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MATHEMATICAL PROGRAMMING PROBLEMS INVOLVING CONTINUUM OF INEQUALITY CONSTRAINTS

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The paper considers mathematical programming problems having a continuum of inequality type constraints and possibly other constraints. These additional constraints can be of quite general nature being described only implicitly by their respective conical approximations. Following the general scheme given in [1], a set of necessary conditions if obtained. It is also shown that then necessary conditions for general static minmax problems with constraints can be derived in a straightforward way.

1. INTRODUCTION

At present time there exists a variety of papers dealing with all possible aspects of mathematical programming problems. Based on assumed structure of a set of constraints, various approaches and methods can be encountered in the literature. It is the opinion of the author that the most general methodology seems to be developed in [1]. However, this fact does not seem to be widely known and used in the past. One reason is that the English version of [1] appeared only recently, but, on the other hand, also somewhat complicated notation and effort of the author to make the presentation as general as possible (in finite dimensional setting) could discourage some potential readers and users.

The aim of this note is to show a straightforward application of the existing results to obtain necessary conditions in a case of infinitely many inequality type constraints preserving general formulation of [1]. The derived conditions are more general than those given in [2–3]. In turn, the obtained conditions enable direct treatment of static minmax problems in a fairly general setting including also the recent results formulated for a case with explicit (equality and/or inequality type) constraints in [4–5].

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2. PROBLEM FORMULATION

Consider the problem of minimizing a function $f : R^n \to R^1$ on a given set

$$(1) \qquad \hat{\Omega} = \Omega \cap \{x \in R^n \mid q(x, y) \leqq 0, \ y \in Y\},$$

where $\Omega \subset R^n$ and Y is a compact subset (index set) of a complete metric space. Here and henceforth it is assumed that all functions are continuously differentiable. To treat a general constraining set Ω the following definition is repeated here for convenience, see [1].

Definition 1. Let $\hat{x} \in \Omega$ and let K be a convex cone with vertex in $\hat{x}$. The cone K is called a conical approximation to the set Ω at $\hat{x}$ if there is an $\varepsilon > 0$ and a continuous mapping $\psi : (K \cap U_\varepsilon(\hat{x})) \to \Omega$, such that $\psi(x) = x + o(x - \hat{x})$, where $U_\varepsilon(\hat{x})$ denotes an ε-neighbourhood of $\hat{x}$, and $o(x) = 0$ for $x = 0$ and $\lim_{\|x\| \to 0} \|o(x)\|/\|x\| = 0$.

The denotation "conical approximation" is preferred by the author to the names translated as "mantle" or "tent". It is not very difficult to see that K, being a conical approximation in the sense of Definition 1, is also a conical approximation (of the second kind) introduced and used throughout [6]. However, in this finite-dimensional setting the both concepts are practically almost equivalent, especially when dealing with more explicit form of Ω, e.g. Ω being convex or given by a system of equalities and inequalities. Further details can be found in the mentioned references [1] and [6].

Moreover, if $K \subset R^n$ is a convex cone with vertex in $\hat{x}$, the $K' = \{a \in R^n \mid \langle x - \hat{x}, a\rangle \leqq 0, \ x \in K\}$ is the dual (polar) cone to K. Finally, let $K_1, \ldots, K_s$ be a family of convex cones in R^n having a common vertex $\hat{x}$. We say that this family possesses a separation property in R^n, if there exist vectors $a_i \in K'_i$, $i = 1, \ldots, s$, with at least one of them being nonzero, such that $a_1 + \ldots + a_s = 0$. For convenience, let us also recall the basic result of [1].

Theorem 1. Let $\Omega_1, \ldots, \Omega_s$ be a family of sets in R^n having a common point $\hat{x}$, and $K_1, \ldots, K_s$ the corresponding conical approximations at this point. Assume that at least one of the cones $K_1, \ldots, K_s$ is not a hyperplane (of any dimension). In order that the intersection $\Omega_1 \cap \ldots \cap \Omega_s$ is just the point $\hat{x}$, it is necessary that the family of cones $K_1, \ldots, K_s$ possesses the separation property.

Fairly involved reasoning and construction was needed in [1] to prove this theorem. However, if in Definition 1 the mapping ψ is assumed to be continuously differentiable, the corresponding version of Theorem 1 for this case of "smooth" conical approximations is obtained in a considerably simpler way as demonstrated in [1] and [7]. Moreover, in all cases of practical interest this less general concept has shown quite satisfactory as follows from [7], where such type of a conical approximation is exclusively used.

3. NECESSARY OPTIMALITY CONDITIONS

Let

$$\Omega_0 = \{x \in R^n \mid q(x, y) \leqq, \ y \in Y\} . \tag{3}$$

Denote as $Y(\hat{x}) = \{y \in Y \mid q(\hat{x}, y) = 0\}$ the active index set at $\hat{x}$. We say that the family of functions $q(\cdot, y)$ $y \in Y$ is nondegenetate at $\hat{x}$, if there is a vector $v \in R^n$ such that

$$\langle q_x(\hat{x}, y), v\rangle < 0 \quad \text{for all} \quad y \in Y(\hat{x}) . \tag{4}$$

Here the lower index denotes the respective partial derivative. If $Y(\hat{x})$ happens to be empty, then there is clearly no active constraint at $\hat{x}$, and R^n is a conical approximation of Ω_0. Otherwise, in an analogical way as in [1] or [7] one has the following result – see also [3] in this respect.

Theorem 2. If the family of functions $q(\cdot, y)$ $y \in Y$, is nondegenerate at $\hat{x}$, the set

$$K(\hat{x}) = \{x \in R^n \mid \langle q_x(\hat{x}, y), x - \hat{x}\rangle \leqq 0 \quad \text{for all} \quad y \in Y(\hat{x})\}$$

is a conical approximation to the set Ω_0, given by (3), at the point $\hat{x}$.

It is a direct application of Theorems 1 and 2 to formulate necessary conditions for the considered problem.

Theorem 3. Let $\hat{x}$ be a minimizing point of f subject to (1) with

$$\Omega = \bigcap_{i=1}^{s} \Omega_i . \tag{5}$$

Denote as $K(\hat{x})$ the conical approximation to the set $\Omega_0(\hat{x})$ and as $K_1, \ldots, K_s$ the conical approximations to $\Omega_1, \ldots, \Omega_s$ at $\hat{x}$, and assume that the family of functions $q(\cdot, y)$, $y \in Y$, is nondegenerate at $\hat{x}$. Then there exists a number $\mu \leqq 0$, a vector $a_0 \in K'(\hat{x})$, and vectors $a_i \in K'_i$, not all zero, such that

$$\mu f_x(\hat{x}) + a_0 + a_1 + \ldots + a_s = 0 .$$

The nondegeneracy assumption was discussed in [3] more in detail when dealing with a conical approximation of the type defined in [6]. For somewhat less general formulation of the studied problem it was shown that without this assumption the obtained necessary conditions can be always satisfied in a trivial way not depending on the optimality of $\hat{x}$. Now it is possible to give variety of theorems dealing with some more concrete cases, e.g., subfamily of cones $K_1, \ldots, K_s$ does not possess the separation property, some constraints are given explicitly as a system of equalities, etc. The interested reader should consult [1] for these and other details.

As also discussed in [3] in some other consequences, one can express vector

$a_0 \in K'(x)$ as

$$a_0 = \sum_{i=1}^{r} v_i \, q_x(\hat{x}, y_i) , \tag{6}$$

with $v_i \geqq 0$, $i = 1, \ldots, r$, $y_i \in Y(\hat{x})$, $i = 1, \ldots, r$, and $1 \leqq r \leqq n$. Then the necessary conditions receive a more familiar form, e.g. see [2].

4. STATIC MINMAX PROBLEMS

The aim is to minimize the function

$$f(x) = \sup_{y \in Y} \phi(x, y) \tag{7}$$

subject to

$$x \in \Omega . \tag{8}$$

Here ϕ is a continuously differentiable function, Y is again a compact subset of a complete metric space, and $\Omega \subset R^n$.

Let us consider the problem of minimizing

$$F(z) = F(z_1, \ldots, z_{n+1}) = z_{n+1} \tag{9}$$

subject to

$$(z_1, \ldots, z_n)^T \in \Omega , \tag{10}$$

and

$$\phi(z_1, \ldots, z_n, y) - z_{n+1} \leqq 0 \quad \text{for all} \quad y \in Y . \tag{11}$$

It is a simple exercise to show that the problems (7)–(8) and (9)–(11) are equivalent in the following sense: if $\hat{z}$ solves (9)–(11), then $\hat{x} = (\hat{z}_1, \ldots, \hat{z}_n)^T$ solves (7)–(8); if $\tilde{x}$ solves (7)–(8), then $\tilde{z} = (\tilde{x}^T, \tilde{z}_{n+1})^T$ solves (9)–(11) with $\tilde{z}_{n+1} = \sup_{y \in Y} \phi(\tilde{x}, y)$. This idea was originally suggested in [8] and explored recently when studying static minmax problems with constraints in [4] and [5]. In the studied case, on applying Theorem 3, one easily obtains a general result identical with that of [9]. All known theorems can be then alternatively formulated as partial cases.

Namely, assume that Ω is given by (5) and that K_i, $i = 1, \ldots, s$, are the respective conical approximations at the optimal point $\hat{x}$. Moreover, let the family of functions $\phi(\cdot, y) - f(\hat{x})$, $y \in Y$, be nondegenerate. Then also the family of functions in R^{n+1} described by (11) is nondegenerate at $\hat{z} = (\hat{x}^T, f(\hat{x}))^T$ as can be easily verified. Therefore the cone

$$\tilde{K}(\hat{z}) = \{z \in R^{n+1} \mid \langle \phi_x(\hat{x}, y), x - \hat{x} \rangle \leqq z_{n+1} - f(\hat{x}), y \in Y(\hat{x})\} , \tag{12}$$

where

$$Y(\hat{x}) = \{y \in Y \mid \phi(\hat{x}, y) = f(\hat{x})\} , \tag{13}$$

is a convex approximation to the set

(14) $$\tilde{\Omega}_0 = \{z \in R^{n+1} \mid \phi(x, y) - z_{n+1} \leqq 0,\ y \in Y\}$$

at the point $\hat{z}$. In addition,

(15) $$\tilde{K}_i = K_i \times R^1, \quad i = 1, \dots, s,$$

are obviously convex approximations to the sets

(16) $$\tilde{\Omega}_i = \Omega_i \times R^1, \quad i = 1, \dots, s,$$

at $\hat{z}$. According to Theorem 3, there exists a number $\mu \leqq 0$, a vector $\tilde{a}_0 \in \tilde{K}'(z)$, and vectors $\tilde{a}_i \in \tilde{K}'_i$, $i = 1, \dots, s$, not all zero, such that

(17) $$\mu F_z(\hat{z}) + \tilde{a}_0 + \tilde{a}_1 + \dots + \tilde{a}_s = 0.$$

From (12) it follows that $\tilde{a}_0 = (a_0^T, a_0^{n+1})^T$, $a_0 \in R^n$, lies in the convex cone generated by vectors of the form $(\phi_x^T(\hat{x}, y), -1)^T$, $y \in Y(\hat{x})$, i.e. a_0 lies in the cone generated by $\phi_x(\hat{x}, y)$, $y \in Y(\hat{x})$. Otherwise speaking, $a_0 \in K'(\hat{x})$, where

(18) $$K(\hat{x}) = \{x \in R^n \mid \langle \phi_x(\hat{x}, y), x - \hat{x} \rangle \leqq 0,\ y \in Y(\hat{x})\},$$

and, in turn, $K(\hat{x})$ is a conical approximation to the set

(19) $$\Omega_0 = \{x \in R^n \mid \phi(x, y) \leqq f(\hat{x}),\ y \in Y\}$$

at the point $\hat{x}$. This is in overall agreement with [9].

Relation (15) implies that $\tilde{a}_i = (a_i^T, 0)^T$, $a_i \in K'_i$, $i = 1, \dots, s$. Now (17) splits into

(20) $$a_0 + a_1 + \dots + a_s = 0,$$

(21) $$-\mu + a_0^{n+1} = 0.$$

If $\mu = 0$, then also $a_0^{n+1} = 0$, and because of the mentioned structure of $\tilde{K}'(\hat{z})$ this is only possible when also $a_0 = 0$, i.e. $\tilde{a}_0 = 0$ in this case. Then necessarily at least one of a_i, $i = 1, \dots, s$, must be nonzero. These results are summarized in the following theorem.

Theorem 4. Let Ω be given by (5) and let $K_1, \dots, K_s$ be the corresponding conical approximations at $\hat{x}$, which point is a solution to the static minmax problem (7)–(8). Suppose that the family of functions $\phi(\cdot, y) - f(\hat{x})$, $y \in Y$, is nondegenerate at $\hat{x}$. Then there exists a vector $a_0 \in K'(\hat{x})$, and vectors $a_i \in K'_i$, $i = 1, \dots, s$, not all zero, such that

$$a_0 + a_1 + \dots + a_s = 0.$$

Again, a_0 can be respectively expressed using an analogy with (6), however, owing to the preceding discussion now $1 \leqq r \leqq n + 1$. Various alternatives, as mentioned earlier, are also in this case at a hand, see [7] and [9].

5. CONCLUSIONS

It was shown that general approach to the solution of mathematical programming problems given in [1] can be applied in a straightforward way to deal with the case of infinitely many inequality type constraints. The obtained necessary optimality conditions include all previous results as special cases. In addition, it was demonstrated that after a suitable transformation also general static minmax problems can be treated using the presented conditions.

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