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FIRST-ORDER AUTOREGRESSIVE PROCESSES WITH TIME-DEPENDENT RANDOM PARAMETERS

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We consider a first-order autoregressive process $\{X_t\}$ with random parameters which are not independent in time. We ask when $\{X_t\}$ is stationary and derive the form of its covariance function and spectral density under the assumption that the random parameters generate a first-order moving-average process. We also construct the best linear prediction.

1. INTRODUCTION

Autoregressive models with random parameters are natural generalizations of classical autoregressive processes. The problem of stationarity of the autoregressive series with independent random coefficients was solved by Anděl (see [1]) and Nicholls and Quinn (see [3]). In some practical situations (for instance in applications to economy) the assumption of independence cannot be accepted and it is suitable to consider some kind of time-dependence among the coefficients. In the simplest case random parameters generate the first-order moving average process. In this paper we investigate conditions of stationarity of such a series, its covariance function and spectral density, the inverse of its variance matrix and we construct the best linear prediction.

We shall assume that the first-order autoregressive series with random parameters is generated from a random variable X_1 with $\mathsf{E}X_1 = 0$ and $\operatorname{Var} X_1 = \sigma^2 > 0$ by

$$X_t = b(t)\, X_{t-1} + a^{-1} Y_t \quad \text{for} \quad t = 2, \ldots, N \tag{1}$$

where $Y_2, \ldots, Y_N$ are independent random variables with zero means, unit variances, and independent of X_1; $a > 0$ is a number and $b(2), \ldots, b(N)$ is a series of random parameters generated by

$$b(t) = \beta_0 Z_t + \beta_1 Z_{t-1} \quad \text{for} \quad t = 2, \ldots, N \tag{2}$$

where $Z_1, \ldots, Z_N$ are independent random variables with zero means and the same variance $\delta^2 > 0$ which are independent of $X_1, Y_2, \ldots, Y_N$, and $\beta_0 \neq 0$, $\beta_1 \neq 0$ are real numbers. Obviously

$$\mathsf{E}b(t) = 0 \quad \text{for} \quad t = 2, \ldots, N \tag{3}$$

and the covariance function $B(t)$ of $\{b(t)\}$ satisfies

$$\begin{aligned} B(0) &= \mathsf{E}b^2(s) = (\beta_0^2 + \beta_1^2)\,\delta^2\,, \\ B(1) &= \mathsf{E}b(s+1)\,b(s) = \beta_0\beta_1\delta^2\,, \\ B(t) &= 0 \quad \text{for} \quad t = 2, \ldots, N-2\,. \end{aligned} \tag{4}$$

2. CONDITIONS FOR STATIONARITY AND COVARIANCE FUNCTION

If we write X_t in the equivalent form

$$\begin{aligned} X_t = b(t)\,b(t-1)\ldots b(2)\,X_1 + a^{-1}\,b(t)\ldots b(3)\,Y_2 + \ldots \\ \ldots + a^{-1}\,b(t)\,Y_{t-1} + a^{-1}Y_t\,, \end{aligned} \tag{5}$$

then it becomes evident that the assumption of independence $Z_1, \ldots, Z_N$ on $X_1, Y_2, \ldots$ $\ldots, Y_N$ implies

$$\mathsf{E}X_t = 0 \quad \text{for all} \quad t\,. \tag{6}$$

The covariance function $R(s, t)$ of $\{X_t\}$ is

$$\begin{aligned} R(s, t) = \mathsf{E}X_sX_t = \mathsf{E}b(s)\ldots b(t+1)\,b^2(t)\ldots b^2(2)\,\sigma^2 + \\ + a^{-2}\mathsf{E}[b(s)\ldots b(t+1)\,b^2(t)\ldots b^3(3) + \ldots + b(s)\ldots b(t+1)] \end{aligned} \tag{7}$$

for $s, t = 2, \ldots, N$, $s \geqq t$ and

$$R(s, 1) = \mathsf{E}[b(s)\,b(s-1)\ldots b(2)]\,\sigma^2\,.$$

Now what we ask is, under which conditions $R(s, t)$ depends only on the difference $s - t$. We first derive a necessary condition for stationarity of $\{X_t\}$.

Lemma 1. Let the variables $Z_1, \ldots, Z_N$ have the same moments $\mathsf{E}Z_t^3$ and $\mathsf{E}Z_t^4$ for all t. If the series $X_1, \ldots, X_N$ is stationary, then

$$\mathsf{E}Z_t^3 = 0\,, \quad \mathsf{E}Z_t^4 = \delta^4 \quad \text{for all} \quad t = 1, \ldots, N \tag{8}$$

and

$$\sigma^2 = \frac{a^{-2}}{1 - (\beta_0^2 + \beta_1^2)\,\delta^2} \quad \text{where} \quad (\beta_0^2 + \beta_1^2)\,\delta^2 < 1\,. \tag{9}$$

Proof. If $X_1, \ldots, X_N$ is stationary then $\operatorname{Var} X_1 = \operatorname{Var} X_2 = \ldots \operatorname{Var} X_N = \sigma^2$ where (by (7))

$$\operatorname{Var} X_s = \mathsf{E}X_s^2 = \mathsf{E}b^2(s) \ldots b^2(2)\,\sigma^2 + a^{-2}\mathsf{E}[b^2(s) \ldots b^2(3) + \ldots \\ \ldots + b^2(s) + 1] \quad \text{for} \quad s = 2, \ldots, N\,. \tag{10}$$

Now from $\operatorname{Var} X_2 = \sigma^2$ we get (9), from $\operatorname{Var} X_3 = \sigma^2$ we get $\mathsf{E}Z_t^4 = \delta^4$ and from $\operatorname{Var} X_4 = \sigma^2$ we get $\mathsf{E}Z_t^3 = 0$. □

Lemma 2. Let $Z_1, \ldots, Z_N$ be independent random variables with $\mathsf{E}Z_t = 0$, $\mathsf{E}Z_t^2 = \delta^2$, $\mathsf{E}Z_t^3 = 0$, $\mathsf{E}Z_t^4 = \delta^4$ for all t and let $b(2), \ldots, b(N)$ be generated by (2). Then

$$\mathsf{E}Z_s^2 b^2(s) \ldots b^2(k) = \beta_0^2 \sum_{j=0}^{s-k-1} \beta_1^{2j} \delta^{2(j+2)} \mathsf{E}b^2(s-j-1) \ldots b^2(k) + \\ + \beta_1^{2(s-k)} \delta^{2(s-k)} \mathsf{E}Z_k^2 b^2(k) \tag{11}$$

for all $2 \leqq s \leqq N$ and $2 \leqq k \leqq s$.

Proof. We use induction. Evidently (11) holds for $s = k = 2$. Now

$$\mathsf{E}Z_s^2 b^2(s) \ldots b^2(k) = \beta_0^2 \mathsf{E}Z_s^4 \mathsf{E}b^2(s-1) \ldots b^2(k) + \\ + \beta_1^2 \mathsf{E}Z_s^2 \mathsf{E}Z_{s-1}^2 b^2(s-1) \ldots b^2(k)$$

and from the assumption that (11) holds for $s - 1$ it follows that it holds for s, too.

Lemma 3. Under the assumptions of Lemma 2 it holds

$$\mathsf{E}b^2(s) \ldots b^2(k) = [\delta^2(\beta_0^2 + \beta_1^2)]^{s-k+1} \tag{12}$$

for $2 \leqq s \leqq N$ and $2 \leqq k \leqq s$.

Proof. We use induction again. Obviously (12) holds for $s = k = 2$. Assume that it holds for $s - 1$. Then

$$\mathsf{E}b^2(s) \ldots b^2(k) = \beta_0^2 \mathsf{E}Z_s^2 \mathsf{E}b^2(s-1) \ldots b^2(k) + \beta_1^2 \mathsf{E}Z_{s-1}^2 b^2(s-1) \ldots b^2(k)\,.$$

Now (12) follows from the induction assumption and Lemma 2. □

Corollary 4. The conditions (8) and (9) imply that $\operatorname{Var} X_s = \sigma^2$ for all $s = 2, \ldots, N$.

Proof follows from (10) and Lemma 3. □

Next we show that the conditions (8) and (9) are sufficient for stationarity of $X_1, \ldots, X_N$. First we prove two auxiliary lemmas.

Lemma 5. Under the assumptions of Lemma 2 it holds

$$\mathsf{E}Z_s b^2(s) \ldots b^2(k) = 0 \tag{13}$$

for all $2 \leqq s \leqq N$ and $2 \leqq k \leqq s$.

Proof. We use induction. It is easy to prove that $\mathsf{E}Z_2 b^2(2) = 0$. Now

$$\mathsf{E}Z_s b^2(s) \dots b^2(k) = \beta_0\beta_1 \mathsf{E}Z_s^2 \mathsf{E}Z_{s-1} b^2(s-1) \dots b^2(k)$$

and it is equal to 0 by the induction assumption. □

Lemma 6. Under the assumptions of Lemma 2 it holds

$$\mathsf{E}b(s) \dots b(k) = \beta_0\beta_1\delta^2 \mathsf{E}b(s-2) \dots b(k) \tag{14}$$

for $2 \leqq k \leqq N-2$ and $k+2 \leqq s \leqq N$ and

$$\begin{aligned} \mathsf{E}b(s) \dots b(t+1)\, b^2(t) \dots b^2(k) = \\ = \beta_0\beta_1\delta^2 \mathsf{E}b(s-2) \dots b(t+1)\, b^2(t) \dots b^2(k) \end{aligned} \tag{15}$$

for all $2 \leqq t \leqq N-2$, $t+2 \leqq s \leqq N$ and $2 \leqq k \leqq t$.

Proof is easy. □

Corollary 7. The covariance function $R(s, t)$ satisfies

$$R(s, t) = \beta_0\beta_1\delta^2 R(s-2, t) \tag{16}$$

for all $1 \leqq t \leqq N-2$ and $t+2 \leqq s \leqq N$.

Theorem 8. The series $X_1, \dots, X_N$ is stationary if and only if (8) and (9) are satisfied. The covariance function $R(t)$ is of the form

$$R(t) = \begin{cases} \sigma^2(\beta_0\beta_1\delta^2)^{t/2} & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases} \quad t = 0, 1, \dots, N-1\,. \tag{17}$$

Proof. From Corollary 4 it follows $R(0) = \sigma^2$. Evidently $R(2, 1) = \mathsf{E}b(2)\,\sigma^2 = 0$. From Lemma 5 we obtain that

$$\mathsf{E}b(s+1)\, b^2(s) \dots b^2(k) = \beta_1 \mathsf{E}Z_s b^2(s) \dots b^2(k) = 0$$

and it implies $R(s+1, s) = 0 = R(1)$. Then we use Corollary 7 and get

$$R(t) = R(s+t, s) = \beta_0\beta_1\delta^2\, R(s+t-2, s) = \beta_0\beta_1\delta^2\, R(t-2) \tag{18}$$

for $t = 2, \dots, N-1$. We use induction to conclude the proof. □

3. SPECTRAL DENSITY

Theorem 9. The spectral density of the series $X_1, \dots, X_N$ exists and it is equal to

$$f(\lambda) = \frac{\sigma^2}{2\pi} \frac{1 - (\beta_0\beta_1\delta^2)^2}{1 - 2\beta_0\beta_1\delta^2 \cos 2\lambda + (\beta_0\beta_1\delta^2)^2} \tag{19}$$

for $\lambda \in \langle -\pi, \pi \rangle$.

Proof. A sufficient condition for existence of the spectral density is

$$\sum_{t=-\infty}^{\infty} |R(t)| < \infty \tag{20}$$

(see [2], p. 43). In our case (20) is equal to $\sigma^2 \sum_{t=-\infty}^{\infty} |\beta_0\beta_1\delta^2|^{|t|}$ which is a geometric series with the quotient $|\beta_0\beta_1\delta^2| < 1$ and so (20) holds. Now the spectral density can be computed by

$$f(\lambda) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} e^{-it\lambda} R(t) \tag{21}$$

(see [2], p. 43). □

4. INVERSE OF VARIANCE MATRIX

Lemma 10. The series $X_1, \ldots, X_N$ has the same variance matrix as the second-order autoregressive series with fixed parameters generated by

$$V_t = \beta_0\beta_1\delta^2 V_{t-2} + c^{-1} Y_t \quad \text{for} \quad t = 3, \ldots, N \tag{22}$$

where V_1, V_2 are random variables with zero means and a covariance matrix

$$\mathbf{D} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

which are independent of $Y_3, \ldots, Y_N$ and

$$c^{-1} = a^{-1} \sqrt{\frac{1 - (\beta_0\beta_1\delta^2)^2}{1 - (\beta_0^2 + \beta_1^2)\,\delta^2}}\,.$$

Proof. Evidently $\mathsf{E}V_t = 0$ and $R(t) = \beta_0\beta_1\delta^2 R(t-2)$ for $t = 2, \ldots, N-1$. For $t = 0$ we get

$$R(0) = \mathsf{E}V_t^2 = (\beta_0\beta_1\delta^2)^2 \,\mathsf{E}V_{t-2}^2 + c^{-2} = (\beta_0\beta_1\delta^2)^2 R(0) + c^{-2}$$

and so

$$R(0) = \frac{c^{-2}}{1 - (\beta_0\beta_1\delta^2)^2} = \frac{a^{-2}}{1 - (\beta_0^2 + \beta_1^2)\,\delta^2} = \sigma^2\,.$$

For $t = 1$ we have

$$R(1) = \mathsf{E}V_{t+1}V_t = \beta_0\beta_1\delta^2 \mathsf{E}V_{t-1}V_t = \beta_0\beta_1\delta^2 R(1)$$

and then $R(1) = 0$. □

Theorem 11. Denote $\mathbf{G} = \mathrm{Var}(X_1, \ldots, X_N)$ where $N \geqq 2$. Then elements h_{st} of the matrix $\mathbf{H} = \mathbf{G}^{-1}$ are:

a) for $N = 2$:

$$h_{11} = h_{22} = \sigma^{-2},\; h_{12} = h_{21} = 0; \tag{23}$$

b) for $N = 3$:

$$h_{11} = h_{33} = \frac{1}{\sigma^2[1 - (\beta_0\beta_1\delta^2)^2]}, \quad h_{22} = \sigma^{-2},$$

(24) $$h_{13} = h_{31} = \frac{-\beta_0\beta_1\delta^2}{\sigma^2[1 - (\beta_0\beta_1\delta^2)^2]},$$

$h_{st} = 0$ in the other cases:

c) for $N = 4$:

$$h_{ss} = \frac{1}{\sigma^2[1 - (\beta_0\beta_1\delta^2)^2]} \quad \text{for} \quad s = 1, \ldots, 4,$$

(25) $$h_{s,s+2} = h_{s+2,s} = \frac{-\beta_0\beta_1\delta^2}{\sigma^2[1 - (\beta_0\beta_1\delta^2)^2]} \quad \text{for} \quad s = 1, 2,$$

$h_{st} = 0$ in the other cases;

d) for $N > 4$:

$$h_{ss} = \frac{1}{\sigma^2[1 - (\beta_0\beta_1\delta^2)^2]} \quad \text{for} \quad s = 1, 2, N - 1, N,$$

$$h_{ss} = \frac{1 + (\beta_0\beta_1\delta^2)^2}{\sigma^2[1 - (\beta_0\beta_1\delta^2)^2]} \quad \text{for} \quad s = 3, \ldots, N - 2,$$

(26) $$h_{s,s+2} = h_{s+2,s} = \frac{-\beta_0\beta_1\delta^2}{\sigma^2[1 - (\beta_0\beta_1\delta^2)^2]} \quad \text{for} \quad s = 1, \ldots, N - 2,$$

$h_{st} = 0$ in the other cases.

Proof. We can use the results for the inverse of the variance matrix of the series $V_1, \ldots, V_N$ (see [2], p. 170–172). □

5. PREDICTION

Assume that $X_1, \ldots, X_N$ are known variables. We shall find the best linear prediction $\hat{X}_{N+t}$ of the variable X_{N+t} based on $X_1, \ldots, X_N$, i.e. $\hat{X}_{N+t}$ will be of the form

(27) $$\hat{X}_{N+t} = c_1X_1 + \ldots + c_NX_N$$

such that

(28) $$\mathsf{E}(X_{N+t} - \hat{X}_{N+t})^2$$

is minimal.

Theorem 12. The best linear prediction of the random variable X_{N+t} based on $X_1, \ldots, X_N$ is

$$\hat{X}_{N+t} = \begin{cases} (\beta_0\beta_1\delta^2)^{t/2} X_N & \text{for } t \text{ even} \\ (\beta_0\beta_1\delta^2)^{(t+1)/2} X_{N-1} & \text{for } t \text{ odd.} \end{cases} \tag{29}$$

The residual variance in both cases is

$$\Delta^2 = \mathsf{E}(X_{N+t} - \hat{X}_{N+t})^2 = \sigma^2[1 - (\beta_0\beta_1\delta^2)^t] \,. \tag{30}$$

Proof. Minimization of (28) leads to normal equations

$$\mathsf{E}(X_{N+t} - c_1X_1 - \ldots - c_NX_N)X_k = 0 \quad \text{for} \quad k = 1, \ldots, N \,. \tag{31}$$

In the matrix form it is

$$\operatorname{Var}(X_1, \ldots, X_N) \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} \mathsf{E}X_{N+t}X_1 \\ \vdots \\ \mathsf{E}X_{N+t}X_N \end{pmatrix}$$

and then

$$\begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \boldsymbol{H} \begin{pmatrix} R(N + t - 1) \\ \vdots \\ R(t) \end{pmatrix} . \tag{32}$$

From (32) and Theorem 11 we get

$$\begin{aligned} &c_1 = \ldots = c_{N-1} = 0 \,, \quad c_N = (\beta_0\beta_1\delta^2)^{t/2} \quad \text{for } t \text{ even} \,, \\ &c_1 = \ldots = c_{N-2} = c_N = 0 \,, \quad c_{N-1} = (\beta_0\beta_1\delta^2)^{(t+1)/2} \\ &\text{for } t \text{ odd.} \end{aligned} \tag{33}$$

The proof (30) is easy. □

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REFERENCES

[1] J. Anděl: Autoregressive series with random parameters. Math. Operationsforsch. Statist. *7* (1976), 735–741.

[2] J. Anděl: Statistická analýza časových řad (Statistical analysis of time series). SNTL, Praha 1976.

[3] D. F. Nichols, B. G. Quinn: Multiple autoregressive models with random coefficients. J. Multivariate Anal. *11* (1981), 185–198.

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