KYBERNETIKA — VOLUME 18 (1982), NUMBER 5

SOME NONLINEAR STATISTICAL PROBLEMS OF A POISSON PROCESS

FRANTIŠEK ŠTULAJTER

Some results of the theory of random vectors with values in linear spaces are used to study the structure of a space of random variables with finite dispersion generated by a Poisson process and the problem of estimation of nonlinear functionals of an intensity measure of a Poisson process.

1. INTRODUCTION

The aim of this paper is to study some nonlinear statistical problems of a Poisson random process. Similar problems are considered for example in [1] or in [3] for double stochastic Poisson processes. We shall study in more details the structure of the space $L^2(P(\lambda))$ of random variables with finite dispersion generated by a Poisson process with an intensity measure λ and the problem of estimation of nonlinear functionals of an unknown intensity measure λ of a Poisson process. It is shown that $L^2(P(\lambda))$ is equal to the orthogonal sum of $L^2_n(P(\lambda))$; $n \geqq 0$, where $L^2_n(P(\lambda))$; $n \geqq 0$ are (mutually orthogonal) subspaces of $L^2(P(\lambda))$. L^2_0 is the space of constants, L^2_1 is "the linear subspace" of $L^2(P(\lambda))$, generated by the centered Poisson process, L^2_2 is "the quadratic subspace" of $L^2(P(\lambda))$, and so on. Generating sets of $L^2_n(P(\lambda))$ for $n = 1, 2, 3$ and 4 are given. The same result is true for the space of random variables with a finite dispersion generated by a Gaussian process with zero mean value and a given covariance function as it is shown in [8]. But the rule according to which we form the generating sets of L^2_n; $n \geqq 0$ for a Gaussian process is different from that derived here for the generating sets of a Poisson process.

In the Part 4 of this paper it is shown that every "polynomial" of a measure λ_f (given by $\lambda_f(A) = \int_A f \, d\lambda_0$) has an unbiased estimate. It is shown that a dispersion of the best unbiased estimate can be calculated by the same way as it is given in [9].

2. PRELIMINARIES REGARDING POISSON PROCESS

There are many possibilities to define a Poisson process. The best way, for our objective, is to define a Poisson process as a random point measure valued vector as it is done in [7], where the following statements can be found.

Let $(T, \mathscr{T})$ be a measurable space; denote by $\mathscr{M}(T, \mathscr{T})$ the vector space of finite measures defined on $(T, \mathscr{T})$ and by $\mathscr{L}_\infty(T, \mathscr{T})$ the space of bounded measurable functions defined on $(T, \mathscr{T})$. Let $\mathscr{C}(\mathscr{M}, \mathscr{L}_\infty)$ be a σ-algebra of subsets of $\mathscr{M}(T, \mathscr{T})$, generated by linear transformations $\mu \to \mu(A)$; $A \in \mathscr{T}$. Then we have: for every fixed finite measure $\lambda \in \mathscr{M}(T, \mathscr{T})$ there exists a unique probability measure $P(\lambda)$ defined on $(\mathscr{M}(T, \mathscr{T}), \mathscr{C}(\mathscr{M}, \mathscr{L}_\infty))$ called the Poisson law with intensity λ on $\mathscr{M}(T, \mathscr{T})$. This measure is a distribution of a Poisson process X transforming a probability space $(\Omega, \mathscr{F}, P_\lambda)$ into $(\mathscr{M}, \mathscr{C})$. Realizations of the random process X have the form $X(\omega) = \sum_{j=1}^{n(\omega)} \delta_{t_j(\omega)}$, where $\{t_1(\omega), \ldots, t_{n(\omega)}(\omega)\}$ is a finite set of points of T and δ is a Dirac measure. The random process X has the following properties: for every $A \in \mathscr{T}$ the random variable

$$\langle X(\omega), \chi_A \rangle = \int_A \mathrm{d}\Big(\sum_{j=1}^{n(\omega)} \delta_{t_j(\omega)}\Big) = N_A(\omega) = \text{the number of points } t_j(\omega) \text{ in the set } A\,,$$

has a Poisson distribution with the parameter $\lambda(A)$. If $f_1, \ldots, f_n$ belong to $\mathscr{L}_\infty(T, \mathscr{T})$ and have disjoint supports, then $\langle X, f_1 \rangle, \ldots, \langle X, f_n \rangle$ are independent random variables, where

$$\langle X, f_i \rangle (\omega) = \langle X(\omega), f_i \rangle = \int f_i \,\mathrm{d}\Big(\sum_{j=1}^{n(\omega)} \delta_{t_j(\omega)}\Big)\,; \quad i = 1, \ldots, n\,.$$

The real Laplace transform of the probability space $(\mathscr{M}, \mathscr{C}, P(\lambda))$ is given by $L_{P(\lambda)}(f) = \int e^{\psi(f)} \,\mathrm{d}P(\lambda)$, where ψ is an isomorphism between the vector space $L_0(T, \mathscr{T}, \lambda)$, consisting of classes of equivalence of real measurable functions defined on $(T, \mathscr{T})$ and the space $L(\mathscr{M}, \mathscr{C}, P(\lambda))$, given by

$$\psi(f)\,\Big(\sum_{j=1}^{n(\omega)} \delta_{t_j(\omega)}\Big) = \sum_{j=1}^{n(\omega)} f(t_j(\omega))\,.$$

We can write $L_{P(\lambda)}(f) = \exp\{\int_T (e^f - 1)\,\mathrm{d}\lambda\}$. The Laplace transform is finite, and so defined, for those functions $f \in L_0(T, \mathscr{T}, \lambda)$ for which a function $g = e^f$ belongs to $L^1(T, \mathscr{T}, \lambda)$; it is the set

$$D = \{f \in L_0 : f = \ln g;\ g \geqq 0,\ g \in L^1(T, \mathscr{T}, \lambda)\}\,.$$

The function $f = 0 \bmod \lambda$ is an inner point of the set D, from which we have that a transformation $\varkappa_P$ defined by

$$[\varkappa_P(Y)]\,(f) = \mathsf{E}_{P_\lambda}[Y \,.\, e^{\psi(f)/2}]\,; \quad Y \in L^2(\mathscr{M}, \mathscr{C}, P_\lambda)\,, \quad f \in D$$

is an isomorphism between the Hilbert spaces $L^2(\mathscr{M}, \mathscr{C}, P(\lambda))$ and a reproducing kernel Hilbert space $H(K_\lambda)$ with the kernel

$$K_\lambda(f, f') = L_{P(\lambda)}\left(\frac{f + f'}{2}\right); \quad f, f' \in D .$$

The problem of equivalence of two Poisson laws $P(\lambda)$ and $P(\lambda_0)$ is solved by the next assertion: let λ and λ_0 be two positive finite measures on $(T, \mathscr{T})$. Then $P(\lambda)$ and $P(\lambda_0)$ are equivalent iff λ and λ_0 are equivalent. In the last case denote by $f_\lambda = \mathrm{d}\lambda/\mathrm{d}\lambda_0$. Then

$$\frac{\mathrm{d}P(\lambda)}{\mathrm{d}P(\lambda_0)} = \exp\left\{\psi(\ln f_\lambda) - \int_T (f_\lambda - 1)\,\mathrm{d}\lambda_0\right\},$$

where ψ is the above mentioned isomorphism restricted to $L^1(T, \mathscr{T}, \lambda_0)$.

Now let $T = [0, T_0]$, $T_0 > 0$ be an interval on the real line. Then $N(t) = \langle X, \chi_{[0,t]}\rangle$; $0 \leqq t \leqq T_0$ is a Poisson process with an intensity measure λ, for which we have:

$$\mathsf{E}_\lambda[N(t)] = \lambda([0, t]); \quad 0 \leqq t \leqq T_0$$

and

$$R_\lambda(s, t) = \mathrm{Cov}_\lambda[N(s), N(t)] = \lambda([0, \min(s, t)]) = (\chi_{[0,s]}, \chi_{[0,t]})_{L^2(\lambda)} .$$

In a special case when λ is Lebesgue measure we get $R_\lambda(s, t) = \min(s, t)$, what is the covariance function of the Gaussian Wiener process, too. In the following section we show how these results can be used to solve some nonlinear statistical problems of a Poisson process. The results obtained, are similar to those valid for a Gaussian random process, described in [8] and [9].

3. THE STRUCTURE OF THE SPACE $L^2(\mathscr{M}, \mathscr{C}\ P(\lambda))$

Let $(T, \mathscr{T})$ be a measurable space, λ a finite measure on it and $P(\lambda)$ a distribution of a Poisson process X with values in $(\mathscr{M}, \mathscr{C})$. To solve statistical problems of nonlinear estimation of random variables (for example problems of nonlinear filtration) based on a Poisson process it is necessary to know the structure of the space $L^2(\mathscr{M}, \mathscr{C}, P(\lambda)) = L^2(P(\lambda))$.

It was mentioned in the Section 1 that the Hilbert space $L^2(\mathscr{M}, \mathscr{C}, P(\lambda))$ is isomorphic with the reproducing kernel Hilbert space $H(K_\lambda)$ with a kernel

$$K_\lambda(f, g) = L_{P(\lambda)}\left(\frac{f + g}{2}\right) = \exp\left\{\int_T (\mathrm{e}^{f/2} \cdot \mathrm{e}^{g/2} - 1)\,\mathrm{d}\lambda\right\},$$

where this kernel is defined on a set $E \times E$ with

$$E = \{f \in L_0 : \mathrm{e}^{f/2} \in L^1(T, \mathscr{T}, \lambda)\} = \{f : f = \ln h;\ h \geqq 0,\ h \in L^2(\lambda)\} .$$

According to this isomorphism, the system of random variables $\{\exp \psi(f); f \in E\}$

generates $L^2(\mathscr{M}, \mathscr{C}, P(\lambda))$. Since it is difficult to characterise the space $H(K_\lambda)$, we use the fact that the set of random variables $\{\exp\{\psi(f) - \int_T (e^f - 1)\, d\lambda\}; f \in E\}$ generates $L^2(P(\lambda))$ too, and according to Lemma 2 of [8] we have the following assertion: the Hilbert space $L^2(P(\lambda))$ is isomorphic with a reproducing kernel Hilbert space $H(M_\lambda)$ of functionals defined on E, with a kernel

$$M_\lambda(f, g) = \mathsf{E}_{P(\lambda)}\left[\exp\left\{\psi(f) - \int_T (e^f - 1)\, d\lambda\right\} \exp\left\{\psi(g) - \int_T (e^g - 1)\, d\lambda\right\}\right] =$$

$$= \exp\left\{\int_T (e^f - 1)(e^g - 1)\, d\lambda\right\}, \quad f, g \in E.$$

Now let $H(N_\lambda)$ be a reproducing kernel Hilbert space with a kernel

$$N_\lambda(h, h') = \exp\left\{\int_T h \,.\, h'\, d\lambda\right\}; \quad h, h' \in F,$$

where

$$F = \{h \in L^2(T, \mathscr{T}, \lambda) : h \geqq -1 \mod \lambda\}.$$

Define a transformation ϑ on a set of generating elements of $H(N_\lambda)$ onto a set of generating elements of $H(M_\lambda)$ by $\vartheta(N_\lambda(., h)) = M_\lambda(., \ln(h + 1))$; $h \in F$. ϑ can be naturally extended to an isomorphism between $H(N_\lambda)$ and $H(M_\lambda)$, because we have:

$$\langle N_\lambda(., h), N_\lambda(., h')\rangle_{H(N_\lambda)} = \langle \vartheta(N_\lambda(., h)), \vartheta(N_\lambda(., h'))\rangle_{H(M_\lambda)} =$$

$$= \langle M_\lambda(., \ln(h + 1)), M_\lambda(., \ln(h' + 1))\rangle_{H(M_\lambda)} = \exp\{(h, h')_{L^2(\lambda)}\};$$

$h, h' \in F$. Thus we have proved the following lemma:

Lemma 3.1. The Hilbert space $L^2(\mathscr{M}, \mathscr{C}, P(\lambda))$ is isomorphic with the reproducing kernel Hilbert space $H(N_\lambda)$ with the kernel $N_\lambda(h, h') = \exp\{\int_T hh'\, d\lambda\}$; $h, h' \in F$.

Now we are able to give the following theorem.

Theorem 3.1. There exist an isomorphism say φ, between the Hilbert space $L^2(\mathscr{M}, \mathscr{C}, P(\lambda))$ and $\exp \odot L^2(T, \mathscr{T}, \lambda) = \bigoplus_{n \geqq 0} L^2(T, \mathscr{T}, \lambda)^{n\odot}$, where $L^2(\lambda)^{n\odot}$ is the n-th symmetric tensor power of the space $L^2(\lambda)$.

Proof. It was proved in Lemma 3.1. that $L^2(P(\lambda))$ is isomorphic with $H(N_\lambda; F)$ where $N_\lambda(h, h') = \exp\{(h, h')_{L^2(\lambda)}\}$ is defined on $F \times F$, F being a subset of $L^2(\lambda)$. It is known from the properties of RKHS (see [5]) that $H(N_\lambda; F)$ is isomorphic with a subspace of RKHS $H(N_\lambda; L^2(\lambda))$ of functionals defined on $L^2(\lambda)$ generated by a set of functionals $\{N_\lambda(., h);\ h \in F\}$. Since $H(N_\lambda; L^2(\lambda))$ is isomorphic with $\exp \odot L^2(\lambda)$, it is enough to show that the set $\{N_\lambda(., h); h \in F\}$ generates $H(N_\lambda; L^2(\lambda))$. Let $f \in H(N_\lambda; L^2(\lambda))$ and let $\langle f, N_\lambda(., h)\rangle_{H(N_\lambda; L^2(\lambda))} = 0$ for all $h \in F$. We have to show that $f = 0$. In our case it holds that $f(g) = \sum_{n \geqq 0} (f_n, g \otimes \ldots \otimes g)_{L^2(\lambda)^{n\otimes}}$, where

$f_n \in L^2(\lambda)^{n\otimes}$. Further we have: $N_\lambda(g, h) = \sum_{n \geqq 0} ((1/n!)\, h \otimes \ldots \otimes h, g \otimes \ldots \otimes g)_{L^2(\lambda)^{n\otimes}}$ and thus $0 = \langle f, N_\lambda(., h)\rangle_{H(N_\lambda)} = \sum_{n \geqq 0} (g_n, h \otimes \ldots \otimes h)_{L^2(\lambda)^{n\otimes}}$ for all $h \in F$, where g_n is a projection of f_n onto the subspace $L^2(\lambda)^{n\odot}$ of $L^2(\lambda)^{n\otimes}$. From the last equality we get that $\sum_{n \geqq 0} t^n(g_n, h^{n\odot})_{L^2(\lambda)^{n\odot}} = 0$ for all $t \geqq 0$ and for all $h \in L^2_+(\lambda) = \{h \in L^2(\lambda) : : h \geqq 0 \bmod \lambda\}$, what is possible only in the case when $(g_n, h^{n\odot}) = 0$ for all $h \in L^2_+(\lambda)$. If we set $h = \sum_{j=1}^{n} c_j h_j$, where $c_1, \ldots, c_n$ are any nonnegative real numbers and $h_1, \ldots \ldots, h_n \in L^2_+(\lambda)$, then we get that $(g_n, (\sum_{j=1}^{n} c_j h_j)^{n\odot})_{L^2(\lambda)^{n\odot}}$ – a polynomial in nonnegative variables $c_1, \ldots, c_n$ is identically equal to zero, from which we get that $(g_n, h_1 \odot \ldots \ldots \odot h_n)_{L^2(\lambda)^{n\odot}}$; $h_1, \ldots, h_n \in L^2_+(\lambda)$ – a coefficient of polynomial by a variable $c_1 \ldots c_n$, is equal to zero. Since the set $L^2_+(\lambda)$ generates $L^2(\lambda)$, the set $\{h_1 \odot \ldots \odot h_n;$ $h_1, \ldots, h_n \in L^2_+(\lambda)\}$ generates $L^2(\lambda)^{n\odot}$ for all $n \geqq 0$, and thus g_n must be zero element for all $n \geqq 0$. □

Now we shall study in more details the special case when $T = [0, T_0]$; $T_0 > 0$, $\mathscr{T} = \mathscr{B}(T)$ and λ is a finite measure on $(T, \mathscr{T})$. From Theorem 3.1. we have

Corollary 3.1. The Hilbert space $L^2(\mathscr{M}, \mathscr{C}, P(\lambda))$ is isomorphic with the Hilbert space $\exp \odot H(R_\lambda)$, where $R_\lambda(s, t)$; $s, t \in T$ is the covariance function of a Poisson process $N(t) = \langle X, \chi_{[0,t]}\rangle$; $0 \leqq t \leqq T_0$.

Proof. It was mentioned in Part 1 that $\langle R_\lambda(., s), R_\lambda(., t)\rangle_{H(R_\lambda)} = R_\lambda(s, t) = (\chi_{[0,s]}, \chi_{[0,t]})_{L^2(\lambda)}$. Since the system of functions $\{\chi_{[0,t]};\ t \in T\}$ generates $L^2(\lambda)$ and the set $\{R_\lambda(\cdot, t); t \in T\}$ generates $H(R_\lambda)$, $L^2(\lambda)$ and $H(R_\lambda)$ are isomorphic. □

It follows from the definition of $\exp \odot H(R_\lambda)$ as a direct sum of Hilbert spaces $H(R_\lambda)^{n\odot}$; $n \geqq 0$ and from the isomorphism between $L^2(P(\lambda))$ and $\exp \odot H(R_\lambda)$, that the same partition to orthogonal components must hold for the space $L^2(P(\lambda))$, too. According to this we can write: $L^2(\mathscr{M}, \mathscr{C}, P(\lambda)) = \bigoplus_{n \geqq 0} L^2_n(\mathscr{M}, \mathscr{C}, P(\lambda))$ where $L^2_n(P(\lambda))$ are orthogonal subspaces of $L^2(\lambda)$. For problems of nonlinear estimation of random variables the following theorems is useful.

Theorem 3.2. Let $T = [0, T_0]$, $T_0 > 0$ and let λ be a finite positive measure on $(T, \mathscr{B}(T))$. Then for any random variable $U \in L^2(\mathscr{M}, \mathscr{C}, P(\lambda))$ we have $U = \bigoplus_{n \geqq 0} U_n$, where

$$U_n = \varkappa(\mathsf{E}_\lambda[U \,.\, \varkappa(R_\lambda(., t_1) \odot \ldots \odot R_\lambda(., t_n))]; t_1, \ldots, t_n \in T) ;$$

$n \geqq 0$, and $\varkappa$ is an isomorphism described in Corollary 3.1.

Proof. Since the set $\{R_\lambda(., t_1) \odot \ldots \odot R_\lambda(., t_n); t_1, \ldots, t_n \in T\}$ generates $H(R_\lambda)^{n\odot}$, the system of random variables $\{\varkappa(R_\lambda(., t_1) \odot \ldots \odot R_\lambda(., t_n)); t_1, \ldots, t_n \in T\}$ generates the Hilbert space $L^2_n(P(\lambda))$; $n \geqq 0$. A symmetric function of n-variables

$(t_1, \ldots, t_n) : \mathsf{E}_\lambda[U \,.\, \varkappa(R_\lambda(\cdot, t_1) \odot \ldots \odot R_\lambda(\cdot, t_n))]$ – an element of $H(R_\lambda^{n\odot})$, we can identify with that element of the space $H(R_\lambda)^{n\odot}$, the image of which by the isomorphism $\varkappa$ is the random variable U_n – a projection of a random variable U on the subspace $L_n^2(P(\lambda))$. (For more details see [8]). □

Now we shall try to clarify how the random variables $\varkappa(\{R_\lambda(\cdot, t_1) \odot \ldots \odot R_\lambda(\cdot, t_n); t_1, \ldots, t_n \in T\})$ – generating elements of $L_n^2(P(\lambda))$ can be found for $n \geqq 0$.

Let φ be the isomorphism from Theorem 3.1. Then we have

$$\varphi(\exp \odot (h - 1)) = \exp \left\{ \psi(\ln h) + \int_T (h - 1) \, \mathrm{d}\lambda \right\}; \quad h \in L_+^2(\lambda),$$

where $\exp \odot h = \sum_{n \geqq 0} 1/\sqrt{n!}\, h^{n\otimes}$ or

$$\varphi(\exp \odot (-f)) = \exp \left\{ \psi(\ln (1 - f)) + \int_T f \, \mathrm{d}\lambda \right\},$$

where f is any function from $L^2(\lambda)$ such that $f \leqq 1 \bmod \lambda$. If we set $f = \sum_{i=1}^n c_i \chi_{[0,t_i]}$, where $0 < t_1 \leqq t_2 \leqq \ldots \leqq t_n$ are any fixed points from the interval $[0, T_0]$, $n \geqq 0$ and $c_1, \ldots, c_n$ are any suitable chosen real numbers such that $\sum_{i=1}^n c_i \chi_{[0,t_i]} \leqq 1$, then we get

$$\varkappa(\exp \odot (-\sum_{i=1}^n c_i R_\lambda(\cdot, t_i))) = \varphi(\exp \odot (-\sum_{i=1}^n c_i \chi_{[0,t_i]})) =$$

$$= \exp \left\{ \psi(\ln (1 - \sum_{i=1}^n c_i \chi_{[0,t_i]})) + \sum_{i=1}^n c_i \int_T \chi_{[0,t_i]} \, \mathrm{d}\lambda \right\}.$$

From the equality

$$\exp \odot (-\sum_{i=1}^k c_i R_\lambda(\cdot, t_i)) = \sum_{n_1=0}^\infty \ldots \sum_{n_k=0}^\infty \frac{c_1^{n_1}}{n_1!} \ldots \frac{c_k^{n_k}}{n_k!} (-R_\lambda(\cdot, t_1))^{n_1\odot} \odot \ldots \odot (-R_\lambda(\cdot, t_k))^{n_k\odot}$$

we have that $(-1)^n R_\lambda(\cdot, t_1) \odot \ldots \odot R_\lambda(\cdot, t_n)$ is a coefficient by a variable $c_1 \ldots c_n$. Since $\{R_\lambda(\cdot, t_1) \odot \ldots \odot R_\lambda(\cdot, t_n); t_1, \ldots, t_n \in T\}$ generates $H(R_\lambda)^{n\odot}$, to find $\varkappa(R_\lambda(\cdot, t_1) \odot \ldots \odot R_\lambda(\cdot, t_n))$, it suffices to find a coefficient by $c_1 \ldots c_n$ in an expansion of the random variable $\exp \{\psi(\ln (1 - \sum_{i=1}^n c_i \chi_{[0,t_i]})) + \sum_{i=1}^n c_i \int_T \chi_{[0,t_i]} \, \mathrm{d}\lambda\}$. To do this, we can proceed as follows: using formally the expression $\ln (1 - x) = -\sum_{k=0}^\infty x^{k+1}/k + 1$ we get

$$\exp \left\{ \psi(\ln (1 - \sum_{i=1}^n c_i \chi_{[0,t_i]})) + \sum_{i=1}^n c_i \int_T \chi_{[0,t_i]} \, \mathrm{d}\lambda \right\} =$$

$$= \exp\left\{-\int_T \sum_{k\geqq 0} \frac{(\sum_{i=1}^{n} c_i\chi_{[0,t_i]})^{k+1}}{k+1}\,\mathrm{d}N(t) + \sum_{i=1}^{n} c_i \int_T \chi_{[0,t_i]}\,\mathrm{d}\lambda\right\} =$$

$$= \exp\left\{\sum_{i=1}^{n} c_i\left(\int_T \chi_{[0,t_i]}\,\mathrm{d}\lambda - N(t_i)\right) - \int_T \sum_{k\geqq 2} \frac{(\sum_{i=1}^{n} c_i\chi_{[0,t_i]})^{k}}{k}\,\mathrm{d}N(t)\right\}.$$

Expanding the function exp into an infinite series we get the coefficient by $c_1 \dots c_n$ of this expandion. We are not able to derive an general expression for this coefficient for any $n \geqq 0$. Here are the first four, derived by this method:

$$\varkappa(R_\lambda(\cdot, t)) = N(t) - \int_T \chi_{[0,t]}\,\mathrm{d}\lambda;\ t \in T$$

Let us denote by $M(t) = N(t) - \int_T \chi_{[0,t]}\,\mathrm{d}\lambda;\ t \in T$. Then

$$\varkappa(R_\lambda(\cdot\cdot t_1) \odot R_\lambda(\cdot, t_2)) = M(t_1)\,M(t_2) - N(\min\{t_1, t_2\});\quad t_1, t_2 \in T.$$

$$\varkappa(R_\lambda(\cdot, t_1) \odot R_\lambda(\cdot, t_2) \odot R_\lambda(\cdot, t_3)) = \prod_{i=1}^{3} M(t_i) - \sum_{i=1}^{3} M(t_i)\,N(\min(T_3 - \{t_i\})) +$$

$$+ 2N(\min T_3)\,,\quad \text{where}\quad T_3 = \{t_1, t_2, t_3\}\ ;\quad t_1, t_2, t_3 \in [0, T_0] = T.$$

$$\varkappa(R_\lambda(\cdot, t_1) \odot \dots \odot R_\lambda(\cdot, t_4)) = \prod_{i=1}^{4} M(t_i) - \sum_{i<j} M(t_i)\,M(t_j)\,N(\min(T_4 - \{t_i, t_j\})) +$$

$$+ 2!\sum_{i=1}^{4} M(t_i)\,N(\min(T_4 - \{t_i\})) - 3!\,N(\min T_4) + \sum_{i=2}^{4} N(\min\{t_1, t_i\})\,.$$

$$.\,N(\min(T_4 - \{t_1, t_i\}))\,,\quad \text{where}\quad T_4 = \{t_1, \dots, t_4\}\ ;\quad t_1, \dots, t_4 \in T.$$

Remark. Setting $t_1 = \dots = t_n = T_0 = 1$, $n = 1, \dots, 4$ and $\lambda = l$ Lebesgue measure, where $l > 0$, we get the first four orthogonal polynomials of a complete orthogonal system of a Poisson distribution on integers with a parameter l:

$$p_0(x) = 1$$
$$p_1(x) = x - l$$
$$p_2(x) = (x - l)^2 - x$$
$$p_3(x) = (x - l)^3 - 3x(x - l) + 2x$$
$$p_4(x) = (x - l)^4 - 6x(x - l)^2 + 8x(x - l) + 3x^2 - 6x\,,$$

where

$$\sum_{x\geqq 0} p_i(x)\,p_j(x)\,\frac{l^x}{x!}\,\mathrm{e}^{-l} = a_j\delta_{ij}\ ;\quad 1, 2, \dots, 4\,.$$

4. ESTIMATION OF FUNCTIONALS OF AN UNKNOWN INTENSITY MEASURE OF A POISSON LAW

The basis for this part is a general theory of locally best unbiased estimates as given in [6] and used for example in [9]. Now we shall apply this theory to the special case of the estimation of functionals of an unknown intensity measure of a Poisson law.

As we mentioned in Part 2, for two Poisson laws with $\lambda \ll \lambda_0$ on $(T, \mathscr{T})$, we have

$$\frac{\mathrm{d}P(\lambda)}{\mathrm{d}P(\lambda_0)} = \exp\left\{\psi(\ln f_\lambda) - \int_T (f_\lambda - 1)\,\mathrm{d}\lambda_0\right\}, \quad \text{where} \quad f_\lambda = \frac{\mathrm{d}\lambda}{\mathrm{d}\lambda_0}.$$

As we have shown in the preceding part, the system of random variables $\{\exp\{\psi(\ln f) - \int_T (f-1)\,\mathrm{d}\lambda_0\}; f \in L^2_+(\lambda_0)\}$ generates $L^2(\mathscr{M}, \mathscr{C}, P(\lambda_0))$. Every random variable of a type $\exp\{\psi(\ln f) - \int_T (f-1)\,\mathrm{d}\lambda_0\}$; $f \in L^2_+(\lambda_0)$ can be regarded as a Radon-Nikodym derivative $\mathrm{d}P(\lambda_f)/\mathrm{d}P(\lambda_0)$ of a measure $P(\lambda_f)$ with respect to the measure $P(\lambda_0)$, where λ_f is defined on $(T, \mathscr{T})$ by $\lambda_f(\mathrm{A}) = \int_A f\,\mathrm{d}\lambda_0$; $f \in L^2_+(\lambda_0)$, $A \in \mathscr{T}$. Thus there exist a one-to-one correspondence between measures λ absolutely continuous with respect to λ_0 and functions (precisely equivalent classes of functions) from $L^2_+(\lambda_0)$.

From a general theory of locally unbiased estimates [6] we have that a functional $F(.)$ defined on a set of measures, which are absolutely continuous with respect to λ_0, or equivalently, on the set $L^2_+(\lambda_0)$, has an unbiased estimate with a finite dispersion at λ_0, if and only if, $F(.)$ belongs to a reproducing kernel Hilber space $H(K_{\lambda_0})$ of functionals defined on $L^2_+(\lambda_0)$ with a kernel

$$K_{\lambda_0}(f, f') = \mathsf{E}_{\lambda_0}\left[\frac{\mathrm{d}P(\lambda_f)}{\mathrm{d}P(\lambda_0)}\,\frac{\mathrm{d}P(\lambda_{f'})}{\mathrm{d}P(\lambda_0)}\right] = \exp\left\{\int_T (f-1).(f'-1)\,\mathrm{d}\lambda_0\right\}; \quad f, f' \in L^2_+(\lambda_0).$$

It was shown in Theorem 3.1 that $H(K_{\lambda_0})$ and $\exp \odot L^2(\lambda_0)$ are isomorphic, from which we get the following characterization of the space $H(K_{\lambda_0})$, suitable for a case of estimation of functionals.

Theorem 4.1. The reproducing kernel Hilbert space $H(K_{\lambda_0})$ consists of functionals of a type $F_g(.)$; $g \in \exp \odot L^2(\lambda_0)$ defined on the space $L^2_+(\lambda_0)$ and such that

$$F_g(f) = \sum_{n \geq 0} (g_n, (f-1)^{n\odot})_{L^2(\lambda_0)^{n\odot}}, \quad \text{where} \quad g = \bigoplus_{n \geq 0} g_n \in \exp \odot L^2(\lambda_0)$$

and

$$\|F_g\|^2_{H(K_{\lambda_0})} = \|g\|^2_{\exp \odot L^2(\lambda_0)}; \quad h^{n\odot} = \frac{1}{\sqrt{n!}}\, h^{n\otimes} \quad \text{for} \quad h \in L^2(\lambda_0).$$

Proof. Setting $g_n = (g-1)^{n\odot}$; $g \in L^2_+(\lambda_0)$ we get, that $F_g(.) = K_{\lambda_0}(., g)$ is an element of $H(K_{\lambda_0})$. Using the definition of the norm for the class of functionals $F_g(.)$ we get that

$$\langle F_g, K_{\lambda_0}(., f)\rangle_{H(K_{\lambda_0})} = \sum_{n \geq 0} (g_n, (f-1)^{n\odot})_{L^2(\lambda_0)^{n\odot}} = F_g(f)$$

for every $g \in \exp \odot L^2(\lambda_0)$, $f \in L^2_+(\lambda_0)$ and the second property of reproducing kernel Hilbert space $H(K_{\lambda_0})$ is proved. $\square$

It was shown in Part 3 that in the case when $T = [0, T_0]$, $T_0 > 0$, the system $\{\varkappa(R(.,t_1) \odot \ldots \odot R(.,t_n));\ t_1, \ldots, t_n \in T\}$ of random variables generates $L^2_n(P(\lambda_0))$ for every $n \geqq 0$. From this we have

$$\mathsf{E}_{\lambda_f}[\varkappa(R(.,t_1) \odot \ldots \odot R(.,t_n))] = \mathsf{E}_{\lambda_0}\left[\varkappa(R[.,t_1) \odot \ldots \odot R(.,t_n))\frac{\mathrm{d}P(\lambda_f)}{\mathrm{d}P(\lambda_0)}\right] =$$

$$= \int_{T^n} \chi_{[0,t_1]} \odot \ldots \odot \chi_{[0,t_n]}(f-1)^{n\odot}\,\mathrm{d}\lambda_0^{n\otimes} =$$

$$= \prod_{i=1}^{n} \int_T \chi_{[0,t_i]}(f-1)\,\mathrm{d}\lambda_0 = \prod_{i=1}^{n} [\lambda_f([0,t_i]) - \lambda_0([0,t_i])]$$

for any $f \in L^2_+(\lambda_0)$ and we see that a random variable $\varkappa(R(.,t_1) \odot \ldots \odot R(.,t_n))$ is an unbiased estimate of a functional $F_g(f) = \prod_{i=1}^{n} [\lambda_f([0,t_i]) - \lambda_0([0,t_i])]$ depending on λ_0.

We are interested in functionals independent of λ_0. Analogically with results given in [9] we can show that any "polynomial" of a measure λ_f has an unbiased estimate. By a "polynomial of a degree p" we mean a functional $P_p(.)$ given by

$$P_p(f) = \sum_{n=0}^{p} \int_{T^n} h_n \cdot f^{n\odot}\,\mathrm{d}\lambda_0^{n\otimes}\ ; \quad f \in L^2_+(\lambda_0),\ h_n \in L^2(\lambda_0)^{n\odot}\,.$$

According to the proof of Lemma 5.1 in [9] we have

$$\int_{T^n} h_n \cdot f^{n\otimes}\,\mathrm{d}\lambda_0^{n\otimes} = \sum_{i=0}^{n} \binom{n}{i} \int_{T^i} \left(\int_{T^{n-i}} h_n\,\mathrm{d}\lambda_0^{(n-i)\otimes}\right)(f-1)^{i\otimes}\,\mathrm{d}\lambda_0^{i\otimes}$$

for any $n \geqq 0$, from which we can derive that any polynomial has an unbiased estimate. For a dispersion of the best unbiased estimate $\tilde{P}_p$ of a polynomial $P_p(f) = \sum_{n=0}^{p} \int_{T^n} h_n f^{n\otimes}\,\mathrm{d}\lambda_0^{n\otimes}$, where $h_n \in L^2(\lambda_0)^{n\odot}$ we have from Lemma 5.1. of [9]:

$$\operatorname{Var}_{\lambda_0}[\tilde{P}_p] = \sum_{n=1}^{p} \sum_{m=1}^{p} \sum_{i=1}^{\min[m,n]} \binom{n}{i}\binom{m}{i} i! \int_{T^i} \left(\int_{T^{n-i}} h_n\,\mathrm{d}\lambda_0^{(n-i)\otimes}\right)\left(\int_{T^{m-i}} h_m \mathrm{d}\lambda_0^{(m-i)\otimes}\right)\mathrm{d}\lambda_0^{i\otimes}.$$

Let us investigate a special case when

$$h_n = g_1 \odot \ldots \odot g_n\,, \quad = \frac{1}{\sqrt{n!}} \sum_{\sigma} g_{\sigma_1} \otimes \ldots \otimes g_{\sigma_n}\,.$$

Then we get:

$$P_n(f) = \prod_{j=1}^{n} \int_T g_j f\,\mathrm{d}\lambda_0 = \int_T h_n f^{n\odot}\,\mathrm{d}\lambda_0^{n\otimes} = \sum_{i=0}^{n} \binom{n}{i} \int_{T^i} \left(\int_{T^{n-i}} \frac{1}{n!} \sum_{\sigma} g_{\sigma_1} \otimes \ldots\right.$$

$$\left.\ldots \otimes g_{\sigma_{n-i}} \mathrm{d}\lambda_0^{(n-i)\otimes}\right) g_{\sigma_{n-i+1}} \otimes \ldots \otimes g_{\sigma_n} \cdot (f-1)^{i\otimes}\,\mathrm{d}\lambda_0^{i\otimes} =$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{1}{n!} \sum_{\sigma} \left(\prod_{j=1}^{n-i} \int_T g_{\sigma_j}\,\mathrm{d}\lambda_0\right)\left(\prod_{j=n-i+1}^{n} \int_T g_{\sigma_j}(f-1)\,\mathrm{d}\lambda_0\right)$$

and

$$\|P_n\|^2_{H(K\lambda_0)} = \mathsf{E}_{\lambda_0}[\tilde{P}_n^2] = \sum_{i=0}^{n} \binom{n}{i}^2 i! \left\| \frac{1}{n!} \sum_{\sigma} \prod_{j=1}^{n-i} \int_T g_{\sigma_j} \, \mathrm{d}_{\lambda_0} \bigotimes_{j=n-i+1}^{n} g_{\sigma_j} \right\|^2_{L^2(\lambda_0)^{i\otimes}} .$$

Example 4.1. Let $n = 2$. Then we get:

$$P_2(f) = \prod_{j=1}^{2} \int_T g_j f \, \mathrm{d}\lambda_0 = \int_T g_1 \, \mathrm{d}\lambda_0 \int_T g_2 \, \mathrm{d}\lambda_0 + \int_T g_1 \, \mathrm{d}\lambda_0 \int_T g_2(f-1) \, \mathrm{d}\lambda_0 +$$

$$+ \int_T g_2 \, \mathrm{d}\lambda_0 \int_T g_1(f-1) \, \mathrm{d}\lambda_0 + \int_T g_1(f-1) \, \mathrm{d}\lambda_0 + \int_T g_2(f-1) \, \mathrm{d}\lambda_0 .$$

The locally best unbiased estimate $\tilde{P}_2$ of P_2 is given

$$\tilde{P}_2 = \prod_{i=1}^{2} \int g_i \, \mathrm{d}\lambda_0 + \int_T g_1 \, \mathrm{d}\lambda_0 \, . \, \varphi(g_2) + \int_T g_2 \, \mathrm{d}\lambda_0 \, . \, \varphi(g_1) + \varphi(g_1 \odot g_2).$$

Setting $g_i = \chi_{[0,t_i]}$; $i = 1, 2$ we get that the random variable $\tilde{P}_2 = N(t_1) . N(t_2) - N(\min\{t_1, t_2\})$ is the best unbiased estimate of the functional

$$P_2(f) = \lambda_f([0, t_1]) \, . \, \lambda_f([0, t_2]) \, ; \quad t_1, t_2 \in T; \quad f \in L^2_+(\lambda_0)$$

with

$$\mathrm{Var}_{\lambda_0}[\tilde{P}_2] = \|P_2\|^2_{H(K\lambda_0)} - P_2^2(1) = \int_T g_1^2 \, \mathrm{d}\lambda_0 \int_T g_2^2 \, \mathrm{d}\lambda_0 + \left(\int_T g_1 g_2 \, \mathrm{d}\lambda_0 \right)^2 +$$

$$+ \int_T \left[g_1 \int_T g_2 \, \mathrm{d}\lambda_0 + g_2 \int_T g_1 \, \mathrm{d}\lambda_0 \right]^2 \mathrm{d}\lambda_0 \, .$$

If $g_i = \chi_{[0,t_i]}$; $i = 1, 2$, then

$$\mathrm{Var}_{\lambda_0}[\tilde{P}_2] = \lambda_0([0, t_1]) \, . \, \lambda_0([0, t_2]) + \lambda_0^2([0, \min\{t_1, t_2\}]) +$$

$$+ \lambda_0^2([0, t_2]) \, \lambda_0[(0, t_1]) + 2\lambda_0^2([0, t_1]) \, \lambda_0[(0, \min\{t_1, t_2\}]) +$$

$$+ \lambda_0^2([0, t_1]) \, \lambda_0([0, t_2]) \, .$$

Setting $t_1 = t_2 = t$, we get

$$\mathrm{Var}_{\lambda_0}[\tilde{P}_2] = 2\lambda_0([0, t]) + 4\lambda_0^3([0, t]) \text{ -- the classical result.}$$

Example 4.2. Let $P_3(f) = (\int g \, . \, f \, \mathrm{d}\lambda_0)^3$: Then

$$P_3(f) = \sum_{i=0}^{3} \binom{3}{i} \left(\int g(f-1) \, \mathrm{d}\lambda_0 \right)^i \left(\int_T g \, \mathrm{d}\lambda_0 \right)^{3-i} ;$$

$\tilde{P}_3$ – the best umbiased estimate of P_3 is given by:

$$\tilde{P}_3 = \sum_{i=1}^{3} \binom{3}{i} \left(\int g \, \mathrm{d}\lambda_0 \right)^{3-i} \varphi(g^{i\odot});$$

$$\mathrm{Var}_{\lambda_0}[\tilde{P}_3] = 6 \, . \, \|g\|^6_{L^2(\lambda_0)} + 18\|g\|^4_{L^2(\lambda_0)} \left(\int_T g \, \mathrm{d}\lambda_0 \right)^2 + 9\|g\|^2_{L^2(\lambda_0)} \left(\int_T g \, \mathrm{d}\lambda_0 \right)^4 .$$

For $g = \chi_{[0,t]}$ we get $\tilde{P}_3 = N(t) N(t) - 1)(N(t) - 2)$ and

$$\mathrm{Var}_{\lambda_0}[\tilde{P}_3] = 6\lambda_0^3([0, t]) + 18\lambda_0^4([0, t]) + 9\lambda_0^5([0, t]),$$

what is again a classical result given in [2].

(Received July 15, 1981.)

REFERENCES

[1] O. Aalen: Nonparametric inference for a family of counting processes. Ann. Statist. *4* (1978), 701—727.

[2] C. L. Chiang: Introduction to Stochastic Processes in Biostatisties. Wiley, New York 1968.

[3] J. Grandell: Doubly Stochastic Poisson Processes. Springer - Verlag, Berlin 1976.

[4] J. Kerstan, K. Matthes, J. Mecke: Unbegrenzt teilbare Punktprozesse. Akademie - Verlag, Berlin 1974.

[5] J. Neveu: Processus Aléatoires Gaussiens. Les Presses de l'Université de Montréal, Montréal 1968.

[6] E. Parzen: Time Series Analysis Papers. Holden-Day, San Francisco 1967.

[7] J. L. Soler: Contribution á l'étude des structures statistiques infinidimensionnelles. Thèse, Grenoble 1978.

[8] F. Štulajter: RKHS Approach to nonlinear estimation of random variables. Transaction of the Eighth Prague Conference, Vol. B, Academia, Prague 1978, 239—246.

[9] F. Štulajter: Nonlinear estimators of polynomials in mean values of a Gaussian stochastic process. Kybernetika *14* (1978), 3, 206—220.

RNDr. František Štulajter, CSc., Katedra teórie pravdepodobnosti a matematickej štatistiky MFF UK (Faculty of Mathematics and Physics, Department of Probability and Statistics — Comenius University), Mlynská dolina, 842 15 Bratislava. Czechoslovakia.