_n z^{-n}} \tag{3}$$

which can be expressed by expansion into ascending powers of z^{-1} as

e) infinite recurrent sequence

$$G = \frac{b}{a} = \gamma_{-k} z^k + \ldots + \gamma_{-1} z + \gamma_0 + \gamma_1 z^{-1} + \ldots; \; \gamma_i \in R; \tag{4}$$

f) the zero-position coefficient of the sequence (4) $\langle G \rangle = \gamma_0$;

g) $\bar{G} = \frac{\bar{b}}{\bar{a}} = \gamma_{-k} z^{-k} + \ldots + \gamma_{-1} z^{-1} + \gamma_0 + \gamma_1 z + \ldots$;

h) causal G if and provided $(a, b) \sim 1$ only if a is causal in (4); then $\gamma_{-k} = \ldots$ $\ldots = \gamma_{-1} = 0$;

i) stable G if and provided $(a, b) \sim 1$ only if a is stable in (4);

j) the squared quadratic norm of a stable G

$$\sigma_G = \|G\|^2 = \sum_{i=0}^{\infty} \gamma_i^2 = \langle \bar{G} G \rangle . \tag{5}$$

Given polynomials a, b and c the linear diophantine equation

$$ax + by = c \tag{6}$$

has a solution x, y if and only if $(a, b) | c$. If the equation (6) is solvable it has an infinite number of solutions.

Let x_0, y_0 be a particular solution of the equation (6). Then

a) the general solution of (6) can be written in the form

$$x = x_0 - \frac{b}{(a, b)} t , \quad y = y_0 + \frac{a}{(a, b)} t \tag{7}$$

where t is any arbitrary polynomial;

b) the minimum degree particular solution with respect to x is given unambiguously as

$$x_1 = v , \quad y_1 = y_0 + \frac{a}{(a, b)} u \tag{8}$$

where u and v are obtained by division $x_0 = b/(a, b) u + v$ according to (2);

c) the particular solution with $\deg x < \deg b$ is not generally unique and follows from the solution (8) as

$$x_2 = x_1 - \frac{b}{(a, b)} t , \quad y_2 = y_1 + \frac{a}{(a, b)} t \tag{9}$$

provided that $\deg t < \deg (a, b)$. If $(a, b) \sim 1$ then $t = 0$ and $x_2 = x_1, y_2 = y_1$ are given unambiguously.

Provided moreover $\deg b > 0$, $\deg a > 0$ in this case we can estimate degrees of x and y in advance such that

$$\deg x = \deg b - 1 , \deg y = \begin{cases} \deg a - 1 & \text{if } \deg a + \deg b > \deg c \\ \deg c - \deg b & \text{if } \deg a + \deg b \leqq \deg c \end{cases} \tag{10}$$

and the unique particular solutions (8) = (9) can be found by comparison of the coefficients at the same powers of z^{-1} in (6).

Now let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be the values of a discrete-time signal at the time instants $0, \tau, 2\tau, \ldots$, respectively; $\tau(\text{sec}) > 0$. We can simply use the causal sequence

$$F = \varphi_0 + \varphi_1 z^{-1} + \varphi_2 z^{-2} + \ldots \tag{11}$$

or the corresponding polynomial fraction to describe the signal provided the powers of z^{-1} in (11) serve as time position-markers only.

If the signal F is applied to a causal, linear, single variable, discrete-time invariant system the response $H = GF$ where causal G stands for the system response on the unit impulse signal $F_1 = 1$ and can be called the system transfer sequence.

Continuously operating systems subjected to a discrete-time input (11) but being observed at the discrete instants of time $0, \tau, 2\tau, \ldots$ only can be analyzed in the same way.

In this paper given systems and signals are assumed to be described by the minimal forms of their mathematical models and therefore

$$(a, b) \sim 1 \quad \text{if} \quad G = \frac{b}{a} \quad \text{is a system or signal description}\,.$$

Moreover strict physical realizability of continuously operating controlled systems is assumed, i.e.

$$z^{-1} \mid b \tag{12}$$

$$\text{if} \quad G = \frac{b}{a} \quad \text{is a controlled system transfer sequence}\,.$$

2. SIMPLE LINEAR DISCRETE-TIME CONTROL SYSTEM

The well-known results ([1], [13]) of the conventional single-variable, discrete-time, linear control system represented by the block diagram in Fig. 1 are mentioned here to serve for further comparison.

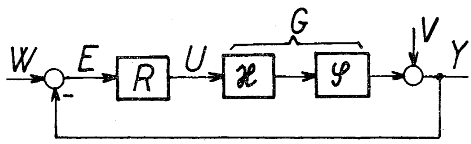

Fig. 1.

Let a controlled system (continuously operating plant $\mathscr{S}$ together with a preceding data reconstructor $\mathscr{H}$) be described by

$$G = \frac{b}{a}, \; (a, b) \sim 1\,,$$

and a controller by

$$R = \frac{m}{n}, \ (n, m) \sim 1 .$$

If a reference signal W is applied to the feedback system which is affected by a disturbance V simultaneously we can write

$$Y = GU + V, \quad U = RE \quad \text{and} \quad E = W - Y$$

where all continuous signals are taken in their discrete-time form, the disturbance moreover being transformed to be additional to the open-loop system output.

1. Closed-loop stability

Closed-loop stability (CLS) is satisfied if

(13) $$an + bm = l, \quad l \text{ stable} .$$

Putting

(14) $$M = \frac{m}{l} \quad \text{and} \quad N = \frac{n}{l}$$

the equation (13) can be rewritten in the other form

(15) $$aN + bM = 1$$

where M and N are stable and N^{-1} causal as

(16) $$R = MN^{-1} \quad \text{must be causal.}$$

2. Optimal control

Let $V = 0$ at first and

$$W = \frac{f}{h}, \quad (h, f) \sim 1 .$$

Then the causal optimal controller can be determined to satisfy both the CLS condition (15) and the control demands in an optimal way.

a) *Time optimal control* (TOC)

Assuming the given sampling period τ the error sequence E must be finite and as short as possible in this case while the control sequence U must be either stable (stable TOC) or finite (finite TOC).

α) Stable TOC is satisfied by the controller (16) with

(17) $$M = \frac{y}{b^+ f^+}, \quad N = \frac{h_0 x}{a_0^+ f^+}$$

where

$$a_0 = \frac{a}{(a, h)}, \quad h_0 = \frac{h}{(a, h)} \tag{18}$$

and x, y is the solution of the equation

$$a_0^- hx + b^- y = f^+ \tag{19}$$

with the minimum degree of a causal x.

The error sequence (polynomial)

$$E = e = a_0^- f^- x \tag{20}$$

and the control sequence

$$U = \frac{a_0 f^- y}{h_0 b^+}. \tag{21}$$

Stable TOC problem is solvable if and only if h_0 is stable. The optimal solution, if it exists, is given under assumption (12) unambiguously.

β) Finite TOC is satisfied by the controller (16) with

$$M = \frac{y}{f^+}, \quad N = \frac{h_0 x}{a_0^+ f^+} \tag{22}$$

where a_0 and h_0 are given by (18) and x, y is the solution of the equation

$$a_0^- hx + by = f^+ \tag{23}$$

with the minimum degree of a causal x.

The error polynomial is given by (20) and the control sequence (polynomial)

$$U = u = \frac{a_0}{h_0} f^- y. \tag{24}$$

The finite TOC problem is solvable if and only if $h_0 \sim 1$. The optimal solution is unique if the condition (12) is valid.

b) *Least squares control* (LSC)

In this case the squared quadratic norm $\sigma_E = \|E\|^2$ of the error sequence E is required to attain its minimum and the control sequence U to be stable.

The optimal controller (16) is given by

$$M = \frac{y}{b^* f^* a_0^{-\sim}}, \quad N = \frac{h_0 x}{b^{-\sim} f^* a_0^*} \tag{25}$$

where a_0 and h_0 are according to (18) and x, y is the solution of the equation

$$a_0^- hx + b^- y = b^{-\sim} f^* a_0^{-\sim} \tag{26}$$

with x causal, $\deg x < \deg b^-$.

The error sequence

(27) $$E = \frac{f^- a_0^- x}{f^{-\sim} a_0^{-\sim} b^{-\sim}},$$

the control sequence

(28) $$U = \frac{a_0 f^- y}{a_0^{-\sim} f^{-\sim} h_0 b^*}$$

and

(29) $$\sigma_{E_{\min}} = \left\langle \overline{\left(\frac{x}{b^-}\right)} \frac{x}{b^-} \right\rangle$$

LSC problem is solvable if and only if h_0 is stable. Assuming (12) the optimal solution is unique.

c) *Disturbance effect*

The formulations and solutions of the given control problems don't change at the presence of a disturbance $V \neq 0$. This fact follows from the transformed block diagram shown in Fig. 2 which is equivalent to Fig. 1.

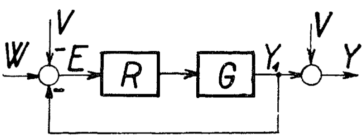

Fig. 2.

Putting

(30) $$W_1 = W - V = \frac{f}{h}$$

all the above relations can be used unchanged.

3. CLOSED-LOOP STABILITY AND CAUSALITY OF A SYSTEM WITH ADDITIONAL CONTROL SIGNAL

If an auxiliary additional control signal U_2 formed by an additional controller R_2 may be applied through a data-reconstructor $\mathscr{H}$ to the selected second part $\mathscr{S}_2$ of a controlled plant according to Fig. 3 the following relations are valid:

(31) $$Y = GU_1 + G_2 U_2 + V, \quad U_1 = R_1 E, \quad U_2 = R_2 E \quad \text{and} \quad E = W - Y$$

where

$$(32)\qquad G = \frac{b}{a}\,, \quad (a, b) \sim 1\,,$$

represents a discrete-time transfer sequence of the whole controlled system (including $\mathscr{H}$, $\mathscr{S}_1$ and $\mathscr{S}_2$),

$$(33)\qquad G_2 = \frac{b_2}{a_2}\,, \quad (a_2, b_2) \sim 1\,,$$

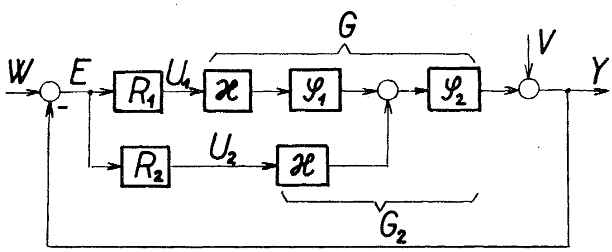

Fig. 3.

a transfer sequence of its second part (including $\mathscr{H}$ and $\mathscr{S}_2$),

$$(34)\qquad R_1 = \frac{m_1}{n_1}\,, \quad (n_1, m_1) \sim 1\,, \quad \text{and} \quad R_2 = \frac{m_2}{n_2}\,, \quad (n_2, m_2) \sim 1\,,$$

are two controllers transfer sequences.

Provided that dynamic modes of $\mathscr{S}_1$ and $\mathscr{S}_2$ cannot be mutually compensated then generally b_2 does not divide b but $a_2 \mid a$ and we can write $a = a_1 a_2$.

Theorem 1. A closed-loop system with ACS pictured in Fig. 3 and described by the relations (31)–(34) is stable and causal (physically realizable) if and only if

$$(35)\qquad R_1 = M_1 N^{-1} \quad \text{and} \quad R_2 = M_2 N^{-1}$$

where M_1, M_2 and N are stable sequences which satisfy CLS equation

$$(36)\qquad aN + bM_1 + b_2 a_1 M_2 = 1$$

and N^{-1} is causal.

Proof.

1. It will be proved at first that closed-loop system is stable if and only if

$$K_{W/Y} = bM_1 + b_2a_1M_2 \tag{37}$$

$$\text{in} \quad Y = K_{W/Y}W + K_{V/Y}V$$

and

$$K_{W/E} = aN \tag{38}$$

$$\text{in} \quad E = K_{W/E}W + K_{V/E}V$$

where M_1, M_2 and N are stable sequences.

a) Only if: Considering (32)–(34) define the controlled system transfer matrix

$$\boldsymbol{G} = [G \ G_2] = a^{-1}[b \ b_2a_1] \tag{39}$$

and the controller transfer matrix

$$\boldsymbol{R} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} m_1n_{20} \\ m_2n_{10} \end{bmatrix} ((n_1, n_2)\, n_{10}n_{20})^{-1} \tag{40}$$

where

$$n_{10} = \frac{n_1}{(n_1, n_2)} \quad \text{and} \quad n_{20} = \frac{n_2}{(n_1, n_2)}.$$

The right-hand sides of (39) and (40) are left coprime factorization of $\boldsymbol{G}$ and right coprime factorization of $\boldsymbol{R}$, respectively ([1], [2], [14]).

Hence closed-loop system pseudocharacteristic polynomial

$$l = a(n_1, n_2)\, n_{10}n_{20} + bm_1n_{20} + b_2a_1m_2n_{10} \tag{41}$$

and closed-loop transfer sequences

$$K_{W/Y} = (bm_1n_{20} + b_2a_1m_2n_{10})\, l^{-1} \tag{42}$$

$$K_{W/E} = a(n_1, n_2)\, n_{10}n_{20}l^{-1}.$$

Denoting

$$M_1 = m_1n_{20}l^{-1}, \quad M_2 = m_2n_{10}l^{-1} \tag{43}$$

$$\text{and} \quad N = (n_1, n_2)n_{10}n_{20}l^{-1}$$

then really $K_{W/Y}$ and $K_{W/E}$ have the form (37) and (38), respectively. A system is stable if and only if its pseudocharacteristic polynomial l is a stable polynomial. Therefore M_1, M_2 and N are stable too.

b) If: Let us assume that pseudocharacteristic polynomial $l = l^+l^-$ with $l^- \sim 1$. If M_1, M_2 and N are stable then according to (43)

$$m_1n_{20} = l^-m_\alpha, \quad m_2n_{10} = l^-m_\beta \quad \text{and} \quad (n_1, n_2)\, n_{10}n_{20} = l^-n$$

is necessary. But

$$(m_1 n_{20}, m_2 n_{10}, (n_1, n_2)\, n_{10} n_{20}) \sim 1$$

and hence $l^- \sim 1$ must be valid.

Note that the remaining closed-loop transfer sequences

(44) $$K_{W/U_1} = -K_{V/U_1} = aM_1\,, \quad K_{W/U_2} = -K_{V/U_2} = aM_2$$

and

$$K_{V/Y} = -K_{V/E} = aN$$

are then stable too.

2. Now using these results we can simply prove the relation (36) since

$$aN + bM_1 + b_2 a_1 M_2 = (a(n_1, n_2)\, n_{10} n_{20} + b m_1 n_{20} + b_2 a_1 m_2 n_{10})\, l^{-1} =$$
$$= l l^{-1} = 1,$$

and (35) since using (44)

$$R_1 = K_{W/U_1} K_{W/E}^{-1} = aM_1 N^{-1} a^{-1} = M_1 N^{-1}$$

and

$$R_2 = K_{W/U_2} K_{W/E}^{-1} = aM_2 N^{-1} a^{-1} = M_2 N^{-1}\,.$$

Obviously causality of N^{-1} is required to ensure causal controllers (35). □

The equation (36) can be rewritten in the polynomial form (41). Properties and solutions of such an equation are investigated in the next section.

4. POLYNOMIAL EQUATION $ax + by + cv = l$ AND ITS SOLUTION

For any polynomials a, b and c we can always find their GCD $d = (a, b, c)$ along with three triplets of polynomials p, q, r; p_1, q_1, r_1 and p_2, q_2, r_2 such that

(45) $$ap + bq + cr = d$$

(46) $$ap_1 + bq_1 + cr_1 = 0 \quad \text{and} \quad ap_2 + bq_2 + cr_2 = 0\,.$$

Note that $d = (a, b, c)$ can also be considered as the special case of a general matrix GCD investigated in [1]. The algorithm for the calculation of d and all other polynomials in (45), (46) is given in the Appendix.

The identities

(47) $$(a, b, c) = ((a, b), c) = (a, (b, c)) = ((a, c,), b)$$

are evident.

Given polynomials a, b, c and l the linear diophantine equation

$$ax + by + cv = l \tag{48}$$

is solved by any triplet x, y, v satisfying (48).

Theorem 2. The equation (48) has a solution if and only if $(a, b, c) \mid l$.

Proof. Only if: Let x_0, y_0, v_0 be a solution of (48) and $a = a_0d$, $b = b_0d$, $c = c_0d$ where $d = (a, b, c)$. Then $d(a_0x_0 + b_0y_0 + c_0v_0) = l$ and consequently $d \mid l$.

If: Let $d \mid l$ and $l = dl_0$. Writting (45) multiplied by l_0 we have $apl_0 + bql_0 + crl_0 = dl_0 = l$ and pl_0, ql_0, rl_0 is a solution of (48). □

General solution of the equation (48) is a linear composition of any particular solution and the general solution of the equation

$$ax + by + cv = 0\,. \tag{49}$$

Let us denote

$$\frac{a}{(a, b)} = a_b\,, \quad \frac{a}{(a, c)} = a_c\,, \quad \frac{b}{(a, b)} = b_a\,,$$

$$\frac{b}{(b, c)} = b_c\,, \quad \frac{c}{(a, c)} = c_a \quad \text{and} \quad \frac{c}{(b, c)} = c_b\,.$$

Obviously

$$ab_a - ba_b = 0\,, \quad ac_a - ca_c = 0 \quad \text{and} \quad bc_b - cb_c = 0\,;$$

hence the triplets b_a, $-a_b$, 0; c_a, $-a_c$, 0 and 0, c_b, $-b_c$ represent solutions of (49). These solutions are the simplest ones because the polynomials in each triplet are not mutually divisable. But only two triplets are independent seeing that

$$\operatorname{rank}\begin{bmatrix} b_a & -a_b & 0 \\ c_a & 0 & -a_c \\ 0 & c_b & -b_c \end{bmatrix} = 2\,.$$

Then the complete general solution of the equation (48) can be written in the form, e.g.,

$$\begin{aligned} x &= x_0 + b_at_1 + c_at_2 \\ y &= y_0 - a_bt_1 \\ v &= v_0 \qquad\quad - a_ct_2 \end{aligned} \tag{50}$$

where x_0, y_0, v_0 is a particular solution of (49) and t_1, t_2 are any arbitrary polynomials.

If $d = (a, b, c)$ and polynomials p, q, r; p_1, q_1, r_1 and p_2, q_2, r_2 satisfying (45) to (46) have been computed the particular solution of (48) can be formed as

$$x_0 = p\,\frac{l}{d}\,, \quad y_0 = q\,\frac{l}{d}\,, \quad v_0 = r\,\frac{l}{d} \tag{51}$$

and the general solution is

$$(52) \qquad x = p\frac{l}{d} + p_1 t_1 + p_2 t_2$$

$$y = q\frac{l}{d} + q_1 t_1 + q_2 t_2$$

$$v = r\frac{l}{d} + r_1 t_1 + r_2 t_2$$

with any arbitrary t_1, t_2.

Special solutions of the equation (48) can be found by the suitable choice of t_1 and t_2 in (50) or (52). The other approach uses the decomposition of the equation (48) into two equations of the type (6). Searching, e.g., for min deg x solution of (48) we put

$$(53) \qquad by + cv = (b, c)\, s$$

and

$$(54) \qquad ax + (b, c)\, s = l\,.$$

Hence provided $(a, b, c) \mid l$ at first the particular min deg x solution x, s of (54) can be determined according to (8), (9) or (10). Having x, s we can find general or a suitable particular solution y, v of the equation (53) which is always solvable.

Three terms in (48) can be conjugated in advance according to our demands, of course.

Note that a special solution of (48) is not generally unique even in the case $(a, b, c) \sim 1$ if the requirement is referred to one of polynomials x, y, v only.

One example is given for illustration.

Example. Let us find a) general solution and b) min deg x solution of the equation

$$(1 - z^{-1})\, x + z^{-1}(1 + 2z^{-1})\, y + z^{-1}v = 1\,.$$

a) Using the GCD algorithm presented in the Appendix we obtain

$$d = (1 - z^{-1}, z^{-1} + 2z^{-2}, z^{-1}) = 1 \quad \text{and} \quad p = 1\,, \quad q = 0\,, \quad r = 1\,,$$

$$p_1 = -z^{-1}\,, \quad q_1 = 1\,, \quad r_1 = -3z^{-1}\,; \quad p_2 = z^{-1}\,, \quad q_2 = 0\,, \quad r_2 = -1 + z^{-1}\,.$$

The given equation is solvable and its general solution can be written according to (52):

$$(55) \qquad x = 1 - z^{-1}t_1 + z^{-1}t_2\,, \quad y = t_1\,, \quad v = 1 - 3z^{-1}t_1 - (1 - z^{-1})\, t_2$$

If the particular solution, e.g., $x_0 = 1 - z^{-1}$, $y_0 = -0{\cdot}5$ and $v_0 = 2{\cdot}5$ and $b_a = b = z^{-1}(1 + 2z^{-1})$, $c_a = c = z^{-1}$, $a_b = a_c = a = 1 - z^{-1}$ are determined then the general solution in the form (50) is

$$x = 1 - z^{-1} + z^{-1}(1 + 2z^{-1})\, t_1 + z^{-1} t_2 \tag{56}$$
$$y = -0{\cdot}5 - (1 - z^{-1})\, t_1$$
$$v = 2{\cdot}5 - (1 - z^{-1})\, t_2\,.$$

b) Starting with the general solution (55) any min deg x solution is obviously with $x = 1$, $\deg x = 0$, for any arbitrary $t_1 = t_2 = t$ in (55). Using the solution (56) and choosing $t_1 = 0$, $t_2 = 1$ we obtain $x = 1$, $y = -0{\cdot}5$ and $v = 1{\cdot}5 + z^{-1}$.

According to the second approach we can decompose the given equation for $(b, c) = z^{-1}$:

$$z^{-1}(1 + 2z^{-1})\, y + z^{-1} v = z^{-1} s$$

and

$$(1 - z^{-1})\, x + z^{-1} s = 1\,.$$

The unique min deg x solution of the second equation is found to be $x = 1$, $s = 1$. Substituting $s = 1$ into the first equation its simplest particular solution is $y_0 = 0$, $v_0 = 1$ and its general solution

$$y = y_0 - t = -t\,, \quad v = v_0 + (1 + 2z^{-1})\, t = 1 + (1 + 2z^{-1})\, t \text{ for any } t\,.$$

Then all triplets $1, 0, 1$; $1, 1, -2z^{-1}$; $1, -0{\cdot}5, 1{\cdot}5 + z^{-1}$; ... represent the min deg x solution of the given equation.

5. TIME OPTIMAL CONTROL WITH ADDITIONAL CONTROL SIGNAL

Let us return to the system with ACS shown in Fig. 3 and formulate the problem of TOC in the same way as for simple control. We shall also distinguish stable and finite TOC with the both control sequences U_1 and U_2 either stable or finite, respectively. The solution is presented and proved in the following theorem.

Theorem 3. Given a discrete-time system with ACS pictured in Fig. 3, described by the relations (31)–(34) and subjected to the inputs

$$V = 0\,, \quad W = \frac{f}{h}\,, \quad (h, f) \sim 1\,,$$

then

a) stable TOC is ensured by the controllers (35) where

$$M_1 = \frac{y}{b^+ f^+}\,, \quad M_2 = \frac{v}{b_2^+ a_1^+ f^+} \quad \text{and} \quad N = \frac{h_0 x}{a_0^+ f^+}\,. \tag{57}$$

Here

(58) $$a_0 = \frac{a}{(a, h)}, \quad h_0 = \frac{h}{(a, h)}$$

and x, y, v is the solution of the equation

(59) $$a_0^- hx + b^- y + b_2^- a_1^- v = f^+$$

with the minimum degree of a causal x.

The error sequence (polynomial)

(60) $$E = e = a_0^- f^- x$$

with

(61) $$\deg x < \deg d^- \quad \text{where} \quad d = (b, b_2)$$

and the control sequences

(62) $$U_1 = \frac{a_0 f^- y}{b^+ h_0} \quad \text{and} \quad U_2 = \frac{a_0 f^- v}{b_2^+ a_1^+ h_0} = \frac{a_1^- a_2 f^- v}{b_2^+ h}.$$

The optimal solution exists if and only if h_0 is stable. Optimal controllers are not given unambiguously while the resulting error polynomial (60) is unique.

b) finite TOC is satisfied by the controllers (35) where

(63) $$M_1 = \frac{y}{f^+}, \quad M_2 = \frac{v}{f^+} \quad \text{and} \quad N = \frac{h_0 x}{a_0^+ f^+}.$$

Here a_0 and h_0 are given by (58) and x, y, v is the solution of the equation

(64) $$a_0^- hx + by + b_2 a_1 v = f^+$$

with the minimum degree of a causal x.

The error polynomial is given in the form (60) with

(65) $$\deg x < \deg d$$

and the control sequences (polynomials) are

(66) $$U_1 = \frac{a_0 f^- y}{h_0} \quad \text{and} \quad U_2 = \frac{a_0 f^- v}{h_0}.$$

The optimal solution exists if and only if $h_0 \sim 1$. Optimal controllers are not given unambiguously but the resulting error is unique.

Proof. Let us write the closed-loop stability equation (36) in the form

(67) $$aN = 1 - bM_1 - b_2 a_1 M_2 = 1 - d(b_0 M_1 + b_{20} a_1 M_2) = 1 - dL$$

where $d = (b, b_2)$, $b = b_0 d$ and $b_2 = b_{20} d$.

Since M_1 and M_2 are required to be stable the sequence

$$L = b_0 M_1 + b_{20} a_1 M_2$$

must be stable too. Multiplying both the sides of (67) by $W = f/h$ then

$$aN\frac{f}{h} = \frac{f}{h} - bM_1\frac{f}{h} - b_2 a_1 M_2 \frac{f}{h} = \frac{f}{h} - dL\frac{f}{h} = E \tag{68}$$

with respect to (42), (43). The error sequence is required to be polynomial $E = e$ and therefore considering (58) the optimal choice of a stable N stands in (57) as well as in (63). The resulting error has the form (60) where a causal polynomial x is undetermined till now. It follows from (68) that $f - he = dfL$ must be a polynomial too.

a) Hence the optimal choice of L is

$$L = \frac{s}{d^+ f^+}$$

and

$$f - he = d^- f^- s$$

or using (60)

$$a_0^- hx + d^- s = f^+ \,. \tag{69}$$

The equation (69) is solvable if and only if $(a_0^- h, d^-) \sim 1$. Since the error polynomial is required to be as short as possible the min deg x solution x, s of (69) is optimal. Considering $z^{-1} \mid d^-$ in accordance with (12) then always $\deg d^- > 0$ and the min deg x solution of (69) is unique; hence the optimal error is unique too.

But L can be realized in the system structure through M_1 and M_2 only in such a way that the equation

$$bM_1 f + b_2 a_1 M_2 f = d^- f^- s$$

resulting directly from (68) must be always solvable for any s. The only choice (57) of stable M_1 and M_2 ensures this property; in this case the equation

$$b^- y + b_2^- a_1^- v = d^- s \tag{70}$$

is obtained which is always solvable. All solutions y, v of (70) are allowed and therefore M_1 and M_2 as well as the controllers R_1 and R_2 are not given unambiguously. Combining the equations (69) and (70) the only equation (59) can be written and solved for min deg x.

The resulting control sequences U_1 and U_2 are given by (62). They are stable for $h_0 \sim h_0^+$ only. In this case the equation (59) as well as (69) is always solvable because $(a_0^- h, b^-, b_2^- a_1^-) \sim (h_0, d^-) \sim 1$.

b) The choice (57) of M_1 and M_2 does not ensure finite U_1 and U_2. Analyzing (62) and (68) the only $h_0 \sim 1$ is allowed in this case and a stable L must be

$$L = \frac{s}{f^+} .$$

Then

$$f - he = df^- s \quad \text{and}$$

(71) $$a_0^- hx + ds = f^+ .$$

Provided $h_0 \sim 1$ the equation (71) is always solvable; its unique min deg x solution x, s must be found.

Substituting L into (68) the equation

$$bM_1 f + b_2 a_1 M_2 f = df^- s$$

is obtained. Since this equation must be always solvable for any s stable M_1 and M_2 are chosen according to (63). The resulting equation is

(72) $$by + b_2 a_1 v = ds .$$

General solution y, v of (72) is allowed and therefore M_1 and M_2 as well as R_1 and R_2 are not given unambiguously. Combining (71) and (72) the only equation (64) is obtained.

The control sequences U_1 and U_2 are given by the relations (66). □

6. LEAST SQUARES CONTROL WITH ADDITIONAL CONTROL SIGNAL

In this case $\sigma_E = \|E\|^2$ is required to attain its minimum and both U_1 and U_2 stable. The solution is formulated and proved in the following theorem.

Theorem 4. Given a discrete-time system with ACS shown in Fig. 3 and described by the relations (31)–(34) with the inputs

$$V = 0 , \quad W = \frac{f}{h} , \quad (h, f) \sim 1 ,$$

LSC is satisfied by the controllers (35) where

(73) $$M_1 = \frac{y}{b^+ d^{-\sim} f^* a_0^{-\sim}} , \quad M_2 = \frac{v}{b_2^+ a_1^+ d^{-\sim} f^* a_0^{-\sim}}$$

and

(74) $$N = \frac{h_0 x}{d^{-\sim} f^* a_0^*} .$$

Here a_0 and h_0 are given by (58),

(75) $$d = (b, b_2)$$

and x, y, v is the solution of the equation

(76) $$a_0^- hx + b^- y + b_2^- a_1^- v = d^{-\sim} f^* a_0^{-\sim}$$

with the causal x, $\deg x < \deg d^-$.

The optimal error sequence has the form

(77) $$E = \frac{f^- a_0^- x}{f^{-\sim} d^{-\sim} a_0^{-\sim}},$$

the corresponding control sequences are

(78) $$U_1 = \frac{a_0 f^- y}{a_0^{-\sim} b^+ d^{-\sim} f^{-\sim} h_0} \quad \text{and} \quad U_2 = \frac{a_0 f^- v}{a_0^{-\sim} b_2^+ a_1^+ d^{-\sim} f^{-\sim} h_0}$$

and the optimal control performance index

(79) $$\sigma_{E_{\min}} = \left\langle \overline{\left(\frac{x}{d^-}\right)} \frac{x}{d^-} \right\rangle .$$

The optimal solution exists if and only if h_0 is stable. Optimal controllers are not given unambiguously while the resulting optimal error sequence is unique.

Proof. Any stable error sequence $E = W - K_{W/Y} W$ and $\sigma_E = \|E\|^2 = \langle \bar{E} E \rangle$ in accordance with (5). Let us denote

(80) $$E^* = W^* - K_{W/Y} W^*$$

where

$$W^* = \frac{f^*}{h} .$$

Then

$$E = E^* \frac{f^-}{f^{-\sim}} \quad \text{and} \quad \bar{E} E = \overline{E^*} E^* .$$

Putting

(81) $$K_{W/Y} = bM_1 + b_2 a_1 M_2 = dL$$

where

(82) $$d = (b, b_2) \quad \text{and} \quad L = \frac{1}{d} (bM_1 + b_2 a_1 M_2)$$

the equation (80) obtains the form

(83) $$E^* = W^* - dLW^* .$$

Hence optimal L ensuring $\sigma_{E_{\min}}$ must be found. Let us write

(84) $$\overline{E^*}E^* = (\overline{W^*} - \overline{dLW^*})(W^* - dLW^*) = (\bar{Z} - \overline{d^*LW^*})(\bar{Z} - d^*LW^*)$$

where a sequence Z simply satisfies the identity (84). Since the identities

(85) $$c\bar{c} = c^*\overline{c^*} = c^{\sim}\overline{c^{\sim}}$$

are valid for any polynomial c the following relations result from the comparison of the multiplied terms in (84):

$$d^*\bar{Z} = d\overline{W^*}\,, \quad \overline{d^*}Z = \bar{d}W^* \quad \text{and} \quad \bar{Z}Z = \overline{W^*}W^*\,.$$

Then (84) can be written in the form

(86) $$\overline{E^*}E^* = \left(\frac{d}{d^*}\overline{W^*} - \overline{d^*LW^*}\right)\left(\frac{\bar{d}}{\overline{d^*}}W^* - d^*LW^*\right) =$$

$$= \left(\frac{d}{d^*}\overline{W^*} - \overline{d^*LW^*}\right)\overline{\left(\frac{c^{\sim}}{c}\right)}\left(\frac{\bar{d}}{\overline{d^*}}W^* - d^*LW^*\right)\frac{c^{\sim}}{c} = \overline{E_0}E_0$$

where

(87) $$E_0 = \frac{\bar{d}c^{\sim}}{\overline{d^*}c}W^* - d^*\frac{c^{\sim}}{c}LW^* = \frac{d^{-\sim}c^{\sim}f^*}{d^-ch} - d^*\frac{c^{\sim}}{c}L\frac{f^*}{h}$$

with a polynomial c undetermined till now.

The following expansion of the first term in (87) into partial polynomial fractions

(88) $$\frac{d^{-\sim}c^{\sim}f^*}{d^-ch} = \frac{x}{d^-} + \frac{s}{ch}$$

results in the equation

(89) $$chx + d^-s = d^{-\sim}f^*c^{\sim}\,.$$

Putting

(90) $$X = \frac{s}{ch} - d^*\frac{c^{\sim}f^*}{ch}L$$

then

(91) $$E_0 = X + \frac{x}{d^-}$$

and

(92) $$\sigma_E = \sqrt{\left\langle\overline{\left(X + \frac{x}{d^-}\right)}\left(X + \frac{x}{d^-}\right)\right\rangle}\,.$$

It is proved by Kučera in [1] (pp. 208) that the expression of the form (92) reaches the minimum value if $X = 0$ provided x, s represent the particular solution of the equation (89) with $\deg x < \deg d^-$. Using this result $\sigma_{E_{\min}}$ stands in (79) and optimal

$$L = \frac{s}{d^* f^* c^\sim} . \tag{93}$$

Substituting (93) into (83) then

$$E^* = \frac{f^*}{h} - \frac{d^- s}{d^{-\sim} h c^\sim} = \frac{f^*}{h} - \frac{d^{-\sim} f^* c^\sim - chx}{d^{-\sim} h c^\sim} = \frac{cx}{c^\sim d^{-\sim}} \tag{94}$$

where the equation (89) has been used.

In a stable system there is at the same time

$$E^* = aNW^* = \frac{a_0 f^*}{h_0} N . \tag{95}$$

The comparison of (95) and (94) results in

$$N = \frac{h_0 cx}{a_0 f^* c^\sim d^{-\sim}} .$$

Hence $c = a_0^-$ must be chosen to ensure stability of N. Then optimal N stands in (74), optimal error in (77), the equation (89) obtains the form

$$a_0^- hx + d^- s = d^{-\sim} f^* a_0^{-\sim} \tag{96}$$

and

$$L = \frac{s}{d^* f^* a_0^{-\sim}} . \tag{97}$$

If L is substituted into the equation (81) then

$$d^* f^* a_0^{-\sim} b M_1 + d^* f^* a_0^{-\sim} b_2 a_1 M_2 = \mathrm{d}s \tag{98}$$

Stable M_1 and M_2 must be chosen in such a way that the equation (98) is always solvable for any s. The choice (73) satisfies this requirement seeing that the resulting equation

$$b^- y + b_2^- a_1^- v = d^- s \tag{99}$$

is always solvable. Combining the equations (99) and (89) the only equation (76) is obtained.

The control sequences result in (78). They are stable if and only if $h_0 \sim h_0^+$. Then the equation (76) as well as (89) is always solvable since $(h, d^-) \sim (h_0, d^-) \sim 1$.

The solution with $\deg x < \deg d^-$ of (89) is unique and identical with min deg x solution. Hence the error (77) is unique too. Any solution y, v of (99) is allowed and therefore R_1 and R_2 are not unique. □

7. OPTIMAL CONTROL AT THE PRESENCE OF A DISTURBANCE

The external disturbance affecting the system through any part of $\mathscr{S}_1$ or $\mathscr{S}_2$ can be transformed to be

$$V = \frac{f_V}{h_V}$$

additional to the open-loop system output (Fig. 3).

Consequently the block diagram can be transformed in the way used in II.2c (Fig. 2) and the only input $W_1 = W - V$ considered for the design.

Putting

$$W_1 = \frac{f}{h}$$

all the results and relations given by Theorem 3 and 4 are valid unchanged.

8. CONCLUSIONS

The natural question whether and when can optimal control be improved by ACS must be discussed and answered.

Three main conclusions concerning solvability, optimality and additional equipment demands follow from the comparison with a simple control system.

1. With regard to solvability of the optimal control problems treated above it must be said that any optimal control problem is not solvable using ACS unless being solvable in simple control system (and on the contrary).

In the both cases the same condition of solvability is valid, namely

$$h_0 \sim h_0^+ \quad \text{for stable TOC and LSC}$$

and

$$h_0 \sim 1 \quad \text{for finite TOC}$$

where

$$h_0 = \frac{h}{(a, h)} \quad \text{if} \quad G = \frac{b}{a} \quad \text{and} \quad W = \frac{f}{h}$$

is the overall controlled system and the input transfer sequence, respectively.

2. Let us consider a general control performance index λ which is minimized by an optimal solution. The minimal values of λ attained by one controller $R_1 = R$ in a simple control system and by two controllers R_1, R_2 in ACS configuration are denoted by λ_1 and λ_2, respectively.

Applying ACS we search the optimal pair R_1, R_2 among all possible pairs including the pairs R_1, 0, i.e., including the simple control configuration.

Therefore

$$(100) \qquad \lambda_2 = \min_{R_1,R_2} \lambda = \min_{R_1,R_2} (\lambda_{R_2 \neq 0}, \lambda_{R_2 = 0}) \leqq \min_{R_1} \lambda_{R_2 = 0} = \lambda_1 \,.$$

Analyzing the relations presented in Sections 2, 5 and 6 we can see that $\lambda_2 = \lambda_1$ if

$$d^- \sim b^- \quad \text{for stable TOC or LSC solution, and}$$

$$d \sim b \quad \text{for finite TOC solution.}$$

In this case the solution x, y, v with $v = 0$ of the equations (59), (64) and (76) always exists among all their optimal solutions and consequently the pair $R_1, 0$ among equivalent optimal R_1, R_2 and therefore ACS cannot bring an effect in optimality. The application of ACS is purposeful if $\lambda_2 < \lambda_1$; it can come only if

$$(101) \qquad d^- \nsim b^- \quad \text{for stable TOC and LSC, and}$$

$$(102) \qquad d \nsim b \quad \text{for finite TOC.}$$

Thus ACS can improve stable TOC or LSC of a non-minimum phase controlled system $G = b/a$ with b^- containing other unstable factors in addition to z^{-1}.

This restriction is not valid for the application of ACS in the case of finite TOC.

3. Additional control signal can be applied in discrete-time control systems almost without a special technical equipment. If the both transfer sequences R_1 and R_2 are realized by computer programs the only additional data reconstructor $\mathscr{H}$ preceding a controlled subsystem $\mathscr{S}_2$ is needed.

Example. The continuous-time controlled subsystems in the block diagram in Fig. 3 are described by their transfer functions (in Laplace transform)

$$\mathscr{S}_1(p) = \frac{e^{-p}}{p}, \quad \mathscr{S}_2(p) = \frac{1}{(p + 0{\cdot}5)^2} \quad \text{and} \quad \mathscr{H}(p) = \frac{1 - e^{-p\tau}}{p} \,.$$

Let us find the time optimal as well as least squares optimal error sequence and the corresponding discrete-time controllers R_1 and R_2 if

$$V = 0 \,, \quad W = \frac{f}{h} = \frac{1}{1 - z^{-1}}$$

and the sampling period $\tau = 1$ sec. The results will be compared with the simple control solution.

At first the discrete-time transfer sequences

$$G = \frac{b}{a} = \frac{0{\cdot}1306z^{-2}(1 + 2{\cdot}9276z^{-1})(1 + 0{\cdot}2071z^{-1})}{(1 - z^{-1})(1 - 0{\cdot}6065z^{-1})^2}$$

and

$$G_2 = \frac{b_2}{a_2} = \frac{0{\cdot}3608z^{-1}(1 + 0{\cdot}7165z^{-1})}{(1 - 0{\cdot}6065z^{-1})^2}$$

are determined.
Hence

$$f^+ = f^- = f = 1\,,\quad h_0 = 1\,,\quad a_0 = a_0^+ = a_2 = (1 - 0{\cdot}6065z^{-1})^2\,,$$

$$a_1 = a_1^- = 1 - z^{-1}\,,$$

$$b^- = z^{-2}(1 + 2{\cdot}9276z^{-1}),\quad b_2^- = z^{-1}\,,\quad d = (b, b_2) = d^- = (b^-, b_2^-) = z^{-1}\,.$$

1. Solving the stable TOC problem we substitute into the equation (59):

$$(1 - z^{-1})\,x + z^{-2}(1 + 2{\cdot}9276z^{-1})\,y + z^{-1}(1 - z^{-1})\,v = 1\,.$$

The min deg x solution is

$$x = 1\,,\quad y = 0{\cdot}2546 - (1 - z^{-1})\,t$$

and $v = 1 + 0{\cdot}7454z^{-1} + z^{-1}(1 + 2{\cdot}9276z^{-1})\,t$ with an arbitrary t.
The error $e = a_0^- f^- x = 1$, $\deg e = \deg x = 0$.

Putting $t = 0$ the simplest pair of the controllers transfer sequences is according to (35) and (57)

$$R_1 = \frac{0{\cdot}2546(1 - 0{\cdot}6065z^{-1})^2}{0{\cdot}1306(1 + 0{\cdot}2071z^{-1})} \quad\text{and}\quad R_2 = \frac{(1 + 0{\cdot}7454z^{-1})(1 - 0{\cdot}6065z^{-1})^2}{0{\cdot}3608(1 + 0{\cdot}7165z^{-1})}\,,$$

the control sequences (62) are $U_1 = R_1$ and $U_2 = R_2$.

Stable TOC in simple system ($R_2 = 0$) solved for the comparison gives the results:

$$e = 1 + z^{-1} + 0{\cdot}7454z^{-2}\,,\quad R = \frac{0{\cdot}2546(1 - 0{\cdot}6065z^{-1})^2}{0{\cdot}1306(1 + 0{\cdot}2071z^{-1})(1 + z^{-1} + 0{\cdot}7454z^{-2})}$$

and

$$U = \frac{0{\cdot}2546(1 - 0{\cdot}6065z^{-1})^2}{0{\cdot}1306(1 + 0{\cdot}2071z^{-1})}\,.$$

Hence $\deg e = 2$ and $\lambda_1 - \lambda_2 = 2$.

2. Solving the finite TOC problem the equation (64) has the form

$$(1 - z^{-1})\,x + 0{\cdot}1306z^{-2}(1 + 2{\cdot}9276z^{-1})(1 + 0{\cdot}2071z^{-1})\,y +$$
$$+ 0{\cdot}3608(1 + 0{\cdot}7165z^{-1})(1 - z^{-1})\,v = 1$$

and its min deg x solution

$$x = 1\,,\quad y = 1{\cdot}9845 - 0{\cdot}3695z^{-1} - 0{\cdot}3608(1 + 0{\cdot}7165z^{-1})(1 - z^{-1})\,t\,,\ \text{and}$$
$$v = 2{\cdot}7716 + 0{\cdot}0674z^{-1} - 0{\cdot}1132z^{-2} +$$
$$+ 0{\cdot}1306z^{-1}(1 + 2{,}9276z^{-1})(1 + 0{\cdot}2071z^{-1})\,t \quad \text{for any } t\,.$$

The error $e = 1$, $\deg e = 0$.

Choosing $t = 0$ the controllers transfer sequences

$$R_1 = (1{\cdot}9845 - 0{\cdot}3695z^{-1})(1 - 0{\cdot}6065z^{-1})^2 \quad \text{and}$$

$$R_2 = (2{\cdot}7716 + 0{\cdot}0674z^{-1} - 0{\cdot}1132z^{-2})(1 - 0{\cdot}6065z^{-1})^2 \,;$$

the control sequences $U_1 = R_1$ and $U_2 = R_2$.

Finite TOC solved for comparison in the simple system ($R_2 = 0$) gives the results:

$$e = 1 + z^{-1} + 0{\cdot}7891z^{-2} + 0{\cdot}1279z^{-3}\,,$$

$$R = \frac{1{\cdot}6148(1 - 0{\cdot}6065z^{-1})^2}{1 + z^{-1} + 0{\cdot}7891z^{-2} + 0{\cdot}1279z^{-3}} \quad \text{and} \quad U = 1{\cdot}6148(1 - 0{\cdot}6065z^{-1})^2\,.$$

Hence $\deg e = 3$ and $\lambda_1 - \lambda_2 = 3$.

3. For LSC solution

$$f^* = 1\,, \quad a_0^{-\sim} = 1 \quad \text{and} \quad d^{-\sim} = (\mathrm{b}, b_2)^{-\sim} = 1 \quad \text{is determined.}$$

The equation (76) is identical with the equation (59) and consequently in this example LSC solution and stable TOC solution are identical too. The control performace index $\sigma_E = 1$. The results of LSC solution in the simple control system are:

$$E = \frac{2{\cdot}9276 + 3{\cdot}9276z^{-1} + 2{\cdot}9276z^{-2}}{2{\cdot}9276 + z^{-1}} = 1 + z^{-1} + 0{\cdot}6584z^{-2} - 0{\cdot}2249z^{-3} + \ldots,$$

$$R = \frac{(1 - 0{\cdot}6065z^{-1})^2}{0{\cdot}1306(1 + 0{\cdot}2071z^{-1})(2{\cdot}9276 + 3{\cdot}9276z^{-1} + 2{\cdot}9276z^{-2})}$$

and

$$U = \frac{(1 - 0{\cdot}6065z^{-1})^2}{0{\cdot}1306(1 + 0{\cdot}2071z^{-1})(2{\cdot}9276 + z^{-1})}\,; \quad \sigma_E = 2{\cdot}49\,.$$

APPENDIX

The following algorithm produces for three given polynomials a, b, c (not all zero) their GCD $d = (a, b, c)$ together with the other polynomials satisfying the equations (45) and (46). It is the simplified special case of general matrix GCD algorithm given in [1] and [2].

The algorithm arranges gradually polynomial terms in row matrix $\boldsymbol{K}$ and square (3×3) matrix $\boldsymbol{Q}$ starting with initial

$$\boldsymbol{K} = [a, b, c] \quad \text{and} \quad \boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}_3\,.$$

1. Set $\boldsymbol{K} = [a, b, c]$ and $\boldsymbol{Q} = \boldsymbol{I}_3$.
2. Determine the position j of least degree nonzero polynomial in $\boldsymbol{K}$; $j \in 1, 2, 3$. If all polynomials are zero stop.
3. If $j \neq 1$ interchange j-th and the first column of both $\boldsymbol{K}$ and $\boldsymbol{Q}$.
4. If both the second and the third columns in $\boldsymbol{K}$ are zero stop.
5. Divide the leading coefficient of the first polynomial in $\boldsymbol{K}$ into the leading coefficients of the second and the third polynomial in $\boldsymbol{K}$, calling the results μ_2 and μ_3, respectively. Subtract the degree of the first polynomial in $\boldsymbol{K}$ from the degree of the second and the third polynomial in $\boldsymbol{K}$, calling the results ν_2 and ν_3, respectively.
6. Subtract $\mu_2 z^{-\nu_2}$ times the first column from the second one and $\mu_3 z^{-\nu_3}$ times the first column from the third one in both matrices $\boldsymbol{K}$ and $\boldsymbol{Q}$.
7. Go to 2.

After finishing the algorithm there are

$$\boldsymbol{K} = [d, 0, 0] \quad \text{and} \quad \boldsymbol{Q} = \begin{bmatrix} p & p_1 & p_2 \\ q & q_1 & q_2 \\ r & r_1 & r_2 \end{bmatrix}.$$

Example. Let us determine GCD of the polynomials $a = 1 - z^{-1}$, $b = z^{-1} + 2z^{-1}$ and $c = z^{-1}$ together with the other polynomials in (45) and (46). Using the given algorithm we have gradually:

$\boldsymbol{K}$	$\boldsymbol{Q}$	
$[1 - z^{-1}; z^{-1} + 2z^{-2}; z^{-1}]$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\mu_2 = -2,\ \mu_3 = -1$ $\nu_2 = 1,\ \nu_3 = 0$
$[1 - z^{-1}; 3z^{-1}; 1]$	$\begin{bmatrix} 1 & 2z^{-1} & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	
$[1; 3z^{-1}; 1 - z^{-1}]$	$\begin{bmatrix} 1 & 2z^{-1} & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\mu_2 = 3,\ \mu_3 = -1$ $\nu_2 = 1,\ \nu_3 = 1$
$[1; 0; 1]$	$\begin{bmatrix} 0 & -z^{-1} & 1 + z^{-1} \\ 0 & 1 & 0 \\ 1 & -3z^{-1} & z^{-1} \end{bmatrix}$	$\mu_2 = 0,\ \mu_3 = 1$ $\nu_2 = 0,\ \nu_3 = 0$
$[1; 0; 0]$	$\begin{bmatrix} 1 & -z^{-1} & z^{-1} \\ 0 & 1 & 0 \\ 1 & -3z^{-1} & -1 + z^{-1} \end{bmatrix}$	

Then

$$d = 1 ; \quad p = 1 , \quad q = 0 , \quad r = 1 , \quad p_1 = -z^{-1} , \quad q_1 = 1 ,$$
$$r_1 = -3z^{-1} , \quad p_2 = z^{-1} , \quad q_2 = 0 , \quad r_2 = -1 + z^{-1} .$$

(Received January 5, 1982.)

REFERENCES

[1] V. Kučera: Algebraic Theory of Discrete-Time Linear Control. Academia, Praha 1978. In Czech.

[2] V. Kučera: Discrete Linear Control — The Polynomial Equation Approach. Wiley, Chichester 1979.

[3] V. Strejc, M. Šalamon, Z. Kotek, M. Balda: Principles of Automatic Control Theory. SNTL, Praha 1958. In Czech.

[4] G. Bleisteiner, W. von Mangold: Handbuch der Regelungstechnik. Springer-Verlag, Berlin—Göttingen—Heidelberg 1961.

[5] W. Oppelt: Kleines Handbuch technischer Regelvorgänge. 4. Ersch., Verlag Chemie GmbH, Weinheim - Bergstr. 1964.

[6] S. Kubík, Z. Kotek, M. Šalamon: Control Theory I, II. SNTL — Alfa, Praha 1968, 1969. In Czech.

[7] G. F. Zajcev, V. K. Steklov: Combined Control Systems. Technika, Kiev 1978. In Russian.

[8] K. Reinisch: Analyse und Synthese kontinuierlicher Steuerungssysteme. VEB Verlag Technik, Berlin 1979.

[9] E. Samal: Verbesserung der Regelgüte durch Verwendung von Hilfsregelgrössen, Störgrössenbeeinflussung oder Reihenschaltung mehrerer Regler. Regelungstechnik *5* (1957), 192—198.

[10] C. Kessler: Ein Beitrag zur Theorie mehrschleifiger Regelungen. Regelungstechnik *8* (1960), 261—266.

[11] K. Havlíček: Multiloop Discrete-Time Control Systems I. Dipl. Eng. Thesis, ČVUT, Praha 1980. In Czech.

[12] M. Lukeš: Multiloop Discrete-Time Control Systems II. Dipl. Eng. Thesis, ČVUT, Praha 1980. In Czech.

[13] V. Kučera: Closed-loop stability of discrete linear single-variable systems. Kybernetika *10* (1974), 2, 146—171.

[14] V. Kučera: Algebraic theory of discrete optimal control for multivariable systems. Kybernetika *10—12*, (1974—76), 1—240. Published in installments.

Ing. Václav Soukup, CSc., katedra řídicí techniky elektrotechnické fakulty ČVUT (Faculty of Electrical Engineering, Department of Automatic Control — Czech Technical University), Karlovo nám. 13, 121 35 Praha 2, Czechoslovakia.