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THE ENERGY DISSIPATION, THE ERROR PROBABILITY AND THE TIME OF DURATION A LOGICAL OPERATION

MILAN MARVAN

On the basis of the physical model of a logical operation we discuss the relations between the energy dissipation E_d, the error probability p and the time of duration of one step of a logical operation t_s. It is shown that for given p there exists the minimal time of duration t_s^* of the step of an operation. Neyman's expression $E_d = kT \ln 1/p$ (k — Boltzman's constant, T — absolute temperature) is interpreted as the expression for energy dissipation in the case of an operation realized in the minimal possible time for given p.

1. INTRODUCTION

The mathematical cybernetics does not take into account the fact that processing of information requires the use of real physical degrees of freedom which are governed by the laws of physics. The logical operation is realized in a material element, which is characterized by physical quantities. The fluctuation of these quantities necessarily produces the errors in the logical operation, which are not considered in mathematical logic.

An engineer trying to build up a computer employs both physical properties of material and physical law in order to achieve good parametres of a logical element. He wants to shorten the time of duration of one elementary logical operation, to get a lower energy dissipation or to get the higher reliability. The effort to get the best parameters of logical elements provokes the interesting question for a physicist: "What are the principal limits on these parametres?"

This question recalls the situation in the half of the last century, when the development of steam engines gave rise to the principal problem of the best (ideal) efficiency of the engines. The determination of the limit of the efficiency have had a great importance in spite of the fact that real engines work with much worse efficiency than the ideal ones up to the present time. The research on this problem stimulated the formulation of the second law of thermodynamics which is one of the fundamental laws of physics.

We believe that the research on the principal limits on the parameters of the information machines can be of a similar importance. We can obtain useful results from the technological point of view as well as regarding the matters of principle.

The first systematic discussion of the concept of information in connection with physical problems was given by L. Brillouin [1] who considered measurement of the physical quantity as the process of gaining information. The character of our problems – connected with logical operations – is similar but not the same. This problem was very well formulated and, in some respects, also solved by R. Landauer and his co-workers [2], [3], [4]. For good reviews on physical limits in digital electronics, see [5]. The most recent approach to the physical limits in information transfer is based on the cosmological consideration [6], [6a].

In the present paper we follow works of R. Landauer and J. W. F. Woo [4] who studied the elementary steps in the logical operation in the case of one possible physical model which is briefly described in Section 2. This disscussion enables us to find the connection between the expression of Landauer and Woo for energy dissipation [4] and the expression given earlier by M. S. Neyman [7]: Neyman's formula expresses the energy dissipation for the step realized in the minimum time (Section 4). We show that the dependence of the energy dissipation on the time t_s of duration of the step (which we are interested in) has two branches (see Fig. 3, Section 5). For the more economical branch" we obtain the result expected: the time and the energy dissipation have the complementary properties, the logical operation running in a shorter time costs more energy (but we cannot make t_s arbitrary small!). The result less expected is the following one: the same time can be realized by two possible different velocities of representative particles. Only one of these velocities corresponds to the "more economical branch" of the energy dissipation.

We find a useful compromise between two demands – between a short time and low energy dissipation; we call the corresponding regime the "optimum regime". At the end of our work we study the relations between the quantities in the "optimum regime".

Although our conclusions are derived from the particular model, we believe that the main results hold more generally.

2. THE ELEMENTARY PHYSICAL MODEL OF A LOGICAL ELEMENT – QUALITATIVE CONSIDERATIONS

Let us consider logical elements which can occur in two alternative states: 0, 1. In our model the logical states 0, 1 are realized by placing a representative "particle" in one of two possible minima of potencial well (Fig. 1). The logical operation is a process in which representative particles are transformed from one minimum to the other minimum.

Before analyzing the elementary model more exactly, an intuitive qualitative consideration is useful. The reliability of elements is connected with the height of the central barrier W (see Fig. 1): the higher barrier corresponds to the smaller probability of random jump across the barrier due to thermal fluctuations. From this

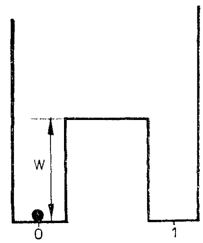

Fig. 1. Logical states 0, 1 are realized by placing a representative "particle" in one of two possible minima of a potencial well.

point of view the high magnitude of W seems to be very favourable, however, the following questions immediately arise: What energy dissipation is connected with transporting "particles" across potential barrier from one to the other state? Does the energy dissipation depend on the height of the barrier W and, as a consequence, is the energy dissipation the function of reliability?

M. S. Neyman [7] has given the formula which can be written in the form:

$$E_d = kT \ln \frac{1}{p} + kT \ln \Delta I \tag{2.1}$$

where p is the error probability due to thermal fluctuations in an "individual measurement". However, according to R. W. Keys and R. Landauer [3]: "... There is some uncertainty in both the definitions and the origin of Neyman's Equation (i.e. (2.1)). Eq. (2.1) has a close formal relationship to Eq. (14.31) of Brillouin's book [1]".

In the equilibrium the probability of the jump of a particle across the barrier is:

$$p \sim e^{-W/kT} \tag{2.2}$$

and the first term of Eq. (2.1) becomes:

$$E_d^{(A)} = W. \tag{2.3}$$

Although such an intuitive consideration is not quite right because the system during the logical operation is not in an equilibrium, it seems clear that Neyman's expression has two different types of terms: the first term is dependent on the height of barrier, but the second is independent. At this point the similar results were given by Landauer and co-workers [2], [3], [4]. They have shown that the total dissipation energy consists of two types of terms, too:

$$E_d = E_d^{(A)} + E_d^B \tag{2.4}$$

in which the first term has the form:

$$E_d^{(A)} = \gamma_A \frac{v}{v_d} W, \tag{2.5}$$

the second term is given by

$$E_d^{(B)} = \gamma_B kT. \tag{2.6}$$

In Eq. (2.5) v is the velocity of representative particle, v_d is the finite velocity which the particle attains in viscous medium (friction coefficient η) due to force K, $v_d = K/\eta$. The quantities γ_A, γ_B in (2.5), (2.6) are the numbers close to 1; for example, $\gamma_B = \ln 2$. In the case $\gamma_A = 1$ and $v = v_d$ Landauer's expression (2.5) goes over to (2.3), i.e. it goes to Neyman's expression which we transformed by means of (2.2). Remembering, however, that the expression (2.2) is not quite exact, the connection between both Neyman's and Landauer's expressions can be considered to be proved qualitatively only. More exact consideration will be given in Section 4. It will be shown that $v = v_d/2$ (and not $v = v_d$) represents the case in which both expressions are the same. We can keep in mind that Neyman's expression is valied for specific velocity v of a representative particle, while (2.5) is more general.

However, there is a drawback in the formula (2.5); namely, $E_d^{(A)}$ is expressed by means of the parametres v, W which are not interesting from the point of view of "good properties of a computer". The aim of our article is to express the energy dissipation by means of more suitable parameters – the error probability p and the time of duration of operation t_s. In order to achieve such a change of variables, we shall use the following expression, in particular (see [4]):

$$p = \frac{\exp[-x_0(K - \eta v)/kT]}{1 + \exp[-x_0(K - \eta v)/kT]} \tag{2.7}$$

which is more general than (2.2).

Now let us give a description of the physical model of a logical element and operation (for a more detailed description, see [4]).

The transport of the particle from the state "0" to the state "1" can be realized by two types of steps:

(A) The potential well which has an oblique bottom (see Fig. 2) is widened (or contracted). We consider the motion of our particle as the motion of a Brownian particle under the constant force K. The energy dissipation in this step is described by $E_d^{(A)}$ (Eq. (2.5)).

(B) The second step consists in the erection (or in the elimination) of the central barrier. The energy dissipation is given by E_d^B (Eq. (2.6)).

In our discussion we shall consider the first type, as far as it will not be said explicitly otherwise.

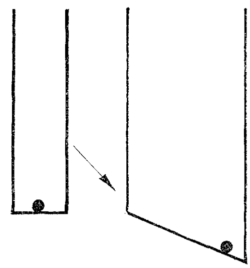

Fig. 2. The first step of a logical operation in the physical model according to R. Landauer and J. W. F. Woo.

3. ENERGY DISSIPATION FOR A GIVEN ERROR PROBABILITY AND FOR A GIVEN VELOCITY OF THE REPRESENTATIVE PARTICLE

In this Section we want to express the energy dissipation as the function of the error probability and of the velocity of the representative particle. First of all we express p (see (2.7)) and $E_d^{(A)}$ (see (2.5)) by means of the same quantities.

If x_0 is the change of x-coordinate of the particle which moves under the influence of the constant force K, the change of the potential energy is:

$$W = x_0 K \,. \tag{3.1}$$

The maximum velocity in viscous medium is given by

$$v_d = \frac{K}{\eta} \,. \tag{3.2}$$

Using (2.7), (3.1), (3.2) we obtain

$$p = \frac{\exp\left[-(W/kT)(1 - v/v_d)\right]}{1 + \exp\left[-(W/kT)(1 - v/v_d)\right]} \,. \tag{3.3}$$

The combination of Eq. (3.3) and Eq. (2.5) gives:

$$E_d^{(A)} = \frac{kT}{v_d/v - 1}\,\mathscr{L} \qquad \text{for} \quad v \neq 0\,,\ v \neq v_d,\ p \neq 0, 1\,, \tag{3.4}$$

where

$$\mathscr{L} = \ln(1/p - 1)\,. \tag{3.4'}$$

The cases $v = 0$, $v = v_d$ are considered separately:

$$\begin{cases} E_d^{(A)} = 0 \\ p = \dfrac{\exp(-W/kT)}{1 + \exp(-W/kT)} \end{cases} \quad \text{for} \quad v = 0\,, \tag{3.5}$$

$$\begin{cases} E_d^{(A)} = W \\ p = \frac{1}{2} \end{cases} \quad \text{for} \quad v = v_d\,. \tag{3.6}$$

Since we are interested in the logical elements with small errors we put:

$$p \ll 1 \tag{3.7}$$

and (3.4), (3.4′) has the form:

$$E_d^{(A)} = \frac{kT}{(v_d/v - 1)} \ln \frac{1}{p}\,. \tag{3.8}$$

In this Section we have replaced W by p. In the following Section we shall substitute the time t_s for velocity and we find the minimum of its magnitude.

4. THE MINIMUM TIME OF ONE STEP OF LOGICAL OPERATION

We shall consider the time of operation in the step A (see Section 2). The velocity of the representative particle moving in the viscous medium under the constant force K (directed along the x-axis) is exponentially increasing at the beginning and, asymptotically, it is going to the magnitude v_d. The potential wall moving in the direction of coordinate x with constant velocity v ($v < v_d$) impedes the free motion of the particle. In this case the velocity of the particle increases only to the value v of the velocity of wall (during the time interval t_1) and then it is maintained on this value during the time interval t_2. If $t_2 \gg t_1$ we can write for time of operation in one step approximately $t_s = t_1 + t_2 \doteq t_2$ and thus we obtain

$$t_s = \frac{x_0}{v}\,, \tag{4.1}$$

or, using (3.1)

$$t_s = \frac{1}{v}\frac{W}{K}\,. \tag{4.2}$$

From (4.2) and (2.5) (we choose $\gamma_A = 1$) follows

$$v = \sqrt{\left(\frac{v_d}{K}\frac{E_d^{(A)}}{t_s}\right)}\,. \tag{4.3}$$

Combining (4.3) with (3.3), we obtain the quadratic equation which has two solutions $(E_d^{(A)})^+$, $(E_d^{(B)})^-$:

$$(4.4) \qquad (E_d^{(A)})^{\pm} = (\tfrac{1}{2}Kv_d t_s - kT\mathscr{L}) \pm \sqrt{(\tfrac{1}{2}Kv_d)}\, t_s \sqrt{(\tfrac{1}{2}Kv_d t_s - 2kT\mathscr{L})}\,.$$

The condition that the solution is real reads:

$$(4.5) \qquad t_s \geqq \frac{4kT}{Kv_d}\mathscr{L}\,.$$

It follows that there *exists the minimum possible time of operation in the step considered*

$$(4.6) \qquad t_s^* = \frac{4kT}{Kv_d}\mathscr{L}\,.$$

If we insert (4.6) into (4.4), we obtain

$$(4.7) \qquad (E_d^{(A)})^* = kT\mathscr{L}$$

which is (for $p \ll 1$) the first term of Neyman's expression (2.1).

We see that *expression of Landauer and Woo is transformed into the Neyman's expression, if the time of operation is equal to its minimium value.*

The next question of interest is what velocity of the particle (or of the wall) corresponds to the regime of the shortest time of operation. From (4.7) and (3.8) it follows that this velocity is given by

$$(4.8) \qquad v^* = \tfrac{1}{2}v_d$$

rather than $v = 0$ as we could naively expect.

From (2.5), (4.7) and (4.8), choosing $\gamma_A = 1$, it follows that the height of the potential barrier (for given p) has necessarily to be:

$$(4.9) \qquad W^* = \tfrac{1}{2}kT\mathscr{L}\,.$$

The more detailed discussion of the dependence of energy on time is in the following Section.

5. THE ENERGY DISSIPATION AS THE FUNCTION OF THE ERROR PROBABILITY

Let us introduce the dimensionless quantities

$$(5.1) \qquad \alpha = \frac{Kv_d t_s}{kT}\,, \quad (\mathscr{E}_d^{(A)})^{\pm} = \frac{(E_d^{(A)})^{\pm}}{kT}\,.$$

Using these quantities, Eq. (4.4) has the form:

$$(5.2) \qquad (\mathscr{E}_d^{(A)})^{\pm} = (\tfrac{1}{2}\alpha - \mathscr{L}) \pm \sqrt{\frac{\alpha}{2}}\sqrt{\left(\frac{\alpha}{2} - 2\mathscr{L}\right)}.$$

The solution is real if

$$\frac{\alpha}{2} \geqq 2\mathscr{L}. \tag{5.3}$$

This equation is equivalent to (4.5). We can introduce ε by relation

$$\frac{\alpha}{2} = (1 + \varepsilon)\, 2\mathscr{L}\,. \tag{5.4}$$

The quantity ε is the measure of the deviation from the shortest time operation regime. This is clear from the fact that Eq. (5.3) can be written in the equivalent form:

$$\varepsilon = \frac{t_s}{t_s^*} - 1\,. \tag{5.3'}$$

Further, we can cast the Eq. (5.2) into the form

$$\frac{(\mathscr{E}_d^{(A)})^{\pm}}{\mathscr{L}} = (1 + 2\varepsilon) \pm 2\sqrt{(\varepsilon(1 + \varepsilon))}\,. \tag{5.4'}$$

From the last equation it follows that the ratio of energy dissipation and the level of the error $\mathscr{L}$ (i.e. the left hand side of (5.4')) is constant for the given ε (*i.e.* for given t_s/t_s^*). The dependence of $\mathscr{E}_d^{(A)}/\mathscr{L}$ on ε is shown in Fig. 3.

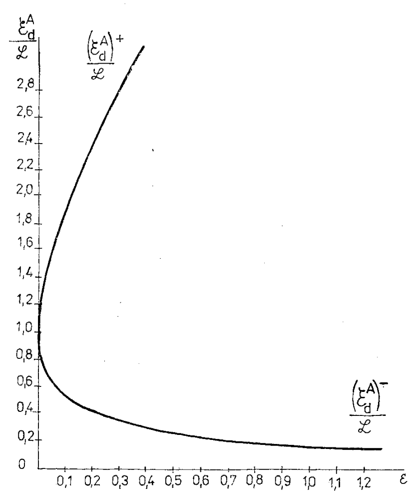

Fig. 3. The dependence of the ratio of energy dissipation and measure of reliability on times. Dimensionless quantities $\mathscr{E}_d = E_d/kT$, $\mathscr{L} = \ln(1/p + 1)$ and $\varepsilon = t_s/t_s^* - 1$ were used.

Two roots $(\mathscr{E}_d^{(A)})^+$, $(\mathscr{E}_d^{(A)})^-$ correspond to two possible velocities v^+, v^- for given p. It can be shown *that* v^+, *satisfying*

$$\frac{1}{2}v_d < v^+ < v_d \tag{5.5}$$

corresponds to $(E_d^{(A)})^+$, *whereas* v^-, *satisfying*

$$0 < v^- < \frac{1}{2}v_d\,, \tag{5.6}$$

corresponds to $(E_d^{(A)})^-$.

From the point of view of energy economy, we prefer the velocity v^- from the interval (5.6).

At the first sight the interval (5.6) would seem advantageous also from the point of view of the short time of the logical operation. But this is not quite true. It will be shown that the both intervals are equivalent with respect to "time economy". From Eq. (2.5) with $\gamma_A = 1$ and from Eqs. (3.4), (4.2) the time t_s can be written as:

$$t_s = \frac{kT}{K}\,\frac{v_d}{v(v_d - v)}\,\mathscr{L}\,. \tag{5.7}$$

It is seen that t_s is the function of v symmetric about the point $v_d/2$, i.e. we obtain the same time t_s both for $v = v_d/2 + \omega$ and $v = v_d/2 - \omega$.

To conclude our discussion, we may *state that there are no reasons for choosing* v *outside the interval* (5.6).

6. THE OPTIMUM REGIME OF THE LOGICAL OPERATION

Until now we have discussed the first type of the step of logical operation (see Section 2). In this step we can choose either short time of operation or, on the other hand, the large energy dissipation with the short time. However, taking into account that in the second type of step we have always $E_d^{(B)} \sim kT$ (see (2.6)), it is clear that the minimalization of E_d^A significantly below the value of kT has no meaning. From this it follows that the optimum regime which is the result of the requirement of both – the small energy dissipation and the short time – can be realized for:

$$\tilde{E}_d^A = kT\,. \tag{6.1}$$

It is easy to show that the corresponding time of the step A of the operation is:

$$\tilde{t}_s = \frac{kT}{Kv_d}(1 + \mathscr{L})^2 = \frac{kT}{K^2}\,\eta(1 + \mathscr{L})^2\,. \tag{6.2}$$

Eq. (6.2) can be written in the dimensionless form:

$$\frac{\tilde{t}_s}{t_s^*} = \frac{1}{4}\,\frac{(1 + \mathscr{L})^2}{\mathscr{L}}\,. \tag{6.3}$$

In the regime under consideration it is necessary to choose the magnitude of the velocity according to

$$\frac{\tilde{v}}{v_d} = \frac{1}{1 + \mathscr{L}} \tag{6.4}$$

and the value of the height of the barrier according to:

$$\tilde{W} = kT(1 + \mathscr{L})\,. \tag{6.5}$$

In Eqs. (6.2)–(6.5) the measure of reliability $\mathscr{L}$ can be choosen, the parameters of the model are temperature T, force K and coefficient of friction η (or v_d).

7. CONCLUSION

Analyzing one of the possible simple models of logical element we have found the dependence of the energy dissipation $E_d^{(A)}$ on the time t_s of the step A of the logical operation and on the error probability. The dependence $E_d^{(A)}$ on the time t_s has two branches which correspond to two different velocity intervals of a representative particle, $(0, v_d/2)$, $(v_d/2, 0)$. The velocity v_d is the physical characteristic of the model, it means the maximum possible velocity in medium with friction coefficient η and in a field of the constant force K. The "more economical" branch shows the decreasing character of the energy dissipation with increasing time t_s.

For a given error probability it is not possible to shorten the time t_s to an arbitrarily low value because there exists the minimal time t_s^* of duration of the step considered.

The energy dissipation of logical elements working in the regime of the minimum time t_s^* is described by expression (4.7) which Neyman considered as the general expression for E_d. Since the energy dissipation of the other steps are in order of magnitude, given by kT independently of the details of the model, the lowering of $E_d^{(A)}$ deep below kT has no practicle importance. Therefore, the regime in which $E_d^{(A)} \sim kT$ can be considered as the optimum regime. The parameters corresponding to this regime were found.

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RNDr. Milan Marvan, CSc., matematicko-fyzikální fakulta University Karlovy (Faculty of Mathematics and Physics — Charles University), V Holešovičkách 2, 180 00 Praha 8. Czechoslovakia.